Part III

Static Games of Incomplete Information

As we have seen so far the basic structure of a game, be it in normal form or extensive form, proved useful in providing a formal structure to which we can apply the game-theoretic solution concept of our choice. If it is a variant of the Prisoner's dilemma, we feel comfortable predicting static Nash equilibrium behavior, and in dynamic settings we have explored the merits of requiring credible, sequentially rational strategies for the players.

In all the examples, and appropriate tools for analysis, we have made an important assumption: that the game played is common knowledge. In particular, we have assumed that the players are aware of who is playing, what the possible actions of each player are, and how outcomes translate into payoffs. Furthermore, we have assumed that this knowledge of the game is itself common knowledge, which gave us the methodological foundation to develop such solution concepts as iterated elimination of dominated strategies, rationalizability, and most importantly, Nash equilibrium and Subgame Perfect equilibrium.

Little effort is needed to convince any experienced person that these ideal situations are not able to capture many interesting strategic interactions. Let's consider one of our first examples, the duopoly market structure. We have analyzed both the Cournot and the Bertrand models of duopolistic behavior, and for each we have a clear and precise, easily understood outcome. One assumption we made is that the payoffs of the firms, like their action spaces, are common knowledge. However, is it reasonable to assume that the production technologies are indeed common knowledge? And if they are, should we believe that the efficiency of workers in each form are known to the other firm? And more generally, that the cost function of each firm is known to their opponent?

Clearly, it is more convincing to think that firm's have a good idea about their opponents costs, but do not know exactly what they are. Similarly, if one thinks of a reasonable variant to the prisoner's dilemma, I may not know how much honor my fellow accomplice has, and whether he will be willing to rat on me for the sake of getting a reduced sentence? And whether my partner in the battle-of-the-sexes game really likes football games or the opera?

Yet, as realistic as these examples are, the tool-box we have developed so far is not adequate to address these situations. How do we think situations in which players have some idea about their opponents' characteristics, but don't know them exactly? A careful thought might lead you to see that this is not so different than the situation in a simultaneous move game: a player does not know what action his opponents are taking, but he has to form an idea in order to choose his best response, and we identified this idea as the player's *belief*. Furthermore, we developed our tools of analysis that required these beliefs, and the appropriate best responses, to be *consistent*, which we called *equilibrium*.

As it turns out, in the mid 1960's John Harsanyi not only realized this similarity, but developed a very elegant, and extremely operational way to capture the ideas of beliefs over not only the action's of one's opponents, but also over their other characteristics, or their *types*, which led him to be the third Nobel laureate to share the prestigious prize with John n Nash and Reinhard Selten in 1994. We call games that incorporate the possibility that players may have different types *games of incomplete information*.

Like with games of complete information, we must require these beliefs to be consistent in order to develop some notion of equilibrium analysis. This is done, following Harsanyi's developments, by making a very strong assuming about the intelligence of our players: we assume that common knowledge reigns over the possible types of players, and over the likelihood that each type prevails.

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Incomplete Information: The Ideas

As before, we have the two "physical" components of a game, which are:

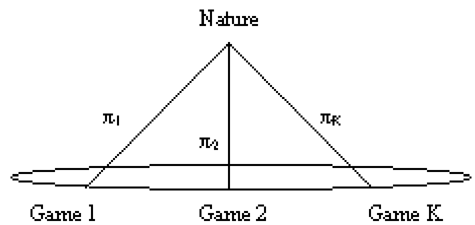
- $N = \{1, 2, ..., n\}$ is the set of players
- $A_i,\ i \in N$, are the actions spaces for each player.

Thus, we are considering an n-player simultaneous-move game. However, the distinction is that Instead of having a single utility function for each player that maps profiles of actions into payoffs (which we had in games of complete information), games of incomplete information will allow players to have one of possibly many utility functions, thus capturing the idea that players' preferences may not be common knowledge.

To capture this idea we assume that first, before the game is played, *Nature* chooses the preferences, or *types* of the different players.[1] A way of thinking about this methodology is that first, Nature chooses one of many games, and then the game is played. Clearly, if Nature is randomly choosing between one of many

[1] As we will see soon, two different types of a player may not necessarily differ in that player's preferences, but may differ in knowledge that the player has about differences in preferences of other players. Since this is a bit more subtle, we leave it for later, when we are more comfortable with the notion of Bayesian games and incomplete information.

possible games, then it must be the case that there is a probability distribution over the different games. For example, in Figure ?? there are K possible games, each game $k \in \{1, 2, ..., K\}$ is played with probability π_k.



Since what we are trying to capture is that players may have different characteristics, but the physical nature of the games is fixed, then the set of players N and the action spaces A_i for each player $i \in N$ will be the same for all the possible games that Nature chooses.

Since players may have different types, we must address the issue of what players know when they play the game. Recall one of the motivating examples of the prisoner's dilemma, in which you may be uncertain about the motives of your accomplice, and possibly, your accomplice may be uncertain about your own motives. For us to be able to discuss optimal behavior, we have to let our players maximize their payoffs given their beliefs about the situation they are in, just as we did with the analysis of games of complete information. Thus, we must assume that players know their own preferences, which in turn will allow us to analyze a player's best response given his assumption on the behavior of his opponents.

So, this leaves us with the final piece of the puzzle: if players know their own preferences, but they do not know the preferences, or types of their opponents, then what must they know for us to introduce a set of tools that will allow for equilibrium-like analysis? In particular, we need to let our players form *rational conjectures* on the preferences and types of their opponents, so that they themselves can for prediction about their opponents behavior. For this reason we assume that although each player does not necessarily know the *actual* preferences of his opponents, he does know the precise way in which Nature chooses these preferences.

That is, each player knows the probability distribution over types, and this itself is common knowledge among the players of the game.

Thus, we have complemented the physical components, payoffs and action spaces, with preferences and information components as follows:

- Nature: a probability distribution over types of players.

- Each player knows his type but not the other players' types.

- The probability distribution over types is common knowledge.

To see how this all fits together, it would be illustrative and useful to consider the following example, which we will call the Modified Prisoner's Dilemma. Imagine that two players play the prisoner's dilemma, but that there is uncertainty about the type of player 1, who can be either *altruistic*, a, or *rational*, r. In contrast, player 2 is known to be a rational type. Thus, we can think of Nature selecting one of the two following games:

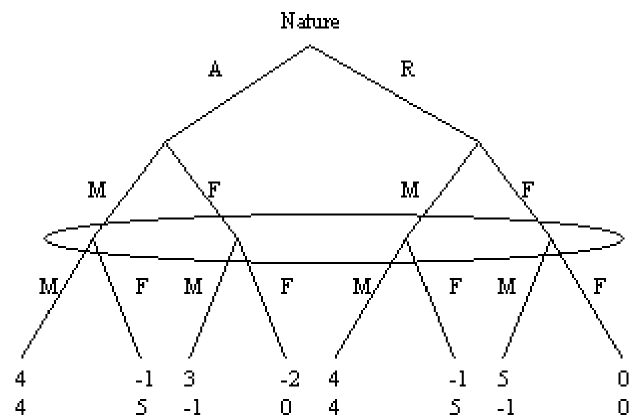
Game 1 (altruistic)

	M	F
M	4,4	-1,5
F	3,-1	-2,0

Game 2 (rational)

	M	F
M	4,4	-1,5
F	5,-1	1,1

We can use the notation T_i to define the set of types of player i, in which case we have, $T_1 = \{a, r\}$, $T_2 = \{r\}$. As we explained earlier, we need a probability distribution over T_1, and in this case we can define this distribution with a single probability, $\pi_a \in [0, 1]$ which is the probability that player 1 is of type a (and with probability $\pi_r = 1 - \pi_a$ he is type r). The extensive form of this game can therefore be described in Figure ??:



Notice an important feature of the extensive form. Namely, player 1 has two information sets whereas player 2 only has one. The reason is precisely that we assume that *player 1 knows his type*, which player 2 only has a (correct) *belief* about the type of player 1. When it is player 1's turn to move, he knows his own type but he does not know the action chosen by player 2 (as a result of the simultaneous move game). When it is 2's turn to move, he neither knows which of the types of player 1 he is facing, nor does he know the choice of player 1.

