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Econ 206 – Solutions for Problem Set 1

Question 1

a) Sufficiency: Suppose that we can write $V^*(\theta) = \sum_i V_i(\theta_{-i})$. Consider the transfer functions of the form

$$t_i(\theta) = \left[\sum_{i \neq j} v_j(k^*(\theta), \theta_j)\right] + h_i(\theta_{-i}),$$

where for all i, $h_i(\theta_{-i}) = -(I-1)V_i(\theta_{-i})$ for all θ_{-i}. By proposition 23.C.4, $(k^*(.), t_1(.), ..., t_I(.))$ is truthfully implementable in dominant strategies. Moreover, for all θ we have,

$$\begin{aligned}
\sum_i t_i(\theta) &= \sum_i \left[\sum_{j \neq i} v_j(k^*(\theta), \theta_j)\right] + \sum_i h_i(\theta_{-i}) \\
&= (I-1)V^*(\theta) - (I-1)\sum_i V_i(\theta_{-i}) = 0
\end{aligned}$$

Necessity: Suppose $(k^*(.), t_1(.), ..., t_I(.))$ is ex-post efficient and truthfully implementable in dominant strategies. Since (23.C.8) is necessary (by assumption) for truthful implementation, this means that there exist functions $(h_i(\theta_{-i}))_{i=1}^I$ such that

$$\begin{aligned}
(I-1)V^*(\theta) + \sum_i h_i(\theta_{-i}) &= \sum_i \left[\sum_{j \neq i} v_j(k^*(\theta), \theta_j)\right] + \sum_i h_i(\theta_{-i}) \\
&= \sum_i t_i(\theta) = 0
\end{aligned}$$

But this implies that by defining $V_i(\theta_{-i}) = \left(\frac{-1}{I-1}\right) h_i(\theta_{-i})$, we can then write $V^*(\theta) = \sum_i V_i(\theta_{-i})$.

b) If $v_i(k, \theta_i) = \theta_i k - \frac{1}{2}k^2$, $\forall i$, then

$$k^*(\theta) = Arg\max_k \left(\sum_i \theta_i\right) k - \frac{3}{2}k^2, \forall\theta$$

and so the FOC implies that $k^*(\theta) = \frac{\sum_i \theta_i}{3}$. Hence,

$$\begin{aligned}
V^*(\theta) &= \sum_{i=1}^{3} \left[\theta_i \left(\frac{\sum_i \theta_i}{3} \right) - \frac{1}{2} \left(\frac{\sum_i \theta_i}{3} \right)^2 \right] \\
&= \left(\frac{\sum_i \theta_i}{3} \right) \sum_i \left[\theta_i - \frac{1}{2} \left(\frac{\sum_i \theta_i}{3} \right) \right] \\
&= \frac{1}{3} (\theta_1 + \theta_2 + \theta_3) \left[\theta_1 + \theta_2 + \theta_3 - \frac{1}{2} (\theta_1 + \theta_2 + \theta_3) \right] \\
&= \frac{1}{6} \left(\sum_i \theta_i \right)^2 \\
&= \frac{1}{6} \left(\theta_1^2 + \theta_2^2 + \theta_3^2 + 2\theta_1\theta_2 + 2\theta_1\theta_3 + 2\theta_2\theta_3 \right)
\end{aligned}$$

We now define,

$$\begin{aligned}
V_1(\theta_2, \theta_3) &= \frac{1}{6} \left(\frac{\theta_2^2 + \theta_3^2}{2} + 2\theta_2\theta_3 \right) \\
V_2(\theta_1, \theta_3) &= \frac{1}{6} \left(\frac{\theta_1^2 + \theta_3^2}{2} + 2\theta_1\theta_3 \right) \\
V_3(\theta_1, \theta_2) &= \frac{1}{6} \left(\frac{\theta_1^2 + \theta_2^2}{2} + 2\theta_1\theta_2 \right)
\end{aligned}$$

and the result then follows from part (**a**) above since

$$V^*(\theta) = V_1(\theta_2, \theta_3) + V_2(\theta_1, \theta_3) + V_3(\theta_1, \theta_2).$$

c) If $V^*(\theta) = \sum_i V_i(\theta_{-i})$, then clearly $\frac{\partial^I V^*(\theta)}{\partial \theta_1 \ldots \partial \theta_I} = 0$.

