ELSEVIER

Journal of Financial Economics 56 (2000) 485–516

JOURNAL OF Financial ECONOMICS

www.elsevier.com/locate/econbase

From cradle to grave: How to loot a 401(k) plan[☆]

Richard Stanton*

Haas School of Business, U.C. Berkeley, 545 Student Services Building #1900, Berkeley, CA 94720-1900, USA

Received 29 May 1997; received in revised form 5 July 1999

Abstract

The regulations governing asset distributions from many retirement plans give participants the option to time retirement or rollovers from the plan strategically. They possess a long-lived put option, whose exercise price resets periodically to the current value of the assets in the plan. I derive a recursive closed-form valuation formula for the option and develop a numerical algorithm for implementing the result. I find that, for reasonable assumptions about volatility and life expectancy, the option's value may approach 40% of the value of the assets in the plan, financed entirely by those still contributing. This wealth transfer can, however, be easily avoided by making a simple change to the current regulations governing valuation and payout of these retirement plans.

JEL classification: G12; G13; G23

Keywords: 401(k) plan; Pension plan; Rollover; Option valuation

☆I thank Jonathan Berk, Richard Lyons, Mark Rubinstein, and the referee for helpful comments and suggestions, Sam Thompson and Joel Vanden for expert research assistance, and Nicky King and the administrators of the Sola Optical 401(k) plan for first making me aware of the subject of this article. I gratefully acknowledge financial assistance from the Berkeley Program in Finance, from the U.C. Berkeley Committee on Research, and from the Fisher Center for Real Estate and Urban Economics.

* Tel.: + 1-510-642-7382; fax: + 1-510-643-1420.

E-mail address: stanton@haas.berkeley.edu (R. Stanton).

PII: S0304-405X(00)00048-9

1. Introduction

The *Wall Street Journal* reported on Thursday October 8, 1998, that American Airlines was having to cancel significant numbers of flights due to a shortage of pilots. In September and October, 1998, more than three times as many pilots retired as during an average month, with the same expected to continue in November. This surge in retirements was occurring because "pilots retiring now can take away retirement distributions based on July's stock-market prices… [A] pilot's lump-sum retirement check would be as much as $300,000 lower if he or she had to take today's stock price" (Wall Street Journal, 1998). Similar accelerated retirements occurred after the stock market crash of 1987. For example, the *Wall Street Journal* reported on Monday November 2, 1987, that over 600 Lockheed Corp. employees had submitted early retirement papers the previous Friday, October 30 (approximately three times the usual monthly figure), because 'employees opting to retire by the last working day of October would have their shares of a stock-based savings plan valued as of September 30 [1987]' (Wall Street Journal, 1987). By retiring early and taking out their assets at a *historical* value, employees were able to avoid the large losses they would otherwise have incurred due to stock market declines. The loss was instead borne by the remaining participants in the plans.

Participants in many 401(k) retirement plans possess a similar option. If they leave their employer (for example, to work for a different company) they may at any time roll their accumulations into another plan. For many plans, the amount transferred on such a rollover is the value of the account on the most recent valuation date (usually quarterly) *prior* to the rollover. For example, in late June 1995 an acquaintance asked to roll over her 401(k) account at her old employer, Sola Optical, into a Keogh account. The 401(k) account was invested in two publicly quoted mutual funds, and the account balance reported on the most recent statement, 3/31/95, was just over $46,000. This was also the amount of the distribution check received at the beginning of July, 3 months later, even though it was simple to determine that the *market value* of the investments on that date was over $6,000 higher. This option allows people who have left the firm to time their rollovers strategically, waiting for a sharp drop in the market before asking to roll over their accounts and thereby avoiding the drop.

This paper shows that both the 'retirement option' and the 'rollover option' described above are long-lived put options, whose exercise price adjusts periodically to the current value of the underlying assets. I derive a recursive, closed-form valuation formula for this option and develop an efficient numerical valuation algorithm, allowing me to value the option and to calculate the optimal rollover/retirement strategy for an investor in one of these plans. I find, for example, that, given reasonable assumptions about life expectancy and volatility, the option to roll over a 401(k) plan strategically may be worth 40% of the value of the underlying assets, financed entirely by those still contributing to the plan. This wealth transfer can, however, be completely avoided by making a simple change to the current regulations governing valuation and payout of 401(k) and other retirement plans.