In a way, it is *as if* player 2 faces two possible players, and nature choose against which player he is playing the game. This is exactly what is captured by the extensive form, and we can translate this extensive form into a normal form game.

Notice that in the normal for representation of this extensive form game, player 1 must have *four* pure strategies: in each of his information sets he has two actions to choose from. Lets define a strategy of player 1 as $xy \in \{MM, MF, FM, FF\}$ where x describes what an altruistic player 1 does, and y what a rational one does. This is a preview of what we will soon see more generally: when we introduce incomplete information, a strategy of a player is now a prescription that tells each *type* of a player what he should do if this is the type that Nature chose for the game.

Once we have pure strategy sets for each player, each pair of pure strategies will give rise to a path of play that starts with Nature's choice, and then follows with the simultaneous actions of both players. In this example, say, if player 1 plays M is he altruistic, and F if he is rational (MF), and if player 2 plays M, then with probability π_a the outcome will yield payoffs of $(4, 4)$, and with probability $(1 - \pi_a)$ the payoffs will be $(5, -1)$. Thus, the pair of payoffs from the pair of strategies (MF, M) is,

$$u_1 = \pi_a 4 + (1 - \pi_a)5, \text{ and } u_2 = \pi_a 4 + (1 - \pi_a)(-1).$$

For concreteness, assume that $\pi_a = \frac{1}{2}$, in which case these payoffs will be $(u_1, u_2) = (4.5, 1.5)$. In a similar way, we can fill in the other nine payoff pairs from the other combinations of pure strategies to get the following matrix form game (verify these!):

	M	F
MM	$4, 4$	$-1, 5$
MF	$4.5, 1.5$	$0, 3$
FM	$3.5, 1.5$	$-1.5, 2.5$
FF	$4, -1$	$-0.5, 0.5$

Now, once we have this new way of modelling the game, we can apply our old tools of equilibrium analysis. Namely, we can apply Nash equilibrium to this game and easily see that (MF, F) is the unique (dominant strategy, in this case) Nash equilibrium.

Note, however, that right now it may be a bit confusing to fully appreciate where we started, and where we ended up. In particular, this new matrix games has two players with expected utilities derived from the probability distribution over the different types of each player (in this case, only player 1 had types). This was John Harsanyi's genius solution: we cannot perform equilibrium analysis unless we assume that each player knows the distribution of his opponents' types, and once he assumes some behavior of these different types, the he can calculate his own expected utility from his different actions. In this way, Harsanyi changed the complex and challenging concept of *incomplete information*, into a well known game of *imperfect information*, in which Nature chooses the players' types, and we then use our standard tools of analysis.

There is a potentially confusing point that is worth clarifying before we go on to the more generalized and formal definitions. We can see how each *type of* player i, say, having beliefs about the other players' types and actions, can calculate his expected utility. But by aggregating the types of all the types of each player into some "meta" player (e.g., player 1 in the example), are we not going a bit too far? Why will this player's optimal response also be the optimal response of each of the different types from which he is composed? This is precisely where the definitions of strategies comes in. When player 1 in the combined matrix game chooses the strategy MM, for example, then the rational type of player 1 is playing a dominated strategy, M instead of F, but the altruistic type is playing his dominant strategy, M. This is precisely why MF in the combined matrix game is better than MM: it gives one type higher payoffs, and thus gives the "meta" player higher expected payoffs. Once again, an elegant part of Harsanyi's solution.

Since we are now finishing up with the ideas of this chapter, and will soon move on to the more general definitions and applications, there is an important note worth thinking about. Harsanyi's elegant solution to the problem of incomplete information is not something that we get for free. We are taking a big leap of faith by assuming that the distribution of types, in the example above given by π_a and π_r, are common knowledge. This is a step beyond the assumptions we made for Nash Equilibrium, where we required players to form conjectures, or beliefs, that in equilibrium have to match the choices of their opponents. Here we are asking for more: all the players agree on the way in which players' types differ from each other, and on the way that Nature chooses among these profiles of types. The alternative approach is not to assume this, and to forgo equilibrium analysis. Thus, when we apply our models and tools to realistic situations, we should be mindful of the strong informational assumptions we are making, which should guide us in putting confidence in our predictions or prescriptions.

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Normal form Representation of Static Bayesian Games

18.1 Players, Actions, Information and Preferences

Recall that we represented a normal form game of complete information in chapter 2 by $\langle N, \{S_i\}_{i=1}^n, \{u_i(\cdot)\}_{i=1}^n \rangle$ where $N = \{1, 2, ..., n\}$ is the set of players, S_i is the action, or strategy space of player i, and $u_i : S \to \Re$ is the utility (payoff) function of player i where $S \equiv S_1 \times S_2 \times \cdots \times S_n$.

Now we want to capture the fact that players know their own payoffs from different actions, but that they do not know the payoffs of other players. For this we introduced three new ideas. First, we said that before the game is actually played, *Nature* chooses *types* for the different players, where each type can be the information that the player has about his own payoffs, or more generally, information he might have about other relevant attributes of the game. Thus, we introduce *type spaces* for each player which represent the sets from which nature chooses these types.

To capture the idea that there is common knowledge about the way in which nature chooses between profiles of types of players, we introduce the notion of a *common prior*, which is the commonly known probability distribution over types. That is, every agent knows his own type, and he uses this prior to form posterior beliefs over the types of other agents. Thus we get:

Definition: The *normal form representation* of an n-player *static Bayesian game* is

$$\langle N, \{A_i\}_{i=1}^n, \{T_i\}_{i=1}^n, \{u_i(\cdot\,; t_i), t_i \in T_i\}_{i=1}^n, \{p_i\}_{i=1}^n\rangle$$

where $N = \{1, 2, ..., n\}$ is the set of players, A_i is the *action set* of player i, $T_i = \{t_i^1, t_i^2, ..., t_i^{k_i}\}$ is the *type space* of player i, $u_i(\cdot; t_i) : A \times T_i \to \Re$ is the *type dependent* utility function of player i, where $A \equiv A_1 \times A_2 \times \cdots \times A_n$, and p_i describes the *belief* of player i with respect to the uncertainty over the other players' types, that is $p_i(t_{-i}|t_i)$ is the (posterior) conditional distribution on t_{-i} (all other types but i) given that i knows his type is t_i.

We assume that the timing of the game is as follows:

1. Nature chooses a profile of types.
2. Each player learns his own type and uses p_i to form beliefs over the other types.
3. Players simultaneously (therefore, *static game*) choose actions from there sets A_i, $i \in N$.
4. payoffs $u_i(a_1, a_2, ..., a_n; t_i)$ are realized for each player.

Note that in our setup we have player's utility $u_i(\cdot; t_i)$ not depend on t_{-i}, which means that one player's payoff does not depend on the types of the other players. We call this setup *private values*, since each type's payoff depends only on his private information. This setup will not capture all the interesting examples we will analyze, and for this reason we will later discuss the case of *common values* where $u_i(a_1, a_2, ..., a_n; t_1, t_2, ..., t_n)$ is allowed for. Yet, for expositional clarity, let's deal with the simpler case first.

18.2 Deriving Posteriors from a Common Prior: A Player's Beliefs

In the definition of a Bayesian game we introduced the idea of a common prior, that is, the distribution over the choices made by Nature. What does it mean for player to use the common prior, and once he knows his type derive a posterior belief about the distribution of the other players' types? This is, not surprisingly, just a simple application of Bayes' rule.

The essence of Bayes' rule is to provide a consistent mathematical rule that derives the way in which you should change your initial (prior) beliefs in the light of new evidence, resulting in a posterior belief. In our application, it allows a player who does not know his type, but knows the initial distribution of all types including his own, to derive new beliefs once he learns his type, and have these beliefs consistent with the prior.

