Solutions for Problem Set 2

Question 1

a) The buyer has three options: sell to the high types, both types or none (leading to zero profits). If the seller sells to both types, then the maximum price that accomplishes this is $\underline{b}$ which yields a profit of $\underline{b}$. The maximum price that sells to only the high type is $\bar{b}$, which yields an expected profit of $\pi = \mu\bar{b}$. By assumption, $\mu\bar{b} > \underline{b}$, thus the seller will sell to only the high type and will set a price of $\bar{b}$.

Optimal Mechanism: By the revelation principle we can restrict ourselves to a truth-telling mechanism where the buyer announces his type $\hat{b}$ and gets $(q(\hat{b}), p(\hat{b})$ where $q(\hat{b}) \in [0, 1]$ is the probability of getting the good and $p(\hat{b})$ is the uncontingent transfer from the buyer to the seller. Let $\bar{q} = q(\bar{b}), \underline{q} = q(\underline{b}), \bar{p} = p(\bar{b})$, and $\underline{p} = p(\underline{b})$. The sellers optimal mechanism will then solve:

$$\begin{array}{cc} \max_{\underline{q},\bar{q},\underline{p},\bar{p}} & \mu\bar{p} + (1-\mu)\underline{p} \\ \text{s.t.} & \bar{q}\bar{b} - \bar{p} \geq \underline{q}\bar{b} - \underline{p} \quad (IC_H) \\ & \underline{q}\underline{b} - \underline{p} \geq \bar{q}\underline{b} - \bar{p} \quad (IC_L) \\ & \bar{q}\bar{b} - \bar{p} \geq 0 \quad (PC_H) \\ & \underline{q}\underline{b} - \underline{p} \geq 0 \quad (PC_L) \end{array}$$

Notice that (IC_H) and (PC_L) imply (PC_H), so we can ignore it. (PC_L) must then be binding (otherwise increase $\bar{p}$ and $\underline{p}$ by an equal small amount, this raises the value of the objective function without violating any constraint). Furthermore, (IC_H) is binding, since otherwise we could increase $\bar{p}$ by a small amount such that (IC_H), (IC_L) and (PC_L) would still be satisfied, and the objective function would be higher. As for (IC_L) there are two cases:

Case 1: (IC_L) binds. Now, by adding (IC_H) and (IC_L) we get $\bar{q} = \underline{q}$. From binding (PC_L), $\underline{p} = \underline{q}\underline{b}$, and from binding (IC_H), $\bar{p} = \underline{p}$. The problem then becomes:

$$\max_{\underline{q},\underline{p}} \ \underline{p} \text{s.t. } \underline{p} = \underline{q}\underline{b} \quad \underline{q} \in [0, 1]$$

which has the solution $\underline{q} = 1, \underline{p} = \underline{b}$, and the expected profit is $\underline{b}$.

Case 2: (IC_L) doesn't binds. Now, from binding (IC_H) and (PC_L) we get $\underline{p} = \underline{q}\underline{b}$, and $\bar{p} = \bar{q}\bar{b} - \underline{q}(\bar{b} - \underline{b})$, and the seller's problem becomes:

$$\max_{\underline{q},\underline{p}} \ \mu[\bar{q}\bar{b} - \underline{q}(\bar{b} - \underline{b})] + (1-\mu)\underline{q}\underline{b} \ = \ \mu\bar{q}\bar{b} + \underline{q}(\underline{b} - \mu\bar{b}) \text{s.t. } \underline{q}, \bar{q} \in [0, 1]$$

Since by assumption $\underline{b} - \mu\bar{b} < 0$, the solution to this program has $\underline{q} = 0, \bar{q} = 1, \underline{p} = 0, \bar{p} = \bar{b}$, and the expected profit is $\mu\bar{b}$, and this is the maximum (higher than case 1) so (IC_L) does not bind. This solution is the same as the "take-it-or-leave-it" equilibrium above.

b) Denote by p_2 the price charged to the buyer who did not buy in period $t = 1$, and by μ_2 the seller's updated beliefs in period $t = 2$, i.e., the updated probability of having a high type. From part **a)** above, it follows that:

$$\mu_2 > \frac{\underline{b}}{\bar{b}} \Rightarrow p_2 = \bar{b}\mu_2 < \frac{\underline{b}}{\bar{b}} \Rightarrow p_2 = \underline{b}\mu_2 = \frac{\underline{b}}{\bar{b}}\text{either} \Rightarrow p_2 = \bar{b}\text{or} \Leftarrow p_2 = \underline{b}$$