2. The option

Even though 401(k) and other retirement plans often invest their funds in assets whose value can be determined every day (such as mutual funds), the vast majority of 401(k) plans are not valued daily. The law merely requires plans to be valued at least once *per year*. Annual, semi-annual, quarterly, and monthly valuation are all common, with quarterly valuation the most widespread among smaller plans.[1]

Apart from retirement, employees who have left a firm have another way to withdraw funds from a 401(k) plan. They can 'roll over' their accumulations at will directly to another eligible plan, such as another employer's 401(k) plan (if it accepts rollovers) or a rollover IRA account. Such rollovers do not result in any liability for taxes or penalties and do not have to occur immediately on separation from the company.

When a rollover or retirement occurs between two valuation dates, it is common (as with the American Airlines, Lockheed, and Sola plans above) for the amount withdrawn to be the balance on the valuation date *prior* to the withdrawal, with some adjustment for contributions/withdrawals made since that date, but *not* adjusted for changes in market value since the last valuation date. While published data on the extent of this practice do not exist, a manager of 401(k) plans told me in January 1997 that approximately 60% of the (mainly small) plans she managed used quarterly valuation, and that, of these, roughly 20% used the last historical balance to determine the amount of any rollover.

Since markets are, on average, rising, the use of a historical account balance to determine the amount withdrawn reduces the expected payoff to an employee who withdraws without conditioning on the recent behavior of asset prices (such as the Sola employee mentioned above). Much more significantly, however, this policy gives plan participants a valuable option to time their retirement or rollovers to another plan strategically. By retiring or rolling over the account just after a sharp drop in the value of the plan's assets (as did the American Airlines employees mentioned above), a 401(k) participant receives the value of the assets on the previous valuation date, completely avoiding the drop. This option is financed entirely by participants still contributing to the plan. Even if only (a very conservative) 1% of all 401(k) plan assets are handled this way, this

[1] The Appendix contains a brief summary of the main regulations governing valuation and distribution of 401(k) plans. Consult Franz et al. (1997) for details.

affects at least \$5 billion, almost double the market capitalization of Apple Computer as of August 1997.

3. Valuing the option

Consider a 401(k) investor who has already left the firm and is considering the optimal time to roll over the money in a 401(k) account. Let S_t be the value of the non-dividend paying assets in the plan. Although the assets in which a 401(k) plan is invested usually do pay dividends, these dividends are reinvested in the plan. Regard S_t as the gross price process, including reinvested dividends. Assume that the plan is revalued only at discrete times 0, Δ, $2\Delta, \ldots$, and, for all $t \geqslant 0$, define $[t]$ to be the last valuation date prior to time t; i.e.,

$$[t] \equiv \max_{\substack{i \in \mathbb{N} \\ i\Delta < t}} i\Delta, \tag{1}$$

If a rollover occurs at time τ, the amount transferred is the balance on the previous valuation date,

$$S_{[\tau]} = S_\tau + (S_{[\tau]} - S_\tau). \tag{2}$$

The total value of the participant's position at time t can thus be written as

$$V_t = S_t + \mathscr{P}_t, \tag{3}$$

where $\mathscr{P}_t$ is the value of an asset whose payout, if exercised at time τ, is

$$\mathscr{P}_\tau = S_{[\tau]} - S_\tau. \tag{4}$$

This is a put option on the underlying assets, whose exercise price adjusts every Δ years to the current value of the underlying asset, S_Δ. It is related to the 'wild card' option embedded in many options and futures contracts,[2] and to the S&P 500 bear market warrant studied by Gray and Whaley (1997). While the exercise price on an S&P 500 bear warrant resets only once, the exercise price on the option described here resets *repeatedly*.

Although this 'rollover put' option plan has no formal expiration date, it does expire with its holder. Suppose there remain i valuation dates until the expiration of the option ($i \geqslant 0$), and write $\mathscr{P}^i(S, K, t)$ for its value, where S is the current asset value, K is the current exercise price (equal to the asset value on the most recent valuation date, $S_{[t]}$), and $0 \leqslant t \leqslant \Delta$.

