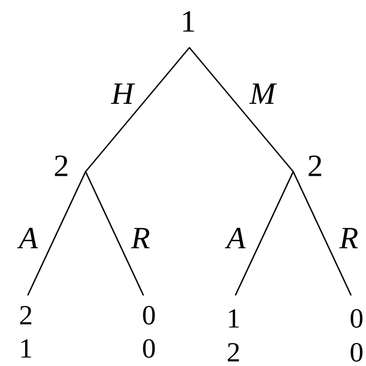
23
Building a Reputation

23.1 Driving a Tough Bargain

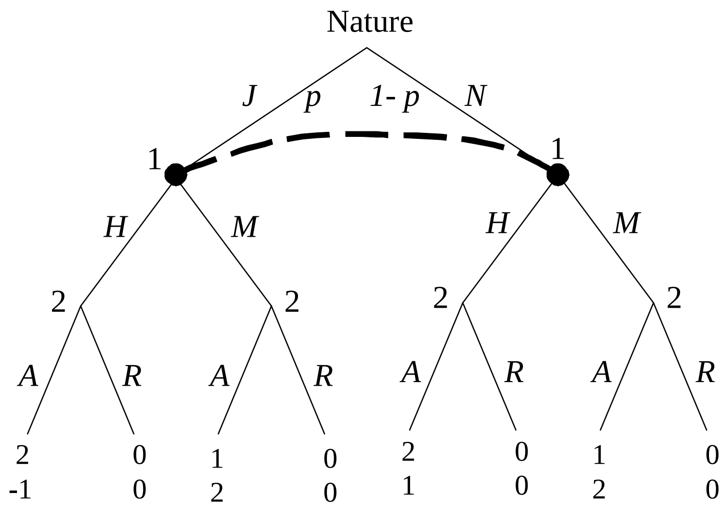
It is very common to use language such as "he has a reputation for driving a tough bargain" or "he's known not to yield and has a reputation of being greedy." What does it really mean to have a reputation of being greedy and ruthless? What can we say about people putting in an effort to build a reputation of being someone they really are not? Using incomplete information, we can shed some light on these questions.

First consider the following perfect information bargaining game, which is admittedly a bit contrived, but captures the main ideas of a bargaining problem: the more I get, the less you get, and we need to reach some agreement. So imagine that a secluded and eccentric rich man dies, and in his will leaves two cars: a shiny new Mercedes sports car (M), and a beat-up Hyundai sedan (H). The will, however, outlines that the final owner of these cars will be determined by a rather unusual bargaining game between his nephew, player 1, and his butler, player 2. Specifically, player 1 first chooses whether to offer player 2 one of the cars, M or H, after which player 2 can choose to accept (A), or reject (R). If player 2 rejects, the cars will be donated to a charity, leaving both players with utility 0. If player 2 accepts, he gets what he was offered, and player 1 is left with the other car. Assuming that a Mercedes is worth 2 and a Hyundai is worth 1, the extensive form is depicted in Figure 5.x.



It is easy to see that this game has a unique Subgame-perfect equilibrium: (H, AA), where following our previous conventions, AA means that player 2 plays A in both his information sets, following either H or M. You should easily be able to convince yourself that this game has Nash equilibria that are not SPE, for example (H, AR), which yields the same outcome as (H, AA) but player 2 is not playing a best response if M were offered. The more interesting one is (M, RA), in which player 2 gets the Mercedes, but this is supported by the incredible threat of rejecting a Hyundai. One can interpret this equilibrium as the one in which player 2 "drives a hard bargain", but since we believe that sequential rationality is an important feature of rational behavior, this equilibrium is not a very convincing one.

Now consider a variation of this game to include some incomplete information. In particular, imagine that player 2 can be "normal" ($t_2 = N$), with payoffs as described above, or he can be a "jerk" ($t_2 = J$), who prefers both players to get nothing over getting the inferior car. Imagine further that nature first chooses the type of player 2, who is a jerk with probability p, and assume that the payoff to the jerk of getting the Hyundai is -1, whereas all other payoffs are the same. Player 1 does not know the type of player 2, but he does know that player 2 is a jerk with probability p. This game is described in figure 5.X below.



As one might expect, since this is a game of incomplete information, we will focus on sequential rationality using Perfect Bayesian equilibria (PBE). For the simpler signaling games we analyzed, we found PBE by first finding the set of BNE in the Bayesian game, given by a matrix, and then checked to see which profiles of BNE strategies can be supported as part of a PBE with appropriate beliefs. Here, we would need a 2 row by 16 column matrix to do this, which is not too demanding, but there is a much simpler way, using backward induction.

To see this, consider player 2 at each of the four nodes that follow an offer from player 1. In any PBE, player 2 must be playing a best response to his beliefs in every information set, which are singletons for these for nodes. thus, for each node player 2 must play a best response *at that node*, which immediately implies that a normal player 2 will accept any offer, while a jerk will accept a Mercedes and reject a Hyundai.

