Solutions for Problem Set 3

Question 1

a) The first-best level of effort solves $\max E(ey) - g(e)$. This is a concave problem, and the first-order condition yields the optimal level of effort $e^* = Ey = 1$. If the project is undertaken, the total surplus is

$$E(e^*y) - g(e^*) - K = \frac{1}{2} - K$$

The project is worth undertaking when this value is non-negative, i.e., $K \leq \frac{1}{2}$. $\square$

b) E solves the following optimal contracting problem:

$$\begin{array}{lll} \max_{e,w(\cdot)} & Ew(ey) - g(e) & \\ \text{s.t.} & E(ey) - Ew(ey) \geq K & \text{(I's IR)} \\ & e \in arg\max_e Ew(ey) - g(e) & \text{(E's IC)} \\ & w(x) \geq 0 \forall x & \text{(LL)} \end{array}$$

Consider a contract of the form $w(x) = \begin{cases} 0 & \text{for } x < a, \\ \overline{w} & \text{for } x \geq a \end{cases}$ for some $a > 0$ and $\overline{w} > 0$. For this contract we have

$$Ew(ey) = \int_{\frac{a}{e}}^{2} \overline{w} f(y)dy = \int_{\frac{a}{e}}^{2} \overline{w}\frac{1}{2}dy = \overline{w}(1 - \frac{a}{2e})$$

Look first at E's Incentive Constraint. His problem now becomes

$$\max_e \ \overline{w}(1 - \frac{a}{2e}) - \frac{1}{2}e^2$$

It is easy to check that the second derivative of the maximand is negative, and the problem is concave. The first-order condition yields

$$\overline{w}\frac{a}{2e^2} - e = 0$$

. We want E to choose the first-best effort $e^* = 1$. Substituting, we get $\overline{w} = \frac{2}{a}$. Now, turn to I's individual Rationality constraint. To maximize E's payoff, we want to make this constraint binding:

$$e - \overline{w}(1 - \frac{a}{2e}) = K$$

Substituting the first-best effort $e^* = 1$, and $\overline{w}$ from the previous expression, we obtain

$$1 - \frac{2}{a}(1 - \frac{a}{2}) = K \leq \frac{1}{2}$$

This implies that $a \leq \frac{4}{3}$ and $\overline{w} \geq \frac{3}{2}$. Therefore, from I's (IR) we have $\overline{w} \geq 2 - k$, and from all the above we get:

$$w(x) = \begin{cases} 0 & \text{for } x < \frac{2}{2-k} \\ 2-k & \text{for } x \geq \frac{2}{2-k} \end{cases}$$

□

c) If E can costlessly destroy and borrow output before it is observed by outsiders, the sharing rule $w(x)$ is effectively constrained to satisfy $0 \leq w'(x) \leq 1$ for every x. E's problem now becomes

$$\begin{array}{lll} \max_{e,w(\cdot)} & Ew(ey) - g(e) & \\ \text{s.t.} & E(ey) - Ew(ey) \geq K & \text{(I's IR)} \\ & e \in arg\max_e Ew(ey) - g(e) & \text{(E's IC)} \\ & w(0) \geq 0 & \text{(LL)} \\ & 0 \leq w'(x) \leq 1 \forall x & \end{array}$$

We will replace E's Incentive Constraint with the corresponding first-order condition, solve the resulting relaxed problem and then show that at a solution E's effort is indeed globally optimal for him. The first-order condition for E's IC is $E(ew'(ey)) - g'(e) = 0$.