Formally, imagine that there are two of many possible states, S (say, it will be sunny) and H (say, the waves will be high), that can occur exclusively or together according to some prior distribution $p(\cdot)$. That is, $p(\cdot)$ describes the probabilities assigned to any combination of these states being true. Let $p(S)$ be the prior probability that it will be sunny, $p(H)$ be the prior probability that the waves will be high, and $p(S \cap H)$ be the prior probability that it will be both sunny and the waves will be high. Let's imagine that you wake up, and see that it is sunny; what can you infer about the probability that the waves are high? It is not necessarily true that it is $p(H)$, because you just learned that it is sunny, and this is new information. this is where Bayes rule comes in handy. It precisely tells you what is the probability of state H given that you know state S happened, and is given by:

$$\Pr\{H|S\} = \frac{p(S \cap H)}{p(S)}$$

The intuition works as follows. If we know that S occurred, there are now two possibilities: either H occurred or not. Thus, we can think of two states: the first being that both S and H occurred, which happens with probability $p(S \cap H)$, and the second that S occurred but H did not occur, which happens with probability $p(S \cap [\text{not-}H])$. Now, what is the probability that S occurs? It must be the sum of the above, $p(S) = p(S \cap H) + p(S \cap [\text{not-}H])$. Now, if I know that S occurred, then *conditional on this knowledge*, the likelihood of H occurring is the *relative likelihood* of both S and H occurring, among all the states in which S occurs, which is,

$$\Pr\{H|S\} = \frac{p(S \cap H)}{p(S \cap H) + p(S \cap [not - H])} = \frac{p(S \cap H)}{p(S)}$$

To bring things close to home, consider the following example. Imagine that there are two players, each having two possible types (say, altruistic and rational) so that t_i^k is type $k \in \{a, r)$ of player $i \in \{1, 2\}$. Natures chooses these types according to a prior over the four possible type combinations, (t_1^k, t_2^l), where $k, l \in \{a, b\}$. Now let the following joint distribution matrix describe Nature's prior:

	t_2^a	t_2^r
t_1^a	1/6	1/3
t_1^r	1/3	1/6

That is, the prior probability that either player 1 is altruistic and player 2 is rational, is equal to the prior probability that player 1 is rational and player 2 is altruistic, and this probability is $\frac{1}{3}$. Similarly, the prior probability that both are altruistic is $\frac{1}{6}$ as is the prior that both are rational. Based on the "common prior" assumption, everybody takes as give that this is how nature distributes the types. Now, when player 1 learns that he is altruistic, what will be his belief about player 2's type? Using Bayes' rule we have,

$$p_1(t_2^a | t_1^a) = \frac{\Pr\{t_1^a \cap t_2^a\}}{\Pr\{t_1^a\}} = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{3}} = \frac{1}{3} ,$$

and similarly,

$$p_1(t_2^r | t_1^a) = \frac{\Pr\{t_1^a \cap t_2^r\}}{\Pr\{t_1^a\}} = \frac{\frac{1}{3}}{\frac{1}{6} + \frac{1}{3}} = \frac{2}{3} .$$

18.3 Strategies and Bayesian Nash Equilibrium

The representation of a game described above has action sets, A_i, for each player $i \in N$. However, as we demonstrated in the example of the modified prisoner's dilemma above, to define *strategies* we need to specify what each type of player will choose when Nature calls upon this type to act. Fir this we have,

Definition: Consider a static Bayesian game $\langle N, \{A_i\}_{i=1}^n, \{T_i\}_{i=1}^n, \{u_i(\cdot\,; t_i), t_i \in T_i\}_{i=1}^n, \{p_i\}_{i=1}^n \rangle$.

A *pure strategy* for player i is a function

$$s_i : T_i \to A_i$$

that specifies a pure action $s_i(t_i)$, which is what agent i will choose to do when his type is t_i. Similarly, a *mixed strategy* is a probability distribution over a player's pure strategies.

This is a convenient way to specify strategies. It is as if players choose their *type-contingent* strategy before they learn their types, and then play according to their strategy. This is useful because it allows us to talk about the beliefs of players over strategies of their opponents, when their opponents can be of different types. As you can now see, the modified prisoner's dilemma we used to demonstrate the central ideas is just a special case of this more general Bayesian game.

To see how the setup we developed gives players the ability to form beliefs over their opponents's behavior, and then use these beliefs to calculate their own expected utility from their actions, consider the example above with two players and two types, $T_i = \{t_i^a, t_i^r\}$, and imagine that each player had two pure actions as in the prisoner's dilemma, $A_i = \{M, F\}$. Now consider the case where player 1 believes that player 2 is using the following strategy

$$s_2(t_2) = \begin{cases} M & \text{if } t_2 = t_2^a \\ F & \text{if } t_2 = t_2^r \end{cases}$$

If player 1 is of type t_1^a, then from his own perspective, his expected utility from playing M will be,

$$\begin{aligned} Eu_1\left(M, s_2\left(\cdot\right); t_1^a\right) &= p_1(t_2^a|t_1^a) \cdot u_1(M, s_2(t_2^a); t_1^a) + p_1(t_2^r|t_1^a) \cdot u_1(M, s_2(t_2^r); t_1^a) \\ &= p_1(t_2^a|t_1^a) \cdot u_1(M, M; t_1^a) + p_1(t_2^r|t_1^a) \cdot u_1(M, F; t_1^a) \end{aligned}$$

- [EXPAND: a pure strategy + nature's choices make player i face a mixed strategy]

- [EXPAND: like in extensive form games, we specify i's strategy for all information sets, even those that are not reached by Nature. the reason is that to allow player j to form beiefs over i's behavior, player j needs to combine the beliefs from his posterior over i's types, with his beliefs over what each type t_i of player i will do.]

Once we defined the Bayesian game and the strategies for each player, we can easily define our newest solution concept, which just builds on the familiar Nash equilibrium as applied to the Bayesian game we just defined. Namely,

Definition: In the static Bayesian game $\langle N, \{A_i\}_{i=1}^n, \{T_i\}_{i=1}^n, \{u_i(\cdot\,; t_i), t_i \in T_i\}_{i=1}^n, \{p_i\}_{i=1}^n\rangle$, a strategy profile $s^* = (s_1^*(\cdot), s_2^*(\cdot), ..., s_n^*(\cdot))$ is a *pure strategy Bayesian Nash Equilibrium* if for every player i, and for each of player i's types, $s_i^*(\cdot)$ solves:[1]

$$\sum_{t_{-i}\in T_{-i}} p_i(t_{-i}|t_i)\cdot u_i(\underbrace{s_i^*(t_i)}_{\substack{\text{action}\\ \text{for } i}}, \underbrace{s_{-i}^*(t_{-i})}_{\substack{n-1 \text{ action for}\\ \text{opponents}}}; t_i) \geq \sum_{t_{-i}\in T_{-i}} p_i(t_{-i}|t_i)\cdot u_i(a_i, s_{-i}^*(t_{-i}); t_i) \text{ for all } a_i \in A_i\,. \tag{18.1}$$

That is, regardless of the type realization, no player wants to change his strategy $s_i^*(\cdot)$.

Once we take a static game of incomplete information and transform it to a Bayesian game as described above, then the "Bayesian" part is in the fact that for each player's type realization, we compute his beliefs about the actions of his opponents using Bayes' rule. Payoffs from strategy profiles are the expected utility that is derived from the strategies played by other players, and the mixing that occurs due to the *randomization of nature* that each player faces through his beliefs $p_i(t_{-i}|t_i)$.

Remark 9 *In the definition of Bayesian Nash equilibrium, we can write the condition (18.1) of playing a best response more generally as follows,*

$$E_{t_{-i}}[u_i(s_i^*(t_i), s_{-i}^*(t_{-i}); t_i)|t_i] \geq E_{t_{-i}}[u_i(a_i, s_{-i}^*(t_{-i}); t_i)|t_i] \textit{ for all } a_i \in A_i.$$

Notice that here we have taken the expectations of player i over the realizations of types t_{-i} when player i knows his own type (hence, the expectation, $E_{t_{-i}}[\cdot|t_i]$, is conditional on t_i). Once we write it down this way, it is more general in the sense that it can apply to a continuum of types for each player*. That is, it may be the case*

[1] Note that on the right-hand-side of the inequality, we can replace a_i with $s_i'(t_i)\forall s_i' \in S_i'$. But it suffices to consider any deviation from $s_i^*(\cdot)$ to an action, a_i, instead of the more complex notion of a type dependent strategy, $s_i'(\cdot)$.

that each player has an infinite number of possible types drawn from an interval $T_i = [\underline{t}_i, \bar{t}_i]$, *with cumulative distribution* $F_i(t_i)$ *and density function* $f_i(t_i) = F_i'(t_i)$. *In this case the expected utility of player* i *will be written as an integral (more precisely,* $n-1$ *integrals) which form the expectations over the realizations of the other players' types, and accordingly, their actions derived from their strategies.*

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Applications of Bayesian Nash Equilibria

19.1 A Game of Aggression

It is not uncommon for two rivals in some form of conflict to face a decision of whether to be aggressive or to cave in and expose weakness. Be it firms in the marketplace, politicians in government, countries at war, or even kids on the playground, the optimal behavior will depend on some combination of each player's tendency to being aggressive, and on his belief about his opponent's tendency to be aggressive.