---

[2] For analysis of the wild card option in Treasury bond futures contracts, see Arak and Goodman (1987), Chance and Hemler (1993), Cohen (1995), Gay and Manaster (1986), and Kane and Marcus (1986). Fleming and Whaley (1994) and Valerio (1992) develop binomial valuation algorithms for option contracts with embedded wild card options.

At a revaluation date, Δ, the option's payoff if exercised is $K - S_\Delta$. If it is *not* exercised, the exercise price resets to the current asset value, S_Δ, and the value of the option (now with $i-1$ revaluations until maturity) is therefore $\mathscr{P}^{i-1}(S_\Delta, S_\Delta, 0)$. The option therefore satisfies the boundary condition

$$\mathscr{P}^i(S_\Delta, K, \Delta) = \max[\mathscr{P}^{i-1}(S_\Delta, S_\Delta, 0), K - S_\Delta], \tag{5}$$

for $i \geqslant 0$, where

$$\mathscr{P}^{-1}(S, K, t) \equiv 0. \tag{6}$$

Given a model for the dynamics of the process S_t (and possibly other variables, such as interest rates or volatility), we can now value this option by solving a standard pricing equation, subject to this boundary condition. In general, this will be computationally very burdensome, as the value of the position depends on the values of both S and K, which change over time. We can, however, reduce the computational burden substantially and obtain more intuition and closed-form solutions by assuming that

A1. The value of a standard European put or call option is homogeneous of degree 1 in the value of the underlying asset and the exercise price.
A2. The only state variable relevant for pricing a put or call option is the current value of the underlying asset, S_t.

These assumptions hold in a Black–Scholes (1973) world, but they also hold more generally, including in worlds in which stock prices follow jump or jump-diffusion processes.

Assume that $\mathscr{P}^{i-1}(S, K, t)$ is homogeneous of degree 1 in S and K (I shall prove this homogeneity later). Eq. (5) then becomes

$$\mathscr{P}^i(S_\Delta, K, \Delta) = \max[S_\Delta \mathscr{P}^{i-1}(1, 1, 0), K - S_\Delta], \tag{7}$$

for $i \geqslant 0$. This is shown, for different values of S_Δ, in Fig. 1. The investor optimally exercises the option if $S_\Delta < \underline{S}_i$, where $\underline{S}_i$ is given by

$$K - \underline{S}_i = \underline{S}_i \mathscr{P}^{i-1}(1, 1, 0), \tag{8}$$

i.e.,

$$\underline{S}_i = K/[1 + \mathscr{P}^{i-1}(1, 1, 0)]. \tag{9}$$

Fig. 1 shows that that an alternative way of achieving exactly the same payoff is to buy, and hold until time Δ, a portfolio consisting of $\mathscr{P}^{i-1}(1, 1, 0)$ units of the underlying asset, and $[1 + \mathscr{P}^{i-1}(1, 1, 0)]$ put options on the asset, with expiration date Δ, and exercise price $\underline{S}_i = K/[1 + \mathscr{P}^{i-1}(1, 1, 0)]$. By arbitrage, the value of this portfolio must equal the value of the option, so

$$\mathscr{P}^i(S, K, t) = \mathscr{P}^{i-1}(1, 1, 0)S + [1 + \mathscr{P}^{i-1}(1, 1, 0)] \\ P(S, K/[1 + \mathscr{P}^{i-1}(1, 1, 0)], \Delta - t), \tag{10}$$