Also, we express the objective function and I's IR in terms of $w(0)$ and $w'(\cdot)$, using integration by parts:

$$\begin{aligned} Ew(ey) &= \int_0^2 w(ey)f(y)dy = w(ey)F(y)|_0^2 - \int_0^2 ew'(ey)F(y)dy = \\ &= w(2e) - \int_0^2 ew'(ey)\frac{y^2}{4}dy = \\ &= w(0) + \int_0^2 ew'(ey)dy - \int_0^2 ew'(ey)\frac{y^2}{4}dy. \end{aligned}$$

Now we can rewrite E's problem as

$$\begin{array}{lll} \multicolumn{3}{l}{\max_{e,w(0),w'(\cdot)} w(0) + \int_0^2 ew'(ey)(1 - \frac{y^2}{4})dy - \frac{1}{2}e^2} \\ \multicolumn{3}{l}{\text{s.t.}} \\ e - w(0) - \int_0^2 ew'(ey)(1 - \frac{y^2}{4})dy \geq K & \text{(I's IR)} & \mu \geq 0 \\ \int_0^2 ew'(ey)\frac{1}{2}dy - e = 0 & \text{(E's FOC)} & \nu \\ w(0) \geq 0 & \text{(LL)} & \lambda \geq 0 \\ 0 \leq w'(x) \leq 1 \ \forall x & & \end{array}$$

The last column contains corresponding Lagrange multipliers. The resulting Lagrangian is

$$L = (1 - \mu + \lambda)w(0) + \int_0^2 ew'(ey)[(1-\mu)(1 - \frac{y^2}{4}) + \frac{1}{2}\nu]dy + (\mu - \nu)e - \frac{1}{2}e^2$$

Differentiating the Lagrangian with respect to $w(0)$, we obtain $1 - \mu + \lambda = 0$, which implies that $\mu = 1+\lambda \geq 1$. Therefore, the expression in square brackets inside the integral is monotone increasing in y. Maximizing the integral with respect to $w'(\cdot)$ pointwise subject to the constraint $0 \leq w'(x) \leq 1$, we get

$$w'(ey) = \begin{cases} 1 \text{ when } [(1-\mu)(1-\frac{y^2}{4}) + \frac{1}{2}\nu] \geq 0, \\ 0 \text{ otherwise.} \end{cases}$$

Since the expression in the square brackets is increasing in y, it must be that for some a:

$$w'(x) = \begin{cases} 1 \text{ when } x \geq a, \\ 0 \text{ otherwise.} \end{cases}$$

Assuming that (LL) binds, we obtain by integrating

$$w(x) = \begin{cases} x - a \text{ when } x \geq a, \\ 0 \text{ otherwise.} \end{cases}$$

Now it is only left to check that given a contract of this form, E's first-order condition will indeed yield the globally optimal effort level. Given such a contract, E's maximization problem becomes

$$\max_e Ew(ey) - g(e) = \int_{\frac{a}{e}}^{2} (ey-a)\frac{1}{2}dy - \frac{1}{2}e^2 = e(4 - \frac{a^2}{e^2}) - \frac{a}{2}(2 - \frac{a}{e}) - \frac{1}{2}e^2$$

If the objective function is expressed in terms of $t = \frac{1}{e}$, it will be concave. Therefore, the objective function has a unique local maximum, which is obtained from the first order conditions. $\square$

d) The contract obtained in (c) is a debt contract.

Question 2

a) Normalize both agents' reservation utilities to zero. Consider the following compensation scheme:

$$v_i(x_1, x_2) = \begin{cases} g(H), \text{ when } x_1 = x_2 \\ g(L) - \delta, \text{ when } x_1 \neq x_2 \end{cases},$$

where $\delta > 0$. In this scheme, (H, H) is a Nash equilibrium. Furthermore, in equilibrium both agents bear no risk, therefore this scheme implements the first best and has the minimum cost. □

b) It is clear from (a) that any least-cost scheme has both agents bear no risk when they choose (H, H), therefore we must have $v_i(0, 0) = v_i(1, 1) = g(H)$. To prevent unilateral deviations from (H, H), we must have

$$\frac{1}{2}v_i(0,1) + \frac{1}{2}v_i(1,0) - g(L) \leq v_i(1,1) - g(H) = 0$$

But this implies that (L, L) is also a Nash equilibrium. Indeed, agent i's utility is $g(H) - g(L)$ when agents play (L, L), and only