Given this behavior of player 2, and the fact that the uninformed player 1 plays after Nature makes its choices, the beliefs of player 1 are immediately pinned down from natures choices and he must believe that $\Pr\{t_2 = J\} = p$. This implies that player 1 will strictly prefer to offer a Hyundai if and only if the following inequality holds:

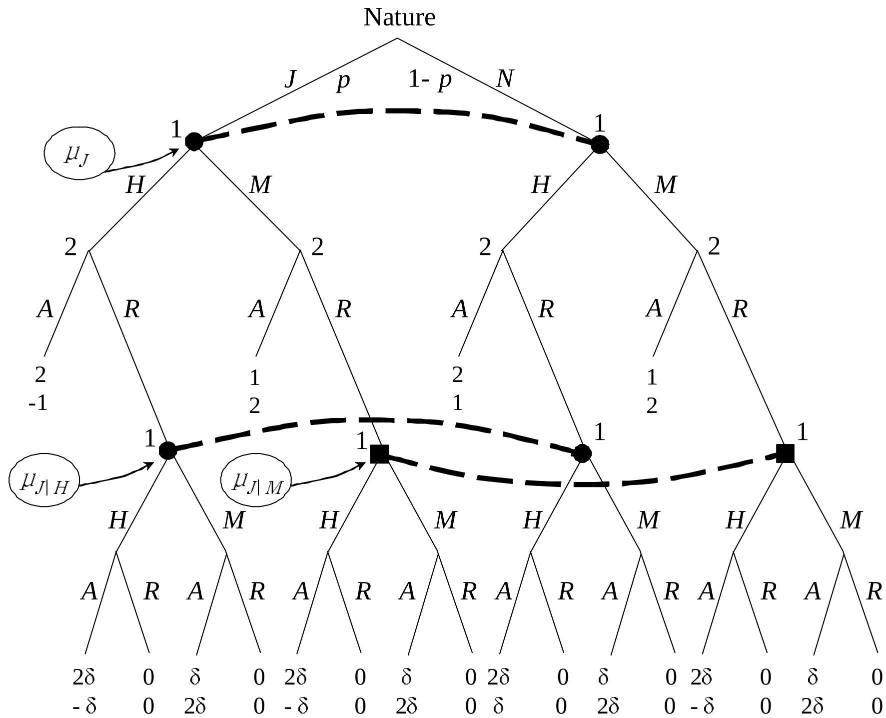
$$\underbrace{p \cdot 0 + (1-p) \cdot 2}_{\text{offer } H} > \underbrace{p \cdot 0 + (1-p) \cdot 2}_{\text{offer } M}$$

or,

$$p < \frac{1}{2} .$$

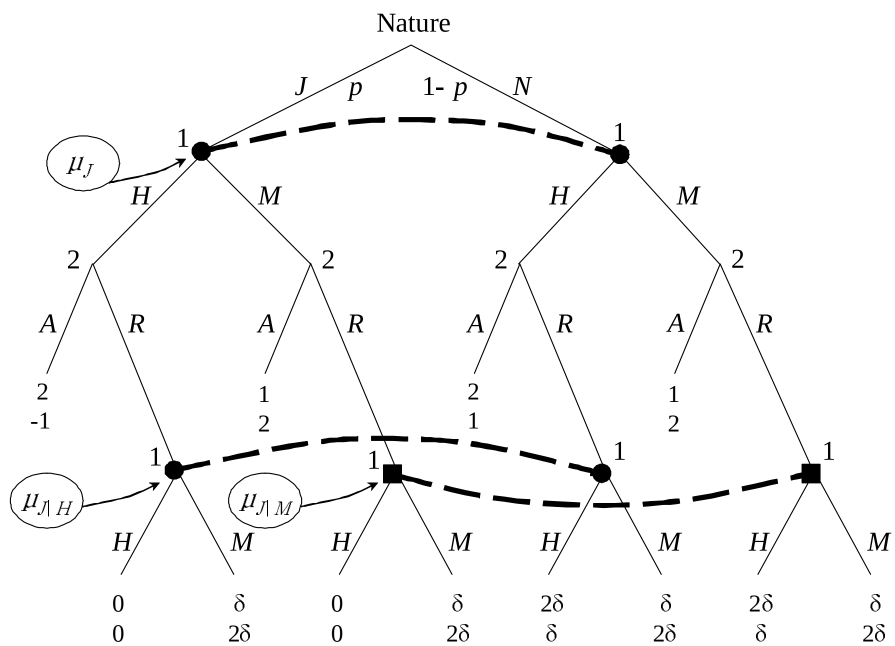
and will prefer to to offer a Mercedes otherwise. The intuition for the effect of p on the unique PBE is simple: if there is a good chance that player 2 is normal ($p < \frac{1}{2}$) then player 1 is better offering the Hyundai and risking a rejection than getting the Hyundai for sure. If, however, player 2 is likely enough to be a jerk ($p > \frac{1}{2}$) then the safe Hyundai is better than the risk of keeping the Mercedes.