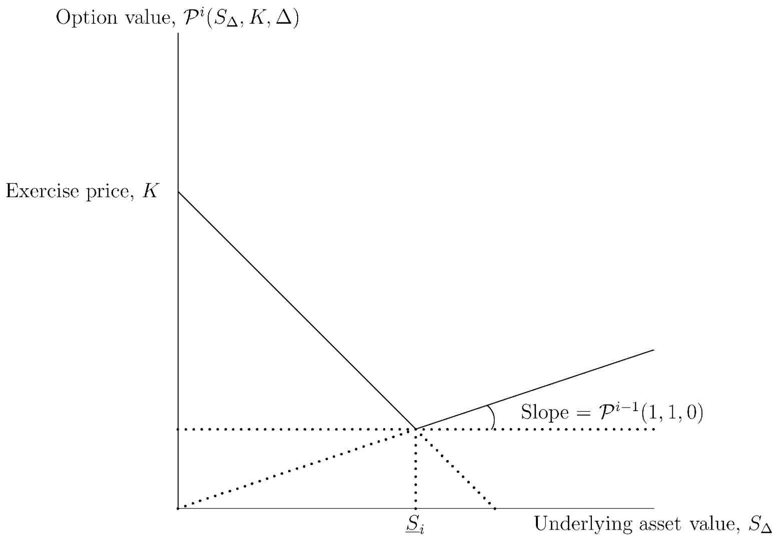

Fig. 1. Value of a rollover put option on a revaluation date. The solid line shows the value of a rollover put option on a revaluation date, with i revaluations left until expiration. If the underlying asset is worth more than $\underline{S}_i$, it is optimal to hold on to the option, now with $i-1$ revaluation periods until expiration. If the underlying asset value falls below the critical value $\underline{S}_i$, it is optimal to exercise the option and receive the payoff $K - S_\Delta$.

for all $i \geqslant 0$, where $P(S,K,t)$ is the value of a standard put option with current asset value S, exercise price K, and time to expiration t. $\mathscr{P}$ is a rollover (periodic reset) put; P is a standard put option. Exactly, the same equation will hold if we allow exercise of the option at any time *between* revaluation dates, except that P will now be the value of an *American* put option.

Note that the right-hand side of Eq. (10) is homogeneous of degree 1 in S and K due to the assumed homogeneity of P. Hence, if $\mathscr{P}^{i-1}$ is homogeneous of degree 1 in S and K, so is $\mathscr{P}^i$. Since

$$\mathscr{P}^0(S,K,t) = P(S,K,\ \Delta - t), \tag{11}$$

which is homogeneous by assumption, I have thus proved by induction that $\mathscr{P}^i$ is indeed homogeneous of degree 1 in S and K for all $i \geqslant 0$.

The value of the investor's overall position at time t is

$$V^i(S,K,t) = S + \mathscr{P}^i(S,K,t). \tag{12}$$

This can obviously be calculated by adding S to the value obtained above for $\mathscr{P}^i(S,K,t)$, but it will be helpful to rederive the results directly in terms of V. From Eq. (12), the boundary condition on a revaluation date, Eq. (7), can be

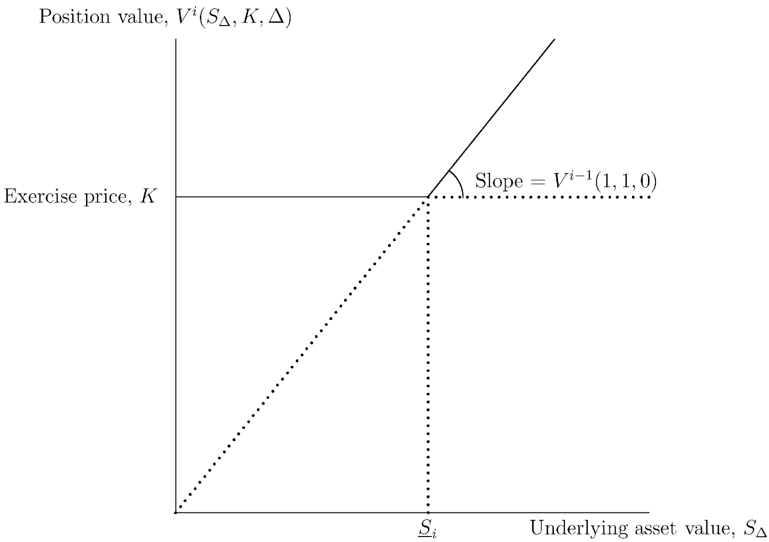

Fig. 2. Value of an investor's position on a revaluation date. The solid line shows the value of the investor's position on a revaluation date, with i revaluations left until expiration. If the underlying asset is worth more than $\underline{S}_i$, it is optimal to keep the money in the plan, now with $i-1$ revaluation periods until expiration. If the underlying asset value falls below the critical value $\underline{S}_i$, it is optimal to exercise the option and withdraw \$$K$.