$$\frac{1}{2}v_i(0,1) + \frac{1}{2}v_i(1,0) - g(L) \leq 0$$

when he deviates. Therefore, each agent does not want to deviate from (L, L) unilaterally, and this is a Nash equilibrium. Moreover, in this equilibrium each agent i gets $g(H) - g(L) > 0$, i.e., more than what he gets in the equilibrium (H, H). □

c) Let agent 1's message space be $\{\hat{H}, \hat{L}\}$. The "extended" compensation scheme $v_i(m, x_1, x_2)$ now also depends on agent 1's message m. Define the compensation scheme as follows:

$$v_i(\hat{H}, x_1, x_2) = \begin{cases} g(H), \text{ when } x_1 = x_2 \\ g(L) - \delta, \text{ when } x_1 \neq x_2 \end{cases},$$

$$v_1(\hat{L}, x_1, x_2) = \begin{cases} v_1(\hat{H}, x_1, x_2) + \epsilon, \text{ when } (x_1, x_2) = (0,0) \\ v_1(\hat{H}, x_1, x_2) - \epsilon, \text{ when } (x_1, x_2) = (1,1) \\ v_1(\hat{H}, x_1, x_2), \text{ when } x_1 \neq x_2 \end{cases},$$

$$v_2(\hat{L}, x_1, x_2) = \begin{cases} g(L) - \gamma, \text{ when } x_1 = x_2 \\ v_2(\hat{H}, x_1, x_2), \text{ when } x_1 \neq x_2 \end{cases},$$

where $\delta > 0$, $\epsilon > 0$ and $\gamma > \delta + (g(H) - g(L))$. This scheme coincides with the scheme in (a) when agent 1 announces $\hat{H}$. When agent 1 announces $\hat{L}$, on the other hand, he is given a lottery, where he gains ϵ when outputs are $(0, 0)$ and loses ϵ when outputs are $(1, 1)$. As for agent 2, he is severely punished by a very large γ when agent 1 announces $\hat{L}$ and their outputs coincide. This scheme induces in the following payoff matrix (denote $a = g(H) - g(L)$):

	H	L
$H\hat{H}$	$0, 0$	$-a-\delta, -\delta$
$L\hat{H}$	$-\delta, -a-\delta$	a, a
$H\hat{L}$	$-\frac{\epsilon}{2}, -a-\gamma$	$-a-\delta, -\delta$
$L\hat{L}$	$-\delta, -a-\delta$	$a+\frac{\epsilon}{2}, -\gamma$

CLAIM. *The only Nash equilibrium in this game is* $(H\hat{H}, H)$.

PROOF. Suppose that agent 2 plays L with a positive probability. Then for agent 1 $H\hat{L}$ is dominated by $H\hat{H}$, and $L\hat{H}$ is dominated by $L\hat{L}$. But if agent 1 only plays $H\hat{H}$ and $L\hat{L}$ with a positive probability, then for agent 2 L is dominated by H. Therefore, we get a contradiction, and agent 2 must play H with probability one. But the unique best response to H is $H\hat{H}$. Thus, we have shown that $(H\hat{H}, H)$ is the unique Nash equilibrium. $\square$

Furthermore, observe that in equilibrium the agents bear no risk and are at their reservation utility levels, which implies that this is a least-cost (in this case, first-best) scheme.

Question 3

Part a

Here the maximization problem is to:

$\max_{\alpha,\beta}\ a_1^{\frac{1}{4}}a_2^{\frac{1}{2}} - \alpha a_1 - \beta$
s.t.

$$(a_1, a_2) = argmax(\alpha a_1 + \beta - C(a_1, a_2) - \tfrac{1}{4}\alpha^2\sigma^2)$$
$$\alpha a_1 + \beta - \tfrac{1}{4}\alpha^2 - C(a_1, a_2)) \geq 0$$