To illustrate this idea, consider a simple game of aggression that is not foreign to many teenagers: the game of *chicken*. The 1955 film, Rebel Without a Cause features James Dean as a juvenile delinquent, an introduced one variant of the game of chicken to the silver screen. In this movie, two teenagers simultaneously drive their cars off a cliff. The first one to jump out is *chicken*, and loses the contest. (In the movie, one dies, and in real life, James Dean died in a hotrodding accident just before the film was released.) Many other films featured variations on chicken, a well known example being the 1978 film *Grease* staring John Travolta and Olivia Newton-John. the following example is a variant of this game.

Two teenagers, named 1 and 2, have borrowed their parents' cars, and decided to play the game of "chicken" as follows: They both drive towards each other on a street, and just before impact they must simultaneously choose whether to be "chicken" (C) and move away to the right side, or continue head-on (H). If both play C, then both gain no respect from their friends, but suffer no losses, thus both get a payoff of 0. If i plays H and $j \neq i$ plays C, then i gains all the respect, which is a payoff of w, and j gets no respect, which is 0. In this case they suffer no additional losses and these are the payoffs. Finally, if both play H, they "split" the respect (since respect is considered to be relative...), but an accident is bound to happen and they will be reprimanded by their parents, which imposes a personal loss of k, so the payoff to each kid is $\frac{w}{2} - k$.

There is, however, a potential difference between these two youngsters: The punishment, k, depends on the type of parents they have. For each kid, parents can be either strict or lenient with equal probability, and the draws are independent. If they are strict, then they will beat the living daylight out of their child, and we denote this by the cost being $k = B$. If they are lenient, then they will give their child a long lecture of why his behavior is unacceptable, and we denote this by the cost being $k = L$. Each kid knows the type of his parents but does not know the type of his opponents parents. The distribution of types is common knowledge.

Remark 10 (The following is in exercise format; replace to flow format)

a. Draw the game tree that represents this game in extensive form.

b. Draw the matrix that represents the normal form of the extensive from you did in (a.) above. Be clear as to your choice of strategy spaces for each player.

c. Now assume that $w = 8$, $B = 16$, and $L = 0$. Solve for the Bayesian Nash equilibria of this game.

d. A preacher, who knows some game theory, decided to use this model to claim that moving to a society in which all parents are lenient will have detrimental effects on the behavior of teenagers. Is this right? (Your answer should be supported with an equilibrium argument!

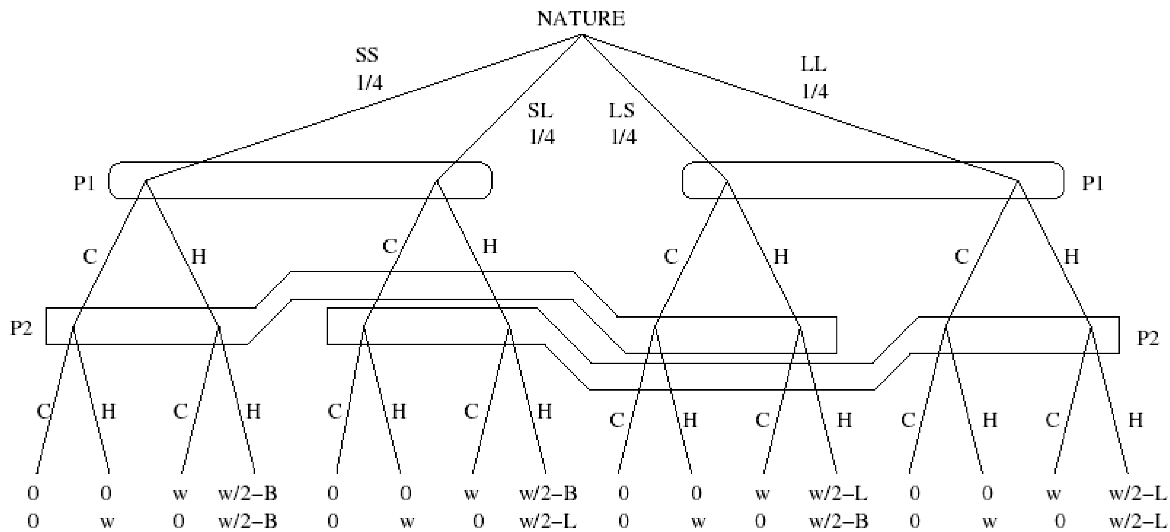

FIGURE 19.1.

		P2			
		CC	CH	HC	HH
P1	CC	$0,0$	$0,\frac{1}{2}w$	$0,\frac{1}{2}w$	$0,w$
	CH	$\frac{1}{2}w,0$	$\frac{3}{8}w-\frac{1}{4}L,\frac{3}{8}w-\frac{1}{4}L$	$\frac{3}{8}w-\frac{1}{4}L,\frac{3}{8}w-\frac{1}{4}B$	$\frac{1}{4}w-\frac{1}{2}L,\frac{3}{4}w-\frac{1}{4}L-\frac{1}{4}B$
	HC	$\frac{1}{2}w,0$	$\frac{3}{8}w-\frac{1}{4}B,\frac{3}{8}w-\frac{1}{4}L$	$\frac{3}{8}w-\frac{1}{4}B,\frac{3}{8}w-\frac{1}{4}B$	$\frac{1}{4}w-\frac{1}{2}B,\frac{3}{4}w-\frac{1}{4}L-\frac{1}{4}B$
	HH	$w,0$	$\frac{3}{4}w-\frac{1}{4}L-\frac{1}{4}B,\frac{1}{4}w-\frac{1}{2}L$	$\frac{3}{4}w-\frac{1}{4}L-\frac{1}{4}B,\frac{1}{4}w-\frac{1}{2}B,$	$\frac{1}{2}w-\frac{1}{2}L-\frac{1}{2}B,\frac{1}{2}w-\frac{1}{2}L-\frac{1}{2}B$

FIGURE 19.2.

Plugging $w = 8$, $B = 16$ and $L = 0$ into the matrix and highlighting best responses gives

		P2			
		CC	CH	HC	HH
P1	CC	$0, 0$	$0, 4$	$0, 4$	$0, 8$
	CH	$4, 0$	$3, 3$	$3, -1$	$2, 2$
	HC	$4, 0$	$-1, 3$	$-1, -1$	$-6, 2$
	HH	$8, 0$	$2, 2$	$2, -6$	$-4, -4$

making (CH, CH) the only Pure-Strategy Bayesian Nash Equilibrium.

In fact, this is the *only* BNE. We can rule out others by strictly dominated strategies: for each player, CC is strictly dominated by CH, and against all strategies other than CC, CH is a strict best response.

FIGURE 19.3.

If all parents were lenient, there would be no uncertainty in this game, and we could model it as a standard two-player simultaneous game:

	C	H
C	$0, 0$	$0, 8$
H	$8, 0$	$4, 4$

In this game, it is easy to see that H is strictly dominant, so the only equilibrium will be (H, H). Thus, the preacher is right – if all parents were lenient, there would be more accidents (an accident would occur every time the game was played, instead of only one quarter of the time.)

FIGURE 19.4.

19.2 Auctions and Competitive Bidding

The use of auctions to sell goods has become as common as apple-pie and french-fries, thanks to the internet auction house eBay® that has invaded many households across the globe. Previously, a common vision of auctions were the sale of a Picasso or Renoir in one of the prestigious auction houses of Sotheby's, that was created in 1744, and Christie's, that was created in 1766. It seems that the use of auctions dates back much further. (For a history of auctions see Ralph Cassidy, Jr. *Auctions and Auctioneering*, University of California Press, 1967).