rewritten as

$$V^i(S_\Delta, K, \Delta) = \max[S_\Delta V^{i-1}(1, 1, 0), K]. \tag{13}$$

Fig. 2 shows this for different values of S_Δ (Fig. 2 is just Fig. 1 with the payoff increased everywhere by S_Δ). The same payoff can be achieved by buying, and holding until time Δ, a portfolio containing K bonds with payoff \$1 each at time Δ, and $V^{i-1}(1, 1, 0)$ call options on the asset, with expiration date Δ and exercise price $\underline{S}_i = K/V^{i-1}(1, 1, 0)$. By arbitrage, the value of the position must equal the value of this portfolio, so

$$V^i(S, K, t) = Ke^{-r(\Delta - t)} + V^{i-1}(1, 1, 0)C(S, K/V^{i-1}(1, 1, 0), \Delta - t), \tag{14}$$

for $i \geqslant 0$, where $C(S, K, t)$ is the value of a standard call option with current asset value S, exercise price K, and time to expiration t, and where

$$V^{-1}(1, 1, 0) \equiv 1. \tag{15}$$

To value the option, we start at the final expiration date of the option and work backwards one revaluation period at a time using Eq. (10) or Eq. (14). These equations hold without needing to assume a specific model for the dynamics of the underlying asset price, but to obtain numerical results we need to make some more specific assumptions. Regardless of the model assumed, given the homogeneity of the problem, the boundary conditions for revaluation period i are determined entirely by the single value $\mathscr{P}^{i-1}(1, 1, 0)$.

3.1. Lognormal asset prices

Assume that S_t follows the process

$$dS_t = \mu S_t \, dt + \sigma S_t \, dZ_t, \tag{16}$$

where μ and σ are constants and Z_t is a standard Brownian motion. Note that this process satisfies the homogeneity assumption A1. Also assume that the continuously compounded riskless interest rate, r, is a constant. These assumptions allow use of the Black and Scholes (1973) results for the (European) option value, P, in Eq. (10),

$$P(S, K, t) = Ke^{-rt}[1 - \Phi(z - \sigma\sqrt{t})] - S[1 - \Phi(z)], \tag{17}$$

where

$$z = \frac{\log(S/K) + (r + \sigma^2/2)t}{\sigma\sqrt{t}}. \tag{18}$$

To value the option, start at maturity ($i = 0$) and apply Eq. (10) repeatedly (using Eq. (17) for the put value) for $i = 0, 1, 2, \dots$, to calculate, in turn, $\mathscr{P}^0(1, 1, 0)$, $\mathscr{P}^1(1, 1, 0)$, $\mathscr{P}^2(1, 1, 0)$, etc. If the plan is investing in securities that can also be traded directly, the investor can, outside the plan, carry out the replicating strategy that leads to the Black–Scholes result.

3.2. Binomial valuation

If we want to allow exercise *between* revaluation dates, we cannot use the Black–Scholes formula, since we need the value of an American put option in Eq. (10). The usual binomial stock price process also satisfies the homogeneity assumption A1, however, and we can use the approach of Cox et al. (1979), as follows:

1. Set up a tree of possible asset value movements over a period of length Δ, starting at the value 1. Set $i = 0$.
2. Using the already known value $\mathscr{P}^{i-1}(1, 1, 0)$, impose the boundary conditions at the end of the tree, given in Eq. (5).
3. Discount back through the tree from period Δ to 0, in the usual way, yielding the value $\mathscr{P}^i(1, 1, 0)$. If exercise of the option between exercise dates is permitted, check at each node whether early exercise is optimal.