Now we take our simple game a step further, to allow for the possibility of reputation building. To do this imagine that the rich uncle's will is modified as follows: If player 1's initial offer is accepted, then the game ends as before. If the initial offer is rejected by player 2, then the cars are put in storage for a year, after which player 1 has another chance to make an offer. After this second offer, the game proceeds as above (rejection causes donation) and the payoffs are the same. However, a year's delay will result in discounted payoffs, with a discount factor of $\delta < 1$ as we had in our previous bargaining models. The extensive form of this game is given in figure 5.X

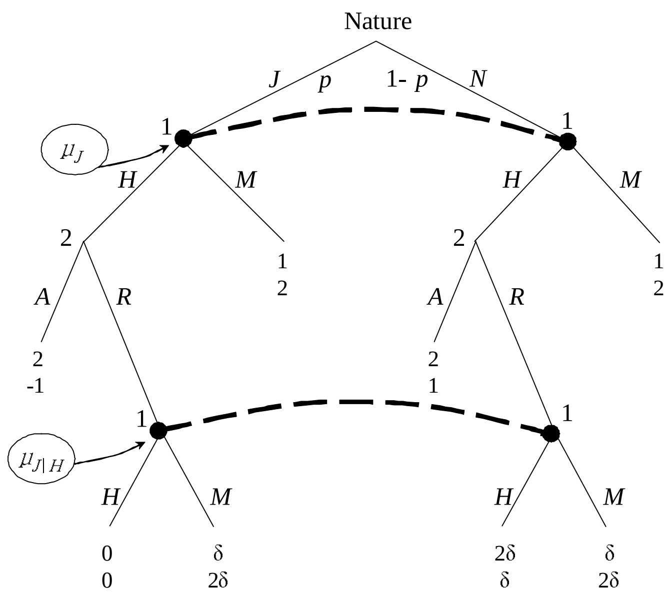


where μ is used to denote the beliefs of player 1 in each of his information sets: $\mu_J \in [0,1]$ is his belief at the beginning of the game that player 2 is a jerk, $\mu_{J|H} \in [0,1]$ is his belief that player 2 is a jerk *conditional on* player 2's rejection of a Hyundai, and $\mu_{J|M} \in [0,1]$ is his belief that player 2 is a jerk *conditional on* player 2's rejection of a Mercedes.

Once again, an attempt to turn this into the normal-form game will lead to an 8-row by 4096-column matrix! (Player 2 has 12 information sets with two actions each, so $2^{12} = 4096$ pure strategies.) However, since we are looking for PBE, we can employ a similar form of backward induction over all the information sets that are singletons. This implies that at the second stage a normal player 2 will accept any offer, and a jerk will accept a Mercedes and reject a Hyundai. Thus, we can perform one stage of backward induction, and taking into account the best response of player 2 at the final stage after a second offer, the game reduces to the one depicted in figure 5.X, which we call "game 2":



If we try to turn this reduced game into its normal form, we still get a rather sizeable matrix with 8 rows and 16 columns, but it seems that we cannot proceed with backward induction due to the information sets of player 1 after a rejection in the first round of bargaining. However, if we think a bit more carefully we can perform another tricky step of backward induction. To do this consider the nodes in the first round at which player 2 has to move *after he is offered to keep the Mercedes*. If he accepts, he gets a payoff of 2, while if he rejects then the game moves into the next stage of bargaining. Note, however, than in the next stage *the most* player 2 can get is $2\delta < 2$, so if he is choosing rationally at these information sets, *he must accept the Mercedes no matter what his type is!* This allows us to reduce the game further since we know that in any PBE player 2 will accept a Mercedes in the first round regardless of his type. thus, the further reduced game appears in figure 5.X, which we call "game 3":



This reduced game is very manageable since it can be represented in its normal form by a 4×4 matrix. To do this we define strategies for both players in this reduced form of the initial game as follows: Let $s_1^1 s_1^2 \in \{HH, HM, MH, MM\}$ be a pure strategy for player 1 where s_1^τ is what player 1 offers in bargaining stage $\tau \in \{1, 2\}$. Similarly, let $s_2^J s_2^N \in \{AA, AR, RA, RR\}$ be a pure strategy for player 2 where s_2^t is what player 2 chooses when offered a Hyundai when his type is $t \in \{J, N\}$.

To complete the matrix with real numbers, however, we need to specify values for δ and p. Let's consider the case where the future maters a lot, with $\delta = 0.9$, and where the likelihood of being a jerk is small, $p = 0.1$. In such a case the pair of expected payoffs from player 1 choosing HM and player 2 choosing AR will be,

$$(Eu_1, Eu_2) = p(2, -1) + (1 - p)(\delta, 2\delta) = (1.01, 1.52) \ .$$

Similarly, we can compute the other combinations to get the following matrix:

		Player 2			
		AA	AR	RA	RR
player 1	HH	$2, 0.8$	$1.82, 0.71$	$1.8, 0.9$	$1.62, 0.81$
	HM	$2, 0.8$	$1.01, 1.52$	$1.89, 1.08$	$0.9, 1.8$
	MH	$1, 2$	$1, 2$	$1, 2$	$1, 2$
	MM	$1, 2$	$1, 2$	$1, 2$	$1, 2$