It also follows from part **a)** above that only (IC_H) binds at $t = 2$. Therefore, given any p_2, at $t = 1$ the seller wants to sell to as many high types as possible subject to the constraint that the resulting belief μ_2 leaves p_2 credible. There are three candidates for PBE:

Case 0: $p_2 = \bar{b}$. The seller can set $p_1 = \underline{b}$, and both types buy in period 1. Since Bayes rule does not change the beliefs, p_2 is credible and the seller's expected profits are:

$$\pi^0 = \underline{b} + \delta\mu\bar{b}$$

Case 1: $p_2 = \bar{b}$. The seller can set $p_1 = \bar{b}$, and the $\bar{b}$ type will have no reason not to buy in both periods if he believes these prices to remain. To sustain $p_2 = \bar{b}$, however, we cannot have all the $\bar{b}$ types buying at $t = 1$. Denote by ρ the proportion of $\bar{b}$ types buying at $t = 1$. The maximum ρ is determined by Bayes rule in the following way:

$$\mu_2 = \frac{(1-\rho)\mu}{(1-\rho)\mu + (1-\mu)} = \frac{\underline{b}}{\bar{b}} \Rightarrow \rho = \frac{\mu - \frac{\underline{b}}{\bar{b}}}{\mu(1 - \frac{\underline{b}}{\bar{b}})} > 0$$

In this case at $t = 2$ all the high types buy, and the low types ($\underline{b}$) never buy. The seller's expected profit is:

$$\pi^1 = \mu(\rho + \delta)\bar{b} = \frac{\bar{b}\mu - \underline{b} + \delta\mu(\bar{b} - \underline{b})}{1 - \frac{\underline{b}}{\bar{b}}}$$

Case 2: $p_2 = \underline{b}$. In this case the seller can sell to all high types at $t = 1$, and charge them $\bar{b}$ at $t = 2$ after they reveal themselves. However, the price that the high types will be charged at $t = 1$ cannot be $p_1 = \bar{b}$. A high type must not find it beneficial to pretend being a low type and waiting for period 2. This is satisfied if:

$$(\bar{b} - p_1) + (\bar{b} - \bar{b}) \geq 0 + \delta(\bar{b} - \underline{b})$$

or,

$$p_1 \leq \bar{b} - \delta(\bar{b} - \underline{b})$$

Obviously, the seller will set p_1 so that the above holds with equality, and the expected profits are:

$$\pi^2 = \mu(\bar{b} - \delta(\bar{b} - \underline{b})) + \delta(\mu\bar{b} + (1-\mu)\underline{b}) = \mu\bar{b} + \delta\underline{b}$$

Finally, we compare the candidates above. It is easy to check that for $\delta < 1$ we get $\pi^2 > \pi^0$. Some simple algebra yields that $\pi^1 > \pi^2$ if and only if

$$\mu > \frac{\underline{b}\bar{b} + \delta\underline{b}\bar{b} - \delta\underline{b}^2}{\underline{b}\bar{b} - \delta\underline{b}\bar{b} + \delta\bar{b}^2}$$

c) First note that a high type buyer is more eager to take a "sale contract" than a low type buyer. Consider an equilibrium where some proportion of the high types take a sale contract at $t = 1$ at a price of q. From **a)** above we know that at $t = 2$ we may have $p_2 = \bar{b}$ or $p_2 = \underline{b}$.

Case 1: $p_2 = \bar{b}$. Then the sale contract can be replicated by two "rental contracts": Charge $p_1 = q - \delta\bar{b}$, and $p_2 = \bar{b}$ regardless of whether the rental contract at $t = 1$ was purchased.

Case 2: $p_2 = \underline{b}$. Then the maximum price q that the high type would pay for the sale contract at $t = 1$ is determined by their utility from rejecting the contract and buying at $t = 2$ at $p_2 = \underline{b}$. This of course implies that $q \leq \bar{b} + \delta\underline{b}$, and the seller will choose q for this to hold with equality. The seller will also find it optimal to have all high types buying at $t = 1$. Now observe that this case can also be replicated by rental contracts. The seller charges $p_1 = \bar{b} - \delta(\bar{b} - \underline{b}) > \underline{b}$ at $t = 1$, and then charge $p_2 = \underline{b}$ if the first contract was rejected, and $p_2 = \bar{b}$ if it was accepted (which replicates case 2 in part **b)**).