d) In this case, $V^*(\theta_1, \theta_2) = v_1(k^*(\theta), \theta_1) + v_2(k^*(\theta), \theta_2)$, therefore,

$$\begin{aligned}
\frac{\partial V^*}{\partial \theta_1} &= \left(\frac{\partial v_1}{\partial k} + \frac{\partial v_2}{\partial k} \right) \frac{\partial k}{\partial \theta_1} + \frac{\partial v_1}{\partial \theta_1} \\
\frac{\partial^2 V^*}{\partial \theta_1 \partial \theta_2} &= \left(\frac{\partial^2 v_1}{\partial k^2} + \frac{\partial^2 v_2}{\partial k^2} \right) \left(\frac{\partial k}{\partial \theta_1} \right) \left(\frac{\partial k}{\partial \theta_2} \right) + \frac{\partial^2 v_2}{\partial k \partial \theta_2} \frac{\partial k}{\partial \theta_1} + \frac{\partial^2 v_1}{\partial k \partial \theta_1} \frac{\partial k}{\partial \theta_2}
\end{aligned}$$

Since, $\frac{\partial v_1}{\partial k} + \frac{\partial v_2}{\partial k} = 0$, we have,

$$\frac{\partial^2 v_i}{\partial k \partial \theta_i} = -\frac{\partial k}{\partial \theta_i} \left(\frac{\partial^2 v_1}{\partial k^2} + \frac{\partial^2 v_2}{\partial k^2} \right)$$

which in turn implies that

$$\frac{\partial^2 V^*}{\partial\theta_1\partial\theta_2} = -\left(\frac{\partial^2 v_1}{\partial k^2} + \frac{\partial^2 v_2}{\partial k^2}\right)\frac{\partial k}{\partial\theta_1}\frac{\partial k}{\partial\theta_2} \neq 0$$

thus proving the statement, since $\frac{\partial k}{\partial\theta_i} > 0$ by the implicit function theorem.

Question 2

a) The efficient trading rule is such that if $\theta_i > \theta_j$ then j should sell the good to i for a price p. To get individual rationality $p \in [\theta_j, \theta_i]$.

b) Let b_i denote agent i's bid. The utilities of the agents conditional on the relative values of b_1 and b_2 are:

$$\begin{aligned} b_1 &\geq b_2 : u_1(b_1, b_2) = 2\theta_1 - b_1, u_2(b_1, b_2) = b_1 \\ b_2 &\geq b_1 : u_1(b_1, b_2) = b_2, u_2(b_1, b_2) = 2\theta_2 - b_2 \end{aligned}$$

Restricting the bids to linear bids, $b_i = a_i + c_i\theta_i$, agent 1 maximizes her expected utility:

$$\begin{aligned} Eu_1 &= E_{\theta_2} u_1 1_{\{b_1 \geq b_2\}} + E_{\theta_2} u_1 1_{\{b_2 \geq b_1\}} \\ &= \int_0^{\frac{b_1 - a_2}{c_2}} (2\theta_1 - b_1)\, d\theta_2 + \int_{\frac{b_1 - a_2}{c_2}}^{1} (a_2 + c_2\theta_2)\, d\theta_2 \end{aligned}$$

Using Leibniz's rule we obtain the FOC:

$$\frac{(2\theta_1 - b_1)}{c_2} - \int_0^{\frac{b_1 - a_2}{c_2}} d\theta_2 - \frac{1}{c_2}\left(a_2 + c_2 \frac{b_1 - a_2}{c_2}\right) = 0$$

which yields (after some simple algebra):

$$b_1 = \frac{a_2}{3} + \frac{2\theta_1}{3} \qquad (i)$$

The symmetric problem for agent 2 yields:

$$b_2 = \frac{a_1}{3} + \frac{2\theta_2}{3} \qquad (ii)$$

(i) and (ii) imply that $a_1 = a_2$ and $b_1 = b_2 = \frac{2}{3}$.