Any Nash equilibrium of this matrix will correspond to a BNE of the reduced Bayesian game, and any PBE of this game together with the best response strategies of player 2 that we have already found will be part of a PBE in the original game. To solve this matrix it is first worthwhile looking for dominated strategies. It is easy to see that for player 1, MH and MM are both strictly dominated by HH. The intuition is simple: For player 1, the strategy HH will replicate the one stage game in which he commits to offer the Hyundai, and then player 2 has to respond with no chance of getting the Mercedes. The *worse that can happen* for player 1 with this strategy is that both jerks and normal players reject initial offers, but at the second stage the normal player 2 will accept. Since the likelihood of a jerk is small, and there is little discounting, this yields a better payoff than getting a Hyundai for sure (1.62 versus 1). If some type of player 2 accepts the initial offer, then things are even better for player 1.[1]

Once we eliminate MH and MM, it is easy to see that for player 2 the strategies AA and AR are strictly dominated by RR. To give intuition, it is actually easier to consider the following set of dominance relations: AA is dominated by RA because given that H is offered in the first stage, the jerk should reject rather than accept. Similarly, AR is dominated by RR. Thus, we are left with the following simple matrix,

[1] Notice that if there were more severe discounting, or if the probability of a jerk was significantly higher, then such dominance of HH would not necessarily hold.

$$
\begin{array}{r|c|c|}
 & \begin{array}{c} q_{RA} \\ \downarrow \\ RA \end{array} & RR \\
\hline
q_{HH} \rightarrow HH & \overline{1.8, 0.9} & \underline{1.62, 0.81} \\
\hline
HM & \underline{1.89, 1.08} & \overline{0.9, 1.8} \\
\hline
\end{array}
$$

for which it is clear there are no pure strategy Nash equilibria.

Since we know any such game must have a mixed strategy Nash equilibrium, to find it we need to find a probability $q_{HH} \in (0, 1)$ of choosing HH by player 1 that will make player 2 indifferent between RA and RR, and a probability $q_{RA} \in (0, 1)$ of choosing RA by player 2 that will make player 1 indifferent between HH and MM. To find q_{HH} we solve:

$$\overbrace{q_{HH} \cdot 0.9 + (1 - q_{HH}) \cdot 1.08}^{\text{2's payoff from choosing } RA} = \overbrace{q_{HH} \cdot 0.81 + (1 - q_{HH}) \cdot 1.8}^{\text{2's payoff from choosing } RR}$$

which yields $q_{HH} = \frac{8}{9}$. Similarly, to find q_{RA},

$$\overbrace{q_{RA} \cdot 1.8 + (1 - q_{RA}) \cdot 1.62}^{\text{1's payoff from choosing } HH} = \overbrace{q_{HH} \cdot 1.89 + (1 - q_{HH}) \cdot 0.9}^{\text{1's payoff from choosing } HM}$$

which yields $q_{RA} = \frac{8}{9}$ as well.

Since in this is the unique BNE of the reduced form game, and since all information sets are reached with positive probability, we know that these mixed strategies are also part of a PBE with the induced beliefs using Bayes rule. Namely, the *unique* PBE in game 3 is as follows:

- **player 1:** play HH with probability $\frac{8}{9}$ and HM with probability $\frac{1}{9}$.
- **player 2:** play AR with probability $\frac{8}{9}$ and RR with probability $\frac{1}{9}$.
- **beliefs:** $\mu_J = 0.1$ and $\mu_{j|H} = \frac{\mathbf{0.1}}{\mathbf{0.1+0.9}\cdot\frac{1}{9}} = 0.5$.[2]

Now we can go back to the original game, and incorporate all that we have analyzed into a PBE. We get the following pair of strategies, together with beliefs.

[2] This follows because the information set of rejection is reached for sure if player 2 is a jerk, hence the 0.1, and is reached with probability $\frac{1}{9}$ if player 2 is normal, hence the $\mathbf{0.9} \cdot \frac{1}{9}$. The rest follows from Bayes' rule.

- **player 1:**
  - In the first stage play H;
  - After rejection following an H offer, play H with probability $\frac{8}{9}$ and M with probability $\frac{1}{9}$;
  - After rejection following an M offer, play H with probability 1.
- **player 2:**
  - If type is J, following H play R and following M play A in any stage.
  - If type is N, then (1) in the first stage following M play A, and following H play R with probability $\frac{1}{9}$ and A with probability $\frac{8}{9}$; (2) in the second stage play A following any offer.
- **beliefs:** $\mu_J = 0.1$; $\mu_{j|H} = \frac{0.1}{0.1+0.9\cdot\frac{1}{9}} = 0.5$; $\mu_{j|M} = \mu^*$ where $\mu^* \in [0, \frac{1}{2}]$

It is important to notice that the combination of backward induction and the analysis of the reduced game left us with freedom to determine what player 1 will do in the information set that occurs after the rejection of a Mercedes. However, as we noticed earlier, nothing that player 1 does in this information will affect the behavior of player 2 when a Mercedes is offered, so all we have to do is assign some belief to player 1 in that information set, and have hi play a best response to that belief. Above, we have him play H in that information set, which implies that he must believe that $\Pr\{t_2 = J|M \text{ rejected}\} < \frac{1}{2}$, and therefore we have the restriction on μ^*. Alternatively, we could have had him play M in that information set, which implies that he must believe that $\Pr\{t_2 = J|M \text{ rejected}\} > \frac{1}{2}$, and therefore we would have had to impose the restriction $\mu^* \in [\frac{1}{2}, 1]$.