The use of game theory to analyze both behavior in auctions, and the design of auctions themselves, was introduced by William Vickrey (1961), another Nobel laureate, and was followed by a large literature that is still expanding. The "big push" of game theoretical research on auctions happened after the successful use of game theory to advise both the US government, and the bidding firms, when the Federal Communications Commission decided to auction off the airwaves for use by telecommunication companies. This was considered so successful, that a reference to the work of many game theorists appears in an article in *The Economist* titled "Revenge of the Nerds," (*The Economist*, July 23, 1994, p. 70).

As we will see, auctions have many desirable properties, and these made them a favorite choice by the US Federal Acquisition Rules as the preferred (by law) form of procurement in the public sector. They are usually very transparent, they have well defined rules, they usually allocate the auctioned good to the party who values it the most, and they are not too easy to manipulate (if well designed).

Generally speaking, there are two common types of auctions. The first type, as we will refer to it, is the *open auction* in which the bidders observe some dynamic price process that evolves until a well defined winner emerges. There are two common forms of the open auction:

- **The "English Auction"**: This is the classic auction we will see in a movie (e.g., *The Red violin*), where the bidders are all in a room (or nowadays sitting by a terminal), and the price of the good goes up as long as someone is willing to bid it up higher. Once the last increase is no longer challenged, the last bidder to increase the price wins the auction and pays that price.

(The price may start at some minimum threshold, which would be the seller's "reservation" price.)

- **The "Dutch Auction"**: In many ways this less common auction is the inverse of the English auction. As with the English auction, the bidders observe the price change before them, but instead of starting low and rising by pressure from the bidders, the price starts at a prohibitively high value, and the auctioneer drops the price gradually. Once a bidder shouts "buy", the auction ends and the bidder gets the good at the last price. This auction was and still is popular in the flower markets of the Netherlands, hence the name of the auction.

The second common type of auctions is the *sealed-bid auction*, where participants write down their bids and submit them without knowing the bids of their opponents. The bids are collected, and the highest bidder wins and pays a price that depends on the auction rules. As with open auctions, there are two common forms of sealed-bid auctions that prevail:

- **The "First-Price Sealed-Bid Auction"**: This very common auction form has each bidder write down his bid, or *willingness to pay* in an envelope, and the envelopes are opened simultaneously. The highest bidder wins, and then pays a price equal to his own bid.

- **The "Second-Price Sealed-Bid Auction"**: As with the First-Price Sealed-Bid Auction, each bidder writes down his bid in an envelope, and the envelopes are opened simultaneously. The difference is that although highest bidder wins here too, he then pays a price equal to the second highest bid, or the *highest losing bid*.

We will start by analyzing the second-price sealed-bid auction because it is the one that lends itself most easily to formal analysis. As we will see, it has a very tight relationship with the English auction.

19.2.1 Second-Price Sealed Bid Auctions: Independent Private Values

We describe our auction setting as a game with $N = \{1, 2, ..., n\}$ being the set of players, and these represent the bidders. We refrain from considering the seller as a player, since once the auction rules are given by him, he plays no role in bidding behavior and the outcomes that follow.

We assume that every player has an *independent private value* of obtaining the good. For example, you and I can both bid on a Beanie-Baby animal on eBay®, and each of us has a value for obtaining the fluffy little critter. However, our values may be one of many, and we each may not know the value of the other, hence their private nature. Furthermore, if we assume away the possibility that we are buying it for resale value and instead only want it for personal use, there is no reason for my value to depend on yours and vice versa, hence the independence.

To capture this idea we introduce the notion of *types*. In particular, assume that the value for player i of obtaining the good is $v_i \geq 0$, and this value can be one of many values given by $v_i \in [\underline{v_i}, \overline{v_i}]$. We further assume that the actual value of v_i is drawn from this interval $[\underline{v_i}, \overline{v_i}]$ according to the cumulative distribution function $F_i(\cdot)$, where $F_i(v') = \Pr\{v_i \leq v'\}$. The utility of a player who does not get the good is normalized to zero, and the utility of a player who gets the good and pays a price $p \geq 0$ is $u_i = v_i - p$.

Since we are using the methodology of Bayesian games, we will assume that every player knows the distribution of all types of the other players, and uses the $n - 1$ cumulative distribution functions $F_j(\cdot)$, $j \neq i$, to form beliefs about the types t_{-i} of the other players.

The rules of the game are as follows. First, players learn their private valuations, but only know the probabilistic information about their opponents. Second, each player submits a bid $b_i \geq 0$, which is the action each player can choose. Finally, the bids are collected, the highest bidder wins, and he pays a price equal to the second highest bid. Thus, we can write the payoff function of each player i as,[1]

$$u_i(b_i, b_{-i}; v_i) = \begin{cases} v_i - b_j^* \text{ if } b_i > b_j \forall j \neq i \text{ and } b_j^* \equiv \max\{b_j\}, j \neq i \\ 0 \text{ if } b_i \leq b_j, \text{for some } j \neq i \end{cases}$$

Given that a player's action is his bid, we know from the development of Bayesian games that a player's strategy must be a mapping that assigns a bid to each of the player's types, in this case, private valuations. Thus, a strategy for player i is a function $s_i : [\underline{v_i}, \overline{v_i}] \to \Re^+$ that assigns a non-negative bid to each of his possible valuations.

Now, we can write down the utility of player i with valuation v_i, as a function of his own bid b_i and the strategies used by the other players $s_j(\cdot)$, $j \neq i$. First, it is worth writing down a short-hand expression for player i's expected utility as follows:

$$E_{t_{-i}}[u_i(b_i, s_{-i}(v_{-i}); v_i)|v_i] = \Pr\{i \text{ wins and pays } p\} \cdot (v_i - p) + \Pr\{i \text{ looses}\} \cdot 0$$

In more specific terms, taking into account the rules of the second-price sealed bid auction, the probability that player i wins is equal to the probability that all the other bids are below b_i, and in this event the price player i pays is the second highest bid. What is the probability that all the other bids are below b_i? This is a relatively easy expression due to the independence assumption on the distribution of valuations.

First notice that if player $j \neq i$ is using the bid function $s_j(v_j)$, then the probability that i's bid is higher than j's bid is clearly equal to the probability that j is a type (has a value) that bids less than b_i, but without knowing exactly what j's strategy is we cannot write this probability down. So, let's make a reasonable assumption on the bidding behavior of player:

Assumption: The higher a player's valuation, the higher is his bid: if $v_j' > v_j''$ then $s_j(v_j') > s_j(v_j'')$.

[1] Notice that we resolve the possibility of two players tied with the highest bid by assuming that then neither gets the good and both pay nothing. This can be easily changed to have them randomly be the winner, and pay the second highest price, which would be equal to their own bid (since the two tied as the highest bid). This would add negligible complications, some that will be mentioned later.

Once we make this reasonable assumption, which we will later see arise naturally in this setting, it implies that j's bid function is *invertible*, meaning that for every bid b that j can make, there is a unique type of player j that makes this bid, and he is $s_j^{-1}(b)$. Now, we have a very simple expression for the probability that i's bid is higher than j's bid, which is,

$$\Pr\{s_j(v_j) < b_i\} = \Pr\{v_j < s_j^{-1}(b_i)\} = F_j(s_j^{-1}(b_i)).$$

Now we will use the fact that the different types are drawn independently, that is, the valuations of the players are independent random variables, so that the probability that i's bid is higher than all other bids is just the multiplication of the probabilities $F_j(s_j^{-1}(b_i))$, $j \neq i$. Thus, the expected utility of player i is,

$$E_{t_{-i}}[u_i(b_i, s_{-i}(v_{-i}); v_i)|v_i] = \overbrace{\prod_{j\neq i}\left[F_j(s_j^{-1}(b_i))\right]}^{\Pr\{i \text{ wins}\}} \cdot (v_i - \overbrace{\max_{j\neq i}\{b_j\}}^{2^{\text{nd}} \text{ highest bid}})$$

Well, after all this we are ready to start looking for a Bayesian Nash equilibrium of the second-price sealed-bid auction, which at this stage may look like quite a daunting task. However, it turns out that the rules of this auction result in a rather remarkable result:

Proposition 15 *In the second-price sealed-bid auction, each player has a weakly dominant strategy which is the bid his true valuation, that is,* $s_i(v_i) = v_i$ *for all* $i \in N$.