Therefore, long term contracts do not help in out case. This is not true for $T > 2$ where the "ratchet effect" plays a role (see Hart-Tirole 1988), and using long term contracts enables the seller to commit not to raise the price. We have seen that for $T = 2$ the seller's concern is to commit not to lower the price. Thus for $T = 2$ the ratchet effect does not occur.

Question 2

part (a)

The principal's problem is to implement $e = 1$ at the lowest cost subject to (IR), (IC) and (LL). It is easy to see that (LL) and (IC) imply (IR), so we have

$$\begin{aligned}
\min_{w(i), i\in\{1,2,3\}} \quad & \textstyle\sum_{i=1}^{3} \pi(i|e=1)w(i) \\
\text{s.t.} \quad & \textstyle\sum_{i=1}^{3} \pi(i|e=1)w(i) - v(1) - \sum_{i=1}^{3} \pi(i|e=0)w(i) \geq 0 \quad (IC) \\
& s(i) \geq 0 \text{ for } i \in \{1,2,3\} \quad (LL)
\end{aligned}$$

and the Lagrangian is:

$$\mathcal{L} = \sum_{i=1}^{3} \pi(i|e=1)w(i) - \lambda \left[\sum_{i=1}^{3} \left[\pi(i|e=1) - \pi(i|e=0)\right] w(i) - v(1) \right] - \sum_{i=1}^{3} \mu_i w(i)$$

where λ is the (IC) multiplier and μ_i is the (LL) multiplier for payment $w(i)$. FOC's w.r.t. the payments are,

$$\pi(i|e=1)w(i) - \lambda \left[\pi(i|e=1) - \pi(i|e=0)\right] - \mu_i = 0, \; i \in \{1,2,3\} \tag{FOC}$$

and complementary slackness conditions for the μ_i's are,

$$w(i)\mu_i = 0, \, i \in \{1,2,3\} \tag{CS}$$

Note that $w(i) > 0$ for some i^* since otherwise (IC) would be violated, and in turn (CS) implies that $\mu_{i^*} = 0$. Then, from (FOC) of payment i^* we have that

$$\lambda = \frac{1}{1 - \frac{\pi(i^*|e=0)}{\pi(i^*|e=1)}} > 1$$

and then for any i (FOC) yields,

$$\frac{\mu_i}{\pi(i|e=0)} = \lambda - (\lambda - 1)\frac{\pi(i|e=1)}{\pi(i|e=0)} \tag{1}$$

From $\frac{\pi(i|e=1)}{\pi(i|e=0)}$ increasing in i (MLRP), and from $\lambda > 1$ together with (1) we have,

$$\frac{\mu_1}{\pi(1|e=0)} > \frac{\mu_2}{\pi(2|e=0)} > \frac{\mu_3}{\pi(3|e=0)} \geq 0 \, .$$

As we have concluded, $\mu_{i^*} = 0$ for some i^*, and these inequalities imply that this can only happen for $i^* = 3$, implying in turn that $\mu_1 > \mu_2 > 0$. By (CS) these conditions imply that $w(1) = w(2) = 0$ and $w(3) > 0$.

Remark: This conclusion can be obtained by a contradiction argument. If $w = (w(1), w(2), w(3))$ is a solution with $w(\widetilde{i}) > 0$ for some $i < 3$ then consider the perturbed contract w' with $w'(\widetilde{i}) = w(\widetilde{i}) - \varepsilon$ and $w'(3) = w(3) - \varepsilon \frac{\pi(\widetilde{i}|e=1) - \pi(\widetilde{i}|e=0)}{\pi(3|e=1) - \pi(3|e=0)} > w(3)$. By construction, (IC) and (LL) are satisfied, and from MLRP the principal's expected costs are lower with contract w', a contradiction.

part (b)

This can be shown with a contradiction argument. The restrictions on payments $\frac{w(2)-w(1)}{2-1} \in [0,1]$ and $\frac{w(3)-w(2)}{3-2} \in [0,1]$ imply that the principal's problem is the above problem with one more constraint:

$$w(2) - w(1) \in [0,1] \text{ and } w(3) - w(2) \in [0,1] \tag{M}$$

The arguments above will imply the contradiction.