The interpretation of this equilibrium is also interesting. It is useful first to consider what *cannot* be an equilibrium. Lets consider three natural cases, and see what's wrong with each as an equilibrium candidate:

case 1: pooling on accept. In this case, following an Hyundai offer of player 1, both types of player 2 pool and choose to accept. This cannot be an equilibrium due to the simple fact that by rejecting, a jerk gets 0 instead of

-1, and may end up getting the Mercedes in the second period, giving an expected utility of at least 0 by rejecting.

case 2: pooling on reject. In this case, following an Hyundai offer of player 1, both types of player 2 pool and choose to reject. If this were part of an equilibrium, then in the second stage player 1 cannot update his prior on the type of player 2, and continues to believe that $\Pr\{t_2 = J\} = 0.1$, in which case he will offer H again. However, if this is the continuation that a normal player 2 faces, he is better off accepting the Hyundai in the first period than in the second because of the discounting. Thus, this case cannot be an equilibrium.

case 3: separating. In this case, following an Hyundai offer of player 1, a normal player 2 accepts the offer while a jerk does not. If this were part of an equilibrium, then in the second stage player 1 updates his prior on the type of player 2, and believes that $\Pr\{t_2 = J\} = 1$, in which case he will offer M. However, if this is the continuation that a normal player 2 believes in, he is better off rejecting the Hyundai in the first period and getting a Mercedes in the second because the discounting is not too severe. Thus, this case cannot be an equilibrium.

Now that we understand the problems of pooling or complete separation, it is easier to understand the mixed strategy equilibrium that we found. In it, the normal player 2 sometimes acts as a jerk, and by doing so causes player 1 to have mixed beliefs about the type of player 2 in stage 2. These beliefs are set to make player 1 indifferent in the second stage, so that he can choose a mixed action that makes the normal type of player 2 indifferent in the first stage. We interpret this as player 2 sometimes pretending to be a jerk, and in this way gaining a chance of getting a Mercedes in the second period.

23.2 Trustworthiness: Saints and Pretenders

It is very common to use language such as "she has a great reputation, you can trust her" or "he's known to be real jerk, don't trust him." What does it really

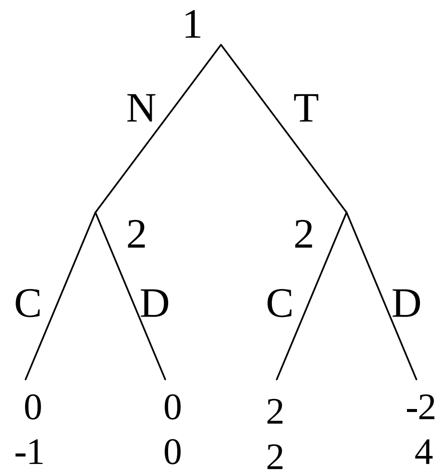


FIGURE 23.1.

mean to have a reputation of being trustworthy, or of being deceptive and self centered? What can we say about people putting in an effort to build a reputation of being someone they really are not? Using incomplete information, we can shed some light on these questions.

Consider the following perfect information trust game. Player 1 first chooses whether to trust (T) player 2 or not to trust him (N), the latter choice giving both players a payoff of zero. If player 1 plays T then player 2 can choose to cooperate (C), giving both players a payoff of 1, or he can defect (D) and get 2, while leaving player 1 with a payoff of (-1). The extensive form is depicted in Figure 5.x.

It is easy to see that this game has a unique Nash equilibrium that is subgame perfect: (N, DD), where following our previous conventions, DD means that player 2 plays D in both his information sets, following either N or T. We can think of this game as a one-sided perfect information version of the Prisoner's dilemma. Namely, both players would like to commit to play (T, DC) (or (T, CC)) but player 2 will rationally deviate to D instead of C following a choice of T by player 1, and anticipating this player 1 will choose not to trust player 2. You should easily be able to convince yourself that if this game is repeated a finite number of times,