If you are not already familiar with this famous result, first discovered by Vickrey, then it is indeed quite remarkable. As you will see, it is also quite straightforward once the analysis is laid out, as we will do now. We will prove this result by showing that for any valuation v_i, bidding v_i weakly dominates a higher bid, and bidding v_i weakly dominates a lower bid.

Consider first a bid $b_i < v_i$. There are two possibilities vis-a-vis the other $n - 1$ bids:

- **case 1:** b_i is greater than all other $n - 1$ bids and i the wins.
- **case 2:** b_i is less than at least one other bid and i does not win.

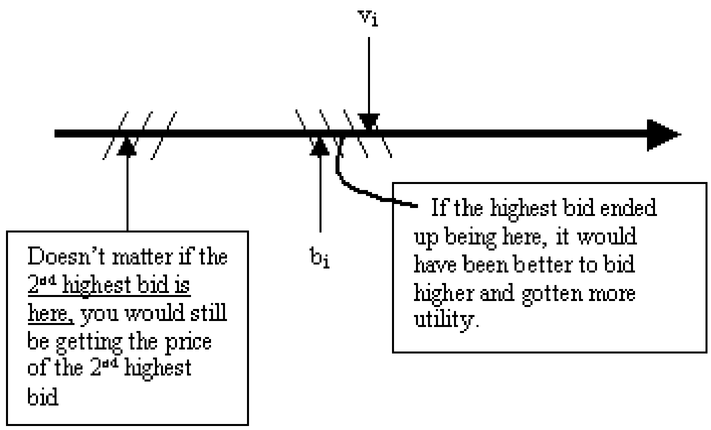

FIGURE 19.5.

In the case 1, bidder i wins and pays $\max_{j \neq i}\{b_j\}$. If instead of bidding b_i he would have bid $v_i > b_i$, then he would still win and pay the same price, making him indifferent between the two options. So, in case 1 bidding his valuation weakly dominates bidding b_i.

In case 2, there are two possibilities. The first is that the highest bid is above player i's valuation. Once again, if instead of bidding b_i he would have bid $v_i > b_i$, then he would still lose the auction since $\max_{j \neq i}\{b_j\} > v_i$, making him indifferent between the two options. So, in this first possibility of case 2, bidding his valuation weakly dominates bidding b_i.

Now we come to the last possible realization, a second possibility of case two in which the second highest bid is above i's bid, but below his valuation: $v_i > \max_{j \neq i}\{b_j\} > b_i$. By bidding b_i, i loses the auction and gets a utility of zero. If, instead, he would have bid v_i, he would have won the auction and received a utility of

$$u_i = v_i - \max_{j \neq i}\{b_j\} > 0,$$

making this a profitable deviation. Thus, we conclude that bidding v_i weakly dominates any lower bid because it is never worse, and sometimes better, as summarized in figure ??.

Exercise 5 *Convince yourself using a similar argument that a bid $b_i > v_i$ will also be weakly dominated by bidding v_i. (Hint: in this case you would either be as well off, or prevent a loss.)*

This fact that every player has a **weakly dominant strategy**, $s_i(v_i) = v_i$, implies the following important corollary:

Corollary 16 *In a second-price sealed-bid auction with independent private values, each player bidding his valuation is a Bayesian Nash Equilibrium in weakly dominant strategies.*

This result is remarkable not only because of its simple recommendation value, that players bid their valuations truthfully in a second-price sealed-bid auction, but also implies three other attractive features of this auction format:

- – **Feature 1:** In a private values setting, bidders in a second-price sealed-bid auction *do not care* about the probability distribution over their opponents' types, and therefore the assumption of common knowledge of the distribution of types can be discarded when such auctions are analyzed. In particular, it means that we can apply this result even when we think that players have no idea about their opponents' valuations.
  - **Feature 2:** in a second-price sealed-bid auction, even if types are correlated, but values are private, then it is a weakly dominant strategy to bid truthfully. This again means that if we are correct about our private values assumption, then even if we incorrectly think that values are independent, the argument that bidding your true valuation is a weakly dominant strategy is still true.
  - **Feature 2:** in a second-price sealed-bid auction, the outcome is Pareto Optimal, as the result will be that the person who values the good most will be the one who gets it.

Thus, we conclude that the second-price sealed-bid auction has many wonderful characteristics, both resulting in prescription for very simple strategies, and outcomes that are Pareto optimal. As will now briefly see, the second-price sealed-bid auction is closely related to the very common English auction.

Remark 11 *We can change the game to be more appealing with respect to the treatment of tied high bids. Assume that if $m \leq n$ bidders tie with the highest bid, then they are each equally likely to win, and they pay the second highest bid, which by definition is equal to their own bid in the case of a tie. Then, the payoff function of each bidder can be written as*

$$u_i(b_i, b_{-i}; v_i) = \begin{cases} v_i - b_j^* \text{ if } b_i > b_j \forall j \neq i \text{ and } b_j^* \equiv \max\{b_j\}, j \neq i \\ \frac{v_i - b_i}{\#\{\text{highest bidders}\}} \text{ if } b_i \geq b_j \forall j \neq i \text{ and } b_i = b_j \text{ for some } j \neq i \\ 0 \text{ if } b_i < b_j, \text{for some } j \neq i \end{cases}$$

You should be able to convince yourself that the arguments leading to $b_i = v_i$ being a weakly dominant strategy are still valid, and this, the analysis above applies.

L14

19.2.2 English Auctions with Independent Private Values

- public sequential bidding: people in a room increase the last bid to a new one, and everyone observes the price progression.
- The last bidder to increase the current price, without having another bidder increase it further is the winner, and he pays his bid.
- How would we describe this as a game? There is a problem: without discrete increments, there is no best response if $p < v_i$ (similar to the problem we had in the Bertrand model, a player may wish to increase the bid, but there is no optimal increase if bids can be from a continuum).
- Two solutions: either (1) Use discrete action spaces such as dollars and cents. This is realistic, but we cannot apply elegant techniques such as derivatives to find optimal behavior, and this makes the analysis very cumbersome; or (2) change the game without losing the spirit of English auctions, which is what Milgrom and Weber (1982) did in their "button-model" auction. It proceeds as follows:

1. As before, there are n players with valuations $v_i \in [\underline{v_i}, \overline{v_i}]$ drawn according to the cumulative distribution function $F_i(\cdot)$, and one auctioneer that announces the current price, starting at $p = 0$ and rising continuously.

2. Each player has a button that is pressed at the beginning of the game when the starting price is $p = 0$. If the button of player i is continuously pressed and the current price is $p > 0$, this means that player i is willing to pay p if everyone else would drop out of the auction now.

3. Once a player releases his button, he is dropped from the auction and cannot re-enter.

4. The winner is the last person to hold on to his button. The price is the posted price at which the before-last player let go of his button.

- Once again, as with the second-price sealed bid auction, we can set up the strategies for each player (which in this case are maps from type, or valuation, to the price at which a player lets go of his button), and we can then derive the expected utility functions of very player given his beliefs about the other players' strategies and types. However, as with the second-price sealed-bid auction, we have the following remarkable, though more anticipated result:

Proposition 17 *In the button-version English auction, each player has a weakly dominant strategy which is to keep holding his button pressed as long as* $p < v_i$ *and to release it one* $p = v_i$*. This results in a weakly dominant Bayesian Nash equilibrium, that is outcome-equivalent to the second-price sealed-bid auction.*

- This follows precisely because of the following simple observations:

  1. if the "current price" is $p < v_i$, then clearly, player i (who is not the currently highest bidder) should continue holding his button, therefore causing the price to increase further.
  2. if $p > v_i$, player i should have dropped out.

- Thus, the solution is just like in the second-price sealed-bid auction: the player with the highest valuation wins, and he pays a price equal to the second highest valuation.

- Note that on eBay®, the way the auctions work is very similar to this button model since eBay® uses a "proxy bidding" system that takes your bid, but if you are the highest bidder then the current price is one increment above the second highest bidder, in a similar fashion to the button model. Interestingly, the instructions on eBay® suggest that you bid truthfully: "When you place a Bid, enter the maximum amount you are willing to pay for the item. eBay will bid on your behalf only if there is a competing bidder and only up to your maximum amount." (http://pages.ebay.com/education/buyingtips/index.html#bid)

19.2.3 Dutch and First-Price Sealed-Bid Auction

- Question: Is it a weakly dominant strategy to bid your valuation in a first-price sealed-bid auction where the highest bidder wins and pays his bid?