c) The social choice function that is implemented is: $\theta_i > \theta_j$ implies that i buys the good from j at price $\frac{2}{3}\theta_i$. The analysis in b) above shows that it is Bayesian incentive compatible, and clearly ex-post efficient. It can be easily shown that it is interim individually rational as well (it is not ex-post IR if $\theta_j \in \left(\frac{2}{3}\theta_i, \theta_i\right)$.) The result differs from the Myerson-Satterthwaite Theorem because of the symmetry of the agents: each agent is a buyer and a seller so there is *always* an efficient trade.

Question 3

a) The social optimum is given by

$$x^*(\theta) = \underset{x}{Arg\max}\left[-C(x,\theta) - D(x)\right]$$

A government with coercive power can give a firm which produces pollution x a transfer $t(x) = k - D(x)$. The firm with type θ then chooses x to maximize

$$t(x) - C(x,\theta) = k - C(x,\theta) - D(x)$$

which yields the social optimum. Like the Groves scheme, the transfer makes the firm pay the net loss of all other parties caused by its action.

b) If the firm may choose not to participate, the scheme above will still work as we simply choose k large enough that all firms will want to participate. If θ is bounded we can use:

$$k = \sup_{\theta}\left[C(x^*(\theta),\theta) + D(x^*(\theta))\right]$$

With a shadow cost of public funds, we must use the mechanism design approach of section 7.3.2.in Fudenberg & Tirole. The problem is

$$\begin{array}{ll} \underset{x(\theta),t(\theta)}{Max} & E_\theta\left[-D(x(\theta)) - (1+\lambda)t(\theta) + t(\theta) - C(x(\theta),\theta)\right] \\ \text{s.t.} & \theta = \underset{\theta'}{Arg\max}\left[t(\theta') - C(x(\theta'),\theta)\right] \quad \text{(IC)} \\ & t(\theta) - C(x(\theta),\theta) \geq 0 \quad \text{(IR)} \end{array}$$

Note first that if (IR) is satisfied for $\theta = \overline{\theta}$ it is automatically satisfied for all other θ (note the difference with respect to the case with a continuum of types in the notes) as

$$t(\theta) - C(x(\theta),\theta) \geq t\left(\overline{\theta}\right) - C\left(x\left(\overline{\theta}\right),\theta\right) \geq t\left(\overline{\theta}\right) - C\left(x\left(\overline{\theta}\right),\overline{\theta}\right) \geq 0$$

Also the (IR) constraint must hold with equality for $\overline{\theta}$ as otherwise we can reduce $t(\theta)$ by an equal amount for all θ, such that all (IR) constraints continue to hold and the (IC) constraints remain unaffected. This would not affect the constraints but improve welfare. Let

$$u_1(\theta) = \underset{\theta'}{Max}\, t(\theta') - C(x(\theta'),\theta)$$

be the indirect utility function. By the envelope theorem, $\frac{\partial u_1}{\partial \theta} = -\frac{\partial C}{\partial \theta}(x(\theta),\theta)$. Hence the (IC) constraint implies that:

$$u_1(\theta) = u_1\left(\overline{\theta}\right) - \int_\theta^{\overline{\theta}} \frac{\partial u_1}{\partial \theta'} d\theta' = 0 + \int_\theta^{\overline{\theta}} \frac{\partial C}{\partial \theta}(x(\theta'),\theta')\, d\theta'$$

and

$$t(\theta) = u_1(\theta) + C(x(\theta), \theta)$$

The problem is now

$$\underset{x(\theta)}{Max} \int_{\underline{\theta}}^{\overline{\theta}} \left[-D(x(\theta)) - \lambda u_1(\theta) - (1+\lambda) C(x(\theta), \theta)\right] p(\theta)\, d\theta$$

subject to $x(\theta)$ is non-increasing.