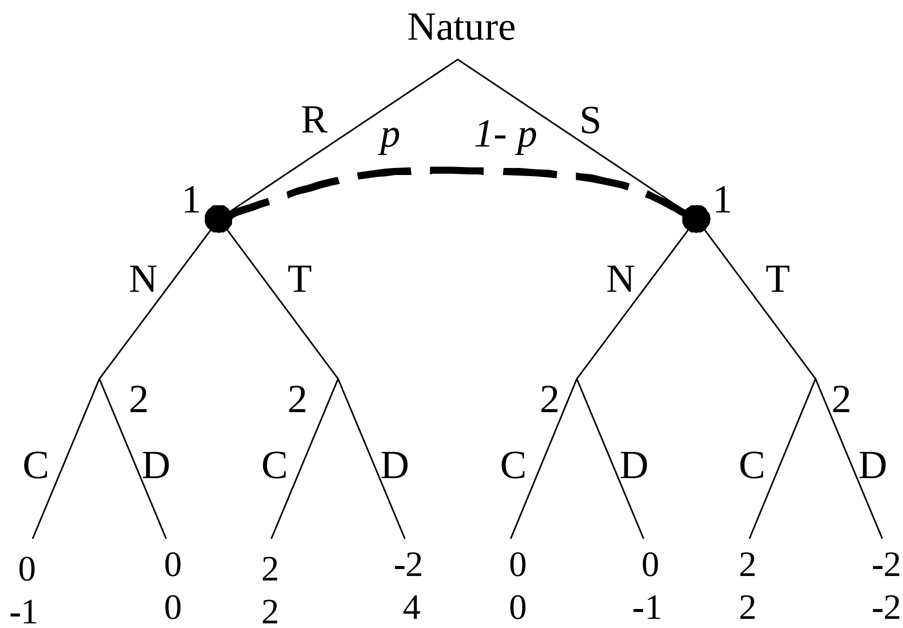


FIGURE 23.2.

then the unique subgame prefect (and unique Nash) equilibrium is for player 1 never to trust, and for player 2 always to deviate if trusted.[3]

Now consider a variation of this game to include some incomplete information. In particular, imagine that player 2 can be "rational" ($t_2 = R$), with payoffs as described above, or he can be a "saint" ($t_2 = S$), who always prefers to cooperate. Imagine further that nature first chooses the type of player 2, who is rational with probability p, and then this game is played twice, with no discounting of payoffs. The one-stage game is described in figure 5.X below.

TO BE COMPLETED

[3]If this game were infinitely repeated, then just as with the regular prisoner's dilemma, if the discount factor is high enough then we can have trust and cooperation with trigger-like strategies.

24

Refinements of Perfect Bayesian Equilibrium

Both in the MBA game and in the entry deterrence game we had a plethora of PBE. This suggests that when we used the *sequential rationality* refinement of PBE over BNE, we did not manage to get rid of many equilibria, and the predictive power of the PBE solution concept is not as sharp as we would like.

Lets consider the MBA game first, and focus attention on the pooling equilibrium in which both types of worker should choose U, and then, regardless of the education choice, the employer assigns the worker to B. Now consider the following deviation, and "speech" that a H type can deliver:

> "I am an H type, and therefore I am deviating to D. If you believe me, and put me in the M job instead of a B job, I will get 8 instead of 6. If I were to be a L type, and the same thing happened, then I would get 5 instead of 6. Therefore, you should believe me because no L type in his right mind would do this."

What should the employer think? The argument makes sense, since if it were an L type, the there is no way he can gain. In contrast, a H type can gain if he is believed by the employer. This logic suggests that the employer should be convinced by this deviation *combined with the speech*. Now, if we take this a step further, the employer can make these kind of logical deductions himself; that is, let me see which type can gain from this deviation. If neither can or if both can, I will keep my out-of-equilibrium-path beliefs as before. But, if only one type of worker can benefit, and other types can only lose, then I should update my beliefs accordingly and act upon these new, more "sophisticated" beliefs.

This logical process is called the *intuitive criterion*, and was developed by David Kreps and In-Koo Cho (1987). It falls under the general category of refinements often called *forward induction*. The reason for this name follows from the logic of the belief process: since player 1 (the one with types) has the potential to signal something to player 2, then for any given set of beliefs, player one can use his action to send a message to player 2 in the spirit of "only a x type would benefit from this move, therefore I am an x type." Formally, the intuitive criterion is a way of ruling out, or *refining equilibria*. That is, take a PBE and see if it survives the intuitive criterion. If it does not, i.e., a player can make a deviation with such a message, then it is ruled out by the intuitive criterion.

If we apply the intuitive criterion to the MBA game above, then only the separating equilibrium we identified satisfies the intuitive criterion. We can apply this logic to the entry deterrence game as well, and not surprisingly, only one separating PBE will survive the intuitive criterion. In particular, this will be the separating equilibrium in which the L cost type produces the quantity that least deviates from his monopoly profits, namely, $q_1^L = 2.618$ with the other components as described above.

25

APPENDIX

25.0.1 Review of Notation, Normal Form, IESDS, Nash Equilibrium

Notation

Γ = "*gamma*" , often used to represent a game; e.g. $\Gamma =< N, S, u >$, or more detailedly, $<N, \{S_i\}, \{u_i\} >$, where:

- $i \in N = \{1, 2, ..., n\} \rightarrow$ "i is a member of N"
- $s_i \in S_i$, $i \in N \rightarrow s_i$ is a strategy in the set of all strategies for player i, represented as S_i
- $u_i : S \rightarrow \Re$, $S = \underbrace{S_1 \times S_2 \times ... \times S_n}_{\text{cross-product of strategy spaces of all the players}}$

$-i = N \setminus \{i\}$, where "$\setminus$" means "not including"

IESDS, Nash Equilibrium, Dominant Strategies...