- The answer is generally no: By bidding your valuation, $Eu = 0$ (if you lose you get 0, and if you win you pay your valuation, thus you are left with 0). If, however, you bid less than your valuation, then $Eu > 0$ (there is some probability that you will win, multiplied by some positive utility of your value less your bid).

- How do we solve this game? What is the best response of a player?

- Unlike the very appealing result of second-price sealed-bid auctions, the players do not have a weakly dominant strategy that maps types to bids, and their best response depends on the belief over other people's bids/type-dependent strategies. Thus, the distributional assumptions will matter, and the common knowledge assumption on a common prior has real bite.

- It turns out that the Dutch and first-price sealed-bid auction are closely related in the sense that they have the same set of Bayesian Nash equilibrium.

- (Intuition: the bid in the first-price sealed-bid auction (or the acceptance price in the Dutch auction) is determined by a marginal-cost-marginal benefit trade-off. By lowering your bid incrementally, you increase your margin when you win, which is the marginal benefit. At the same time, the probability of winning becomes lower, which is the marginal cost.

- **(add an example to complete this section)**

19.2.4 Common Values and the Winner's Curse

The private values scenario was one where each player's utility depended on the profile of actions of all players, but only on his own type. This setting is useful in describing some scenarios like how much different people value a hamburger, or a bag of chips, but in many cases the utility of one player will depend on the private information of other players.

Consider, for instance, a house that is on the market – how much would you be willing to pay for it? The answer will depend on two major components: first, your own private value of living in that house, and second, what you expect to get for the house if you choose to sell it at a later date. The valuation of *other people* will therefore enter into your willingness to pay for a house, and this will generally be their own private information. This same argument will apply to a piece of art, a car, or even a movie – you may value a movie more if you think other people value it more, so you can later talk to them about it and all agree on how good (or bad) it was.

We refer to such scenarios as having a *common values* element. To illustrate an extreme example of common values, imagine that two identical oil firms are considering the purchase of a new oil field. It is common knowledge that the amount of oil is either small, worth 10 million dollars of net profits, medium, worth 20 million dollars of net profits, or large, worth 30 million dollars of net profits. Thus, the oil field has one of three values, $v \in \{10, 20, 30\}$. Imagine that it is also common knowledge that these values are distributed so that it is equally likely that the amount is small or large, and twice as likely that it is medium, so that

$$\Pr\{v = 10\} = \Pr\{v = 30\} = \frac{1}{4}, \text{ and } \Pr\{v = 20\} = \frac{1}{2} \ .$$

Now assume that the government, who currently owns the field, will auction it off in a second-price sealed-bid auction, and that before the auction each firm will perform a (free) exploration that will provide some signal about the quantity of oil in the field. Specifically, each player receives a low or high signal, $s_i = \{L, H\}$, which is correlated with the amount of oil as follows:

- if $v = 10$, then $s_1 = s_2 = L$;
- if $v = 30$, then $s_1 = s_2 = H$;
- if $v = 20$, then either $s_1 = L$ and $s_2 = H$, or $s_1 = H$ and $s_2 = L$, each event occurring with equal probability;

Thus, the probabilities of each pair of signals is given by the following table,

$s_1 \backslash s_2$	L	H
L	1/4	1/4
H	1/4	1/4

and the signal outcomes are *not independent* – they are correlated with the actual amount of oil.

We can associate each player's signal with his *type*, to the extent that given a signal, a player can form expectations about the signal of his opponent, and about the quantity of oil. Namely, if player i observes a low signal L, then he knows that the probability that his opponent got a low signal is equal to the probability he received a high signal, which is $\frac{1}{2}$. Similarly, $\Pr\{s_j = H | s_i = H\} = \Pr\{s_j = L | s_i = H\} = \frac{1}{2}$.

Given these updated probabilities that each type has, we can calculate the expected amount of oil in the field conditional on the signal a player has. If player i observes a low signal, he knows that with probability $\frac{1}{2}$ the other signal is low, and $v = 10$, and with probability $\frac{1}{2}$ the other signal is high, and $v = 20$. Therefore,

$$E[v_i | s_i = L] = \frac{1}{2} \cdot 10 + \frac{1}{2} \cdot 20 = 15,$$

and similarly,

$$E[v_i | s_i = H] = \frac{1}{2} \cdot 20 + \frac{1}{2} \cdot 30 = 25.$$

At this stage we have identified the way types map into expectations over the amount of oil, and therefore, they map into expected utilities from owning the field. However, it is not true that one player's type alone determines the value of obtaining the oil field – there is valuable information in the type of the other player, which as we will now see, makes the previously attractive second-price sealed-bid auction somewhat less straightforward.

To see this, let's first consider the following question: is it a Bayesian Nash equilibrium for both players to submit truthful bids equal to there expected valuation? Formally, is $b_i = Ev_i$ for $i \in \{1, 2\}$ a Bayesian Nash equilibrium? To check this let's assume that player 2 is playing in this truthful fashion, and check whether bidding truthfully is a best response of player 1. When $s_1 = H$ and he bids 25, then with probability $\frac{1}{2}$ his opponent also received a high signal and bids 25, in which case they win with equal probability and pay the second highest bid of 25 (they tied). With probability $\frac{1}{2}$ player 2 has a low signal and player 1 wins and pays player 2's bid of 15. Therefore,his expected utility is

$$Eu_i = \frac{1}{2}[\frac{1}{2} \cdot (30 - 25)] + \frac{1}{2}[20 - 15] = 3.75$$

Similarly, when $s_1 = L$ and player 1 bids 15, then with probability $\frac{1}{2}$ his opponent also bids 15, in which case they win with equal probability and pay the second highest bid of 15. With probability $\frac{1}{2}$ player 2 has a high signal and player 1 loses, giving him a payoff of zero. Therefore, his expected utility is,

$$Eu_i = \frac{1}{2}[\frac{1}{2} \cdot (10 - 15)] + \frac{1}{2}[0] = -1.25$$

Why is it that when they each bid their valuation, they receive expected negative payoffs some of the time? This is a result of the common values and the correlated types. When a player wins the oil field because his opponent's bid was lower, then this is "bad news" to the extent that the opponent's low signal means that the quantity of oil is lower than the player thinks it is.

This a phenomenon which occurs with common values, and is known as the "Winner's Curse": a player wins when his signal is the most optimistic, which due to the common value element means that he has over estimated the value of the good, and is overpaying.

This phenomenon exists in first- and second-price sealed-bid auctions, and has important economic consequences as to which auctions may perform better in allocating a common value good efficiently.

(to be completed...)

19.3 Inefficient Trade and Adverse Selection

One of the main conclusions of competitive market analysis in economics is that markets allocate goods to the people who value them the most. The simple intuition behind this conclusion works as follows: If a good is misallocated so that some people who have it value it less than people who do not, then so called "market pressures" will cause the price of that good to increase to a level where the current owners will prefer to sell it, rather than hold on to it, and the people who value it more will be willing to pay that price. The determination of such a price is not clearly specified, but various mechanisms such as bargaining, auctions, or market intermediaries may obtain it.

This powerful argument is based on some assumptions, one of which is that the value of the good is easily understood by all market participants, or in our terminology, there is *perfect information* about the value of the good. This is, of course, an assumption that applies to an ideal world, one that often departs from the reality in which we live. Yet, under this assumption we not only obtain this amazing result, but we also get what is known as the "Coase Theorem", named after its contributor Ronald Coase. It argues that in a world with perfect information and no market frictions (referred to as "transactions costs"), the allocation of property rights will not affect economic efficiency. That is, even if for some reason goods are allocated to the people who do not value them the most, then with perfect information and well functioning abilities to exchange goods, these goods will end up in the hands of those who value them the most.

It is therefore important to understand the extent to which these arguments stand or fall in the face of incomplete information, where some people are more informed about the value of goods than others. To address this question we will develop a simple example that follows in the spirit of an important contribution made by George Akerlof (1970), a contribution that introduced the idea of *adverse selection* into economics, and earned its author a Nobel prize.