The first equation is the incentive compatibility constraint. I've reduced it by factoring out the noise term out of the utility function. (Another way of saying it is that I've computed the certainty equivalent.) The second is the participation constraint. We can use the first-order approach here since it satisfies the conditions given in class (MLRP and the distribution functions are convex). Hence we can solve for the agent's optimum, and set the participation constraint binding. The agent's problem is to optimize:

$$\alpha a_1 + \beta - (a_1 + a_2)^2 + 4(a_1 + a_2) - \frac{1}{4}\alpha^2$$

Hence the first order conditions give (obviously) that

$$(a_1 + a_2) \leq 4$$

and

$$(a_1 + a_2) \leq 4 + \alpha$$

Hence if $\alpha > 0$, $(a_1, a_2) = (4 + \alpha, 0)$. If $\alpha = 0$ the agent will put forth any combination fo effort whose sum is 4. If $\alpha < 0$ then[1] $(a_1, a_2) = (0, 4 + \alpha)$. Then β is given by d

Thus if the principal choose $\alpha \neq 0$ the firm gets no output. Hence the firm will chose $\alpha = 0$. In this case, the participation constraint gives us the β which is equal to

$$\beta^* = \frac{1}{4}\alpha^2 - 8$$

Part b

Again the maximization problem is:

$\max_{\alpha,\beta}\ 18ln(a_1) + 6ln(a_2) - \alpha a_1 - \beta$
s.t.

$$(a_1, a_2) = argmax(\alpha a_1 + \beta - C(a_1, a_2) - \tfrac{1}{4}\alpha^2\sigma^2)$$
$$\alpha a_1 + \beta - \tfrac{1}{4}\alpha^2 - C(a_1, a_2)) \geq 0$$

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[1] It's not unreasonable to consider this case. In this problem the agent gets a particular pleasure in working at the firm. Hence the principle might be able to "charge" him for it.

The agent's problem hasn't changed so he'll still decide the same action based on α. Since $ln(a_1)$ goes to infinity as $a_1 \to 0$, it follows that $\alpha \geq 0$.

Assume $\alpha > 0$. Let's plug in for α into the maximization and solve. I'm ignoring the participation constraint for now, it can be made to hold ex post by using β. The maximand then becomes:

$$18ln(4+\alpha) + 6ln(1) - \frac{(\alpha+4)^2}{2} + 4(\alpha+4) - \frac{1}{4}\alpha^2$$

Taking the derivative with respect to alpha gives:

$$\frac{18}{\alpha+4} = \frac{\alpha}{2}$$

We can solve this to get that(one of the two solutions to the quadratic is negative):

$$\alpha^* = 2$$

with

$$a_1^* = 6$$

Finally profits are:

$$18ln(6) + 5$$

The case where $\alpha = 0$ is simple because then the agent chooses action such that $a_1 + a_2 = 4$. Thus we can solve for the optimal split of this for the principal who has payoff

$$18ln(a_1) + 6ln(1+a_2) + C$$

where C is some constant independent of a_i. Setting up the Lagrangian and solving gives that

$$(a_1^*, a_2^*) = (\frac{18.5}{24}, \frac{6.5}{24})$$

Thus we get the payoff of

$$18ln(\frac{18.5}{24}) + 6ln(\frac{30.5}{24}) + 8$$

But computation gives that the profit from $\alpha = 2$ is bigger, so we get that $\alpha = 2$.

Part c

The strange thing is that we have $\alpha > 1$ which is counterintuitive. It violates the intuitive constraint we have that the incentive scheme should have derivative less than 1. More surprising is that this incentive scheme has slope greater than 1 everywhere. In a traditional principal-agent problem this would correspond to selling the firm many times over.

Part d

Holmstrom and Milgrom show that if four conditions are satisfied you get $\alpha = 0$:

1. $C(t_1, t_2) = C(t_1 + t_2)$; and the cost function is strictly convex in $t_1 + t_2$. Moreover the cost function is u-shaped; it has a minimum at $\bar{t}$.
2. Action 1 is measurable with noise, while 2 is not contractible.
3. $B(t_1, t_2)$ is increasing in t_i.
4. $B(0, t_2) = 0$.

1-3 are satisifed here. But 4 is not since $18ln(0) = \infty$. This is why we don't get $\alpha = 0$