Note that, contrary to what we did in class, we are now requiring $x(\theta)$ to be non-increasing. This is just due to the fact that we are dealing with a different objective function and constraints. With this on mind, we can just replicate the proof we did in class and obtain the non-increasing condition.

Now, using integration by parts and Leibniz's rule,

$$\begin{aligned}
\int_{\underline{\theta}}^{\overline{\theta}} u_1(\theta)\, p(\theta)\, d\theta &= \int_{\underline{\theta}}^{\overline{\theta}} \int_{\theta}^{\overline{\theta}} \frac{\partial C}{\partial \theta}(x(\theta'), \theta')\, d\theta' p(\theta)\, d\theta \\
&= \int_{\theta}^{\overline{\theta}} \frac{\partial C}{\partial \theta}(x(\theta'), \theta')\, d\theta' P(\theta) |_{\underline{\theta}}^{\overline{\theta}} - \int_{\underline{\theta}}^{\overline{\theta}} \left(-\frac{\partial C}{\partial \theta}(x(\theta), \theta)\right) P(\theta)\, d\theta \\
&= \int_{\underline{\theta}}^{\overline{\theta}} \frac{P(\theta)}{p(\theta)} \frac{\partial C}{\partial \theta}(x(\theta), \theta)\, p(\theta)\, d\theta
\end{aligned}$$

Ignoring the monotonicity constraint for now, we wish to maximize

$$\int_{\underline{\theta}}^{\overline{\theta}} \left[-D(x(\theta)) - \lambda \frac{P(\theta)}{p(\theta)} \frac{\partial C}{\partial \theta}(x(\theta), \theta) - (1+\lambda) C(x(\theta), \theta)\right] p(\theta)\, d\theta$$

Assuming that $D(\cdot)$ is concave and the derivatives of $C(\cdot)$ are as in the problem, we can simply use the FOC for a maximum at each θ. This gives

$$\frac{\partial D}{\partial x}(x(\theta)) + (1+\lambda) \frac{\partial C}{\partial x}(x(\theta), \theta) + \lambda \frac{P(\theta)}{p(\theta)} \frac{\partial^2 C}{\partial x \partial \theta}(x(\theta), \theta) = 0$$

Let $x^*(\theta)$ be the solution to this equation. If $x^*(\theta)$ is in fact non-increasing, we have found the solution. Firms announce their true type θ, production is $x^*(\theta)$ and a transfer of $t(\theta)$ given by the earlier formula is used.

c) The d'Aspremont-Gerard-Varet scheme is described in Fudenberg & Tirole section 7.4.3. It consists of announcements $\theta_1', ..., \theta_I'$ and prescribes $x(\theta_1', ..., \theta_I') = x^*(\theta_1', ..., \theta_I')$ where $x^*(\theta_1', ..., \theta_I')$ is the social optimum given types $\theta_1', ..., \theta_I'$, i.e.

$$x^*(\theta_1', ..., \theta_I') = \arg\max_x \left[-\sum_i C_i(x, \theta_i) - D(x)\right]$$

The transfer is

$$t_i(\theta') = \varepsilon(\theta_i') + \tau_i\left(\theta'_{-i}\right)$$

where $\varepsilon(\theta_i)$ is the expected externality when player i announces θ_i',

$$\varepsilon(\theta_i') = E_{\theta_{-i}}\left[-\sum_{j\neq i} C_j\left(x_j\left(\theta_i', \theta_{-i}\right), \theta_j\right) - D\left(x_j\left(\theta_i', \theta_{-i}\right)\right)\right]$$

and $\tau_i\left(\theta'_{-i}\right) = -\frac{1}{I-1}\sum_{j\neq i}\varepsilon\left(\theta_j'\right)$. See section 7.4.3 in Fudenberg and Tirole for an explanation of why this does induce the social optimum.