Imagine a scenario where player 1 owns an orange grove. The yield of fruit depends on the quality of soil and other local conditions, and assume that through his experience only player 1 knows the quality of land. Local geological surveys conclude that the quality of land can be poor, mediocre or good, each with equal probability of $\frac{1}{3}$, with monetary-equivalent values for player 1, $v_1 \in \{10, 20, 30\}$. Thus, we can think of the knowledge of player 1 as his *type*, $T_1 = \{t_1^L, t_1^M, t_1^H\}$, where each type has a different value for the land so that his utility from owning the land is $v_1(\text{own}|t_1)$ as described above.

Now imagine that player 2 is a potential soy-bean grower, who is considering the purchase of this land for his production. Player 2's family expertise of growing soy-beans is very profitable, but also depends on the quality of the land. In particular, for poor, mediocre or good land, player 2's monetary-equivalent values are $v_2 \in \{14, 24, 34\}$. The problem, however, is that player 2 only knows that the quality id distributed equally among the three options (the geological survey's results) but does not know which of the three it is.

Consider the following game: player 2 makes a take-it-or-leave-it offer to player 1, after which player 1 can accept (A) or reject (R) the offer, and the game ends with either a transfer of land for the suggested price, or no transfer. A strategy for player 2 is therefore a single price offer, p, and a strategy for player 1 is a mapping from his type space T_1 to a response, $s_1 : T_1 \to \{A, R\}$. What can emerge as an equilibrium of this game?

The assumptions on payoffs imply that from an efficiency point of view, it is player 2 who should own the land. Indeed, if the quality of land were common knowledge then there are many prices for which both player 1 and player 2 would be happy to trade the property. For example, if the quality is known to be low, then in a similar way to the one-period bargaining model, the unique subgame perfect equilibrium would have player 2 offering player 1 a price of 10, and player 1 accepting. Similarly, any price between 10 and 14 would be supported by some Nash equilibrium.[2] We will now see what kind of trade could occur in equilibrium, when there is asymmetric information as described.

Let's consider the value that player 2 places on the land. He knows that with equal probability it is worth one of the values $v_2 \in \{14, 24, 34\}$, so on average it is worth 24. He also knows that on average it is worth 20 to player 1. It would seem

[2] Namely, for any $p^* \in [10, 14]$, a strategy for player 1 accepting any offer $p \geq p^*$ and a strategy for player 2 of offering p^* would be a Nash equilibrium when the quality is low.

that the natural equilibrium candidate would be to offer the lowest price at which player 2 thinks that player 1 will accept, and such an offer would be $p = 20$.

But what would then be player 1's response? Recall that player 1 *knows* the quality of land, in which case player 1 would accept the offer only if his type is t_1^L or t_1^M, meaning that player 2 would get a parcel of land that is of low or mediocre with equal probability, and get an expected value of 19, and he would make a loss![3]

Notice that for any offer $p \in [20, 30)$ player 1 would accept the offer only if his type is t_1^L or t_1^M, and player 2's value is still 19 on expectation, making such a trade impossible in equilibrium. Thus, if player 2 is to take into account the best response of player 1, he knows that he will get all types of player 1 to agree to sell *only if* player 2 offers 30, but in this case he would only get a value of 24, which is not profitable. Therefore, we conclude that no trade can occur at a price greater than 20.

This implies that if trade is to occur, it will occur at a price less than 20, which in turn implies that player 1 will agree to such a trade only if his type is t_1^L. Taking this into account, player 2 should offer a price $p = 10$, commensurate with player 1 trading if his type is indeed t_1^L. This logic yields the following result:

Proposition 18 *Trade can occur in Bayesian Nash Equilibrium only if it involves the lowest type of player 1 trading. Furthermore, any price $p^* \in [10, 14]$ can be supported as a Bayesian Nash equilibrium.*

The reason that only the lowest type can trade in equilibrium follows from our analysis above. To see that any price $p^* \in [10, 14]$ can be supported as a Bayesian Nash equilibrium consider the following strategies: player 2 offers a price p^*, and the strategy for player 1 is:

$$s_1(t_1) = \begin{cases} A \text{ if and only if } p \geq p^* \text{ and } R \text{ otherwise} & \text{when } t_1 = t_1^L \\ A \text{ if and only if } p \geq 20 \text{ and } R \text{ otherwise} & \text{when } t_1 = t_1^M \\ A \text{ if and only if } p \geq 30 \text{ and } R \text{ otherwise} & \text{when } t_1 = t_1^H \end{cases}$$

In this case, the two strategies are mutual best responses, and therefore they constitute a Bayesian Nash equilibrium.

[3] This follows from Bayes' rule: if only t_1^L or t_1^M types ell, then *conditional* on a sale, each of these types occurs with probability $\frac{1}{2}$.

The conclusion of this result is that trade will occur only if the quality of land is the lowest. The reason this happens is because of what is called "adverse selection". When the buyer is willing to pay a price equal to his average value, then the type of seller who is willing to sell at this price is *below average*, because the best types choose not to participate at an average price, hence the adverse selection of lower than average sellers. In the example, this unravelling causes traded quality to drop to its lowest level, preventing the market from implementing efficient trade outcomes.

It is also worth mentioning that this scenario falls into the category of *common values*, since the type of player 1 affects the payoff of player 2. It is precisely this that causes the adverse effects of equilibrium when there is this kind of asymmetric information.

(add the fact that "cheap talk" would not help here, and allude to the possibility of costly signalling)

19.4 Mixed Strategies Revisited: Harsanyi's Interpretation♠♠

Recall the static game of matching pennies,

	H	T
H	$1,-1$	$-1,1$
T	$-1,1$	$1,-1$

and recall that the unique mixed strategy equilibrium has each player playing heads with probability $\frac{1}{2}$. One reason this solution may be somewhat unappealing is that players are indifferent between H and T, yet they are prescribed to randomize between these strategies in a unique and particular way for this to be a Nash equilibrium. Does it make sense to ask for this requirement when a player is indifferent?

This question has caused some discomfort with the notion of mixed strategy equilibria. However, John Harsanyi (1973) offered a twist on the basic model of behavior to resolve this problem and alleviate, to some extent, the indifference problem. His idea works as follows. Imagine that each player may have some slight preference to choosing heads over tails, or choosing tails over heads. This is done in such a way as to "break" the indifference of a player's best response if he believes that the probability of his opponent playing heads is exactly $\frac{1}{2}$.

To do this, imagine that the payoffs are given by this perturbed matching pennies game,

	H	T
H	$1+\varepsilon_1, -1+\varepsilon_2$	$-1+\varepsilon_1, 1$
T	$-1, 1+\varepsilon_2$	$1, -1$

and imagine that both ε_1 and ε_2 are independent and uniformly distributed on the interval $[-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$ but small. This means that if $\varepsilon_i > 0$ is realized, then player i has a strict preference to choosing H over T when he believes his opponent is choosing H with probability $\frac{1}{2}$.

Now consider this as a Bayesian game where each player i knows his type, ε_i, but his opponent only knows the distribution of ε_i. A pure strategy for each player is therefore a mapping $s_i : [-\varepsilon, \varepsilon] \to \{H, T\}$ that assigns a choice to every type of player i.

Proposition 19 *In the Bayesian game of perturbed matching pennies, there is a unique pure strategy Bayesian Nash Equilibrium where $s_i(\varepsilon_i) = H$ if and only if $\varepsilon_i \geq 0$, and $s_i(\varepsilon_i) = L$ if and only if $\varepsilon_i < 0$. This equilibrium converges in outcomes and payoffs to the matching pennies game when $\varepsilon \to 0$.*

It is quite easy to see that the proposed strategies are a Bayesian Nash equilibrium. If they are followed by player i, then from the distribution of ε_ι, with probability $\frac{1}{2}$ player i is playing H, in which case the strategy of player j is a best response. To see that this is the unique Bayesian Nash equilibrium requires more work, but is not too hard. This proposition is known as "Harsanyi's Purification theorem", following the idea that we are using incomplete information to "purify" the mixed strategy equilibrium of a game of complete information.

What is the interpretation of this result? It implies that if people are somewhat heterogeneous in the way monetary payoffs and actions are related, then we can have uncertainty over the types of players who are playing *pure strategies*, but the distribution of types makes a player have beliefs *as if* he were facing a player that is playing a mixed strategy. Harsanyi argues that using mixed strategy equilibria in simple games of complete information can be thought of as a solution to the more complex games of incomplete information, in which players do not randomize but rather have strict best responses.