Proposition 21 *If $s^* \in S$ is a DS equilibrium then it is unique.*

Lemma 22 *If s^* is a DS equilibrium, then $u_i(s_i^*, s_{-i}) > u_i(s_i', s_{-i}) \forall s_i' \in S_i \diagdown \{s_i^*\}, \forall s_{-i} \in S_{-i}$. Suppose s^* is not unique, then $\exists\ \hat{s} \in S_i \diagdown \{s_i^*\}$ that is a DS equilibrium. Thus,*

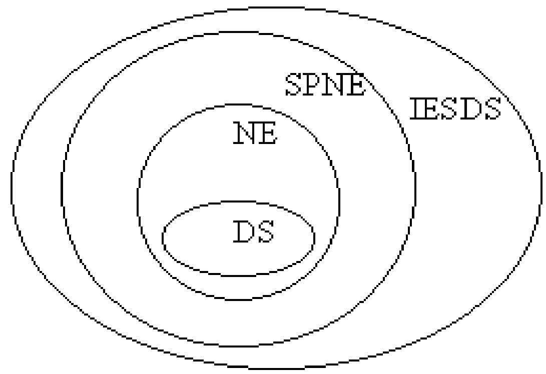


FIGURE 25.1.

$u_i(\hat{s}_i, s_{-i}) > u_i(s_i', s_{-i}) \forall s_i' \in S_i \diagdown \{\hat{s}_i\}, \forall s_{-i} \in S_{-i} \forall i \in N$. *Choose* i *s.t.* $\hat{s}_i \neq s_i^*$, *then* $u_i(\hat{s}_i s_{-i}) > u_i(s_i^*, s_{-i})$ *[set* $s_i' = s^*$*] and* $u_i(s_i^* \; s_{-i}) > u_i(\hat{s}_i, s_{-i})$ *[set* $s_i' = \hat{s}$*].*

Nash Equilibrium (NE)

Definition 35

1. *NE is a vector of mutual BRs*
2. *NE is a strategy profile from which* $\nexists$ *("there does not exist") profitable (i.e. make strictly better off) unilateral deviations.*
3. *NE is intersection of best response correspondences*

25.0.2 IEWDS, IESDS

IEWDS-Problems, etc

In general, we will not use IEWDS in this class. There are many weaknesses of the IEWDS:

For example, consider the following example:

	0	1
0	0,0	0,0
1	0,0	0,0

There is no strict response, no answer$\rightarrow$existence problem

The main problem with IEWDS is that it doesn't give the same answer when we start with player 1 than when we start with player 2.

We need to see whether path dependent or path-independent (whether it matters who goes first), for example:

	a	b	c
x	10,0	5,1	4,-200
y	10,100	5,0	0,-100

25.0.3 Multi-Stage Game-Sequential Bargaining

Single-Stage Deviation Principle

If s^* in a multi-stage game Γ has no single-stage profitable unilateral deviations and

1. actions are observed in each stage

2. if Γ is infinite then payoffs are discounted with $\delta < 1$ and stage game payoffs are uniformly bounded[1]

then s^* is a SPE.

Example

		2		
		L	M	H
1	L	10,10	3,15	0,7
	M	15,3	$\boxed{7,7}$	-4,5
	H	7,0	5,-4	-15,-15

G : NE=(M, M) by IESDS

$G(T)$:SPE=$(M, M)\forall\ h^t \forall\ t$ for $T < \infty$

What is the SPE for $G(\infty)$?

- Worst punishment: (H, H)
- Consider a strategy, s^*:
  - Play L if (L, L) always
  - Play H otherwise

[1]By "uniformly bounded", we mean that we can find one positive number, β, where $\beta > |a_t| \forall t$, no payoff will exceed that number. (β is the same for each stage)

Equilibrium Path: $\underbrace{\frac{10}{1-\delta}}_{\substack{\text{if} \\ \text{cooperate}}} \geq 15 + \underbrace{\delta\frac{-15}{1-\delta}}_{\substack{\text{punishment} \\ \text{forever}}} \Rightarrow \delta \leq \frac{1}{4}$

Punishment: $\underbrace{\frac{-15}{1-\delta}}_{\substack{\text{in punishing} \\ \text{phase}}} \geq \underbrace{0}_{\substack{\text{deviate} \\ \text{today}}} + \underbrace{\delta\frac{-15}{1-\delta}}_{\substack{\text{get punishment} \\ \text{forever}}} \Rightarrow \text{not SPE}$

- So s^* is not SPE

- Consider strategy, s^* :
  - Play L if (L, L) always
  - Play M otherwise

    Equilibrium Path: $\underbrace{\frac{10}{1-\delta}}_{\substack{\text{if} \\ \text{cooperate}}} \geq 15 + \underbrace{\delta\frac{7}{1-\delta}}_{\substack{\text{punishment} \\ \text{forever}}} \Rightarrow \delta \leq \frac{5}{8}$

    Punishment: ✓ $\Rightarrow$ do not need to check because it is a Nash Equilibrium

- So this <u>is</u> a SPE.

$\Rightarrow s^*$: play L if (L, L) always

1. if player 1 deviates, follow $(M, H), (L, M), (L, M), ...$
2. if player 2 deviates, follow $(H, M), (M, L), (M, L), ..$
3. if both deviate at the same time, follow (M, M) forever ✓

Now, show that you would not want to deviate from cooperation phase:

- First, hold player 2 fixed:

  .Equilibrium Path: $\underbrace{\frac{10}{1-\delta}}_{\substack{\text{if} \\ \text{cooperate}}} \geq 15 + \underbrace{\delta(-4) + \delta^2\frac{3}{1-\delta}}_{\substack{\text{punishment} \\ \text{forever}}} \Rightarrow \delta \gtrsim 0.295$

  Punishment Phase:

- Period 1:
  - Punished: $-4 + \delta\frac{3}{1-\delta} \geq 0 + \delta(-4) + \delta^2(\frac{3}{1-\delta}) \Rightarrow \delta \geq \frac{4}{7}$
  - Punisher: $5 + \delta\frac{15}{1-\delta} \geq 7 + \delta(-4) + \delta^2(\frac{3}{1-\delta}) \Rightarrow \delta \geq 0.098$
- Period 3:
  - Punished: $\frac{3}{1-\delta} \geq 7 + \delta(-4) + \delta^2(\frac{3}{1-\delta}) \Rightarrow \delta \geq \frac{4}{7}$ and $\delta < \frac{5}{8}$
  - Punisher: $\frac{15}{1-\delta}$ ✓ (best possible) $\Rightarrow$ player 2 will not want to deviate

- This is a SPE.

25.0.4 Auctions: A Uniform Distribution Example

Consider the following uniform distribution game for a 2nd price auction and 1st price auction:

- $N = \{1, 2\}$
- $a_i \in A_i = [0, 1]$
- $t_i \in T_i = [0, 1]$
- $p_i(t_{-i} < \bar{t} \mid t_i) = \bar{t}$

2PA

$$u_i(a, t_i) = \begin{cases} t_i - a_{-i} \text{ if win} \\ 0 \text{ if lose} \\ \frac{1}{2}(t_i - a_{-i}) \text{ if tie} \end{cases}$$

1PA

$$u_i(a, t_i) = \begin{cases} t_i - a_{-i} \text{ if win} \\ 0 \text{ if lose} \\ \frac{1}{2}(t_i - a_{-i}) \text{ if tie} \end{cases}$$

BNE

2PA

- assume $a_{-i}(t_{-i}) = kt_{-i}$

- $$\max_{a_i} E_{t_{-i}}(u_i(a,t)|t_i) = E_{t_{-i}}((t_i - a_{-i}(t_i))\overbrace{I(a_i > a_{-i}(t_{-i})) + \frac{1}{2}I(a_i = a_{-i}(t_{-i})))}^{\text{indicator function: takes value 1 if true, 0 if false}}$$
$$= E_{t_{-i}}((t_i - a_{-i}(t_i))I(a_i > kt_{-i}) \text{ given we win}$$

$$= \underbrace{\Pr(a_i > kt_{-i})}_{\substack{=\Pr(t_{-i} < \frac{1}{k}a_i) \\ =\frac{1}{k}a_i}} E_{t_{-i}}((t_i - kt_i | a_i > kt_{-i})$$

$= \frac{1}{k}a_i(t_i - k(\frac{1}{2k}a_i))$; we want to maximize and take FOC.

– FOC: $\frac{1}{k}(t_i - a_i) = 0 \Rightarrow a_i^* = t_i, k = 1.$

<u>1PA</u>

- $\underset{a_i}{max} E_{t_{-i}}(u_i(a,t)|t_i) = E_{t_{-i}}(u_i(a_i, \overbrace{a(t_i)}^{=kt_{-i}}, t_i)|t_i)$

$$= E_{t_{-i}}((t_i - a_{-i}(t_i))I(a_i > kt_{-i}) + \overbrace{\frac{1}{2}I(a_i = a_{-i}(t_{-i}))}^{=0})$$given we win

$$= E_{t_{-i}}((t_i - a_{-i}|a_i > kt_{-i}) \Pr(\tfrac{1}{k}a_i > t_{-i})$$

$$= \frac{1}{k}a_i(t_i - a_i)$$

– FOC: $\frac{1}{k}(t_i - 2a_i) = 0 \Rightarrow a_1^* = \frac{1}{2}t_i, k = \frac{1}{2}$

25.0.5 Perfect Bayesian Equilibrium: Joint-Venture Example

Fact in any PBE:

1. I:F iff $\beta \leq \frac{1}{3}$, I:NF if $\beta < \frac{1}{3}$
2. E2:A

25.0.6 Pooling and Separating Equilibrium: A Dynamic Game Example