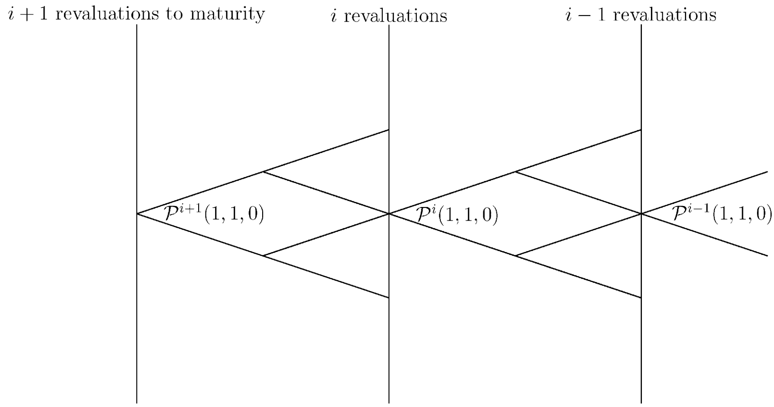

Fig. 3. Binomial valuation of a rollover put option. To calculate the value of a rollover put immediately after a revaluation date, with current asset value 1, exercise price 1, and i revaluation periods until expiration, $\mathscr{P}^i(1, 1, 0)$, start at the final revaluation date ($i = 0$), impose the terminal boundary condition from Eq. (7),

$$\mathscr{P}^0(S_\Delta, 1, \Delta) = \max[0, 1 - S_\Delta],$$

and discount back through a tree of length Δ to calculate $\mathscr{P}^0(1, 1, 0)$. Use this value to impose the terminal conditions for the prior revaluation date,

$$\mathscr{P}^1(S_\Delta, 1, \Delta) = \max[S_\Delta \mathscr{P}^0(1, 1, 0), 1 - S_\Delta],$$

and discount back through another tree of length Δ to calculate $\mathscr{P}^1(1, 1, 0)$. Repeat for $i = 2, 3, \ldots$, at each date imposing the terminal boundary condition,

$$\mathscr{P}^i(S_\Delta, 1, \Delta) = \max[S_\Delta \mathscr{P}^{i-1}(1, 1, 0), 1 - S_\Delta].$$

4. Increase i by 1, and go back to step 2. Continue for as many revaluation periods as desired.

Fig. 3 shows this process. The valuation proceeds from right to left, as usual, but rather than generating one large binomial tree, as in the case of a standard put option, we here generate only a sequence of smaller trees, each of total length Δ. This significantly reduces the amount of calculation required.

3.3. Valuation results

Fig. 4 shows the value of the rollover put option just after a revaluation (so $K = S$) as a fraction of the value of the underlying assets, for annual volatility, $\sigma = 10\%, 20\%, 30\%$, plotted against the time to expiration. The continuously compounded riskless interest rate is assumed to be 5%, and asset revaluation occurs once per year. I calculate the values using the Black–Scholes put option

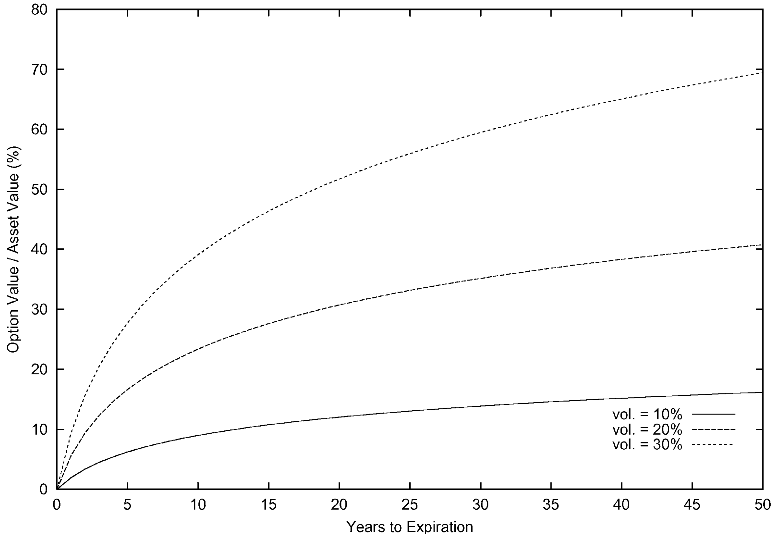

Fig. 4. Rollover put option value vs. volatility. Values are calculated via repeated applications of Eq. (10), using the Black–Scholes put option formula. For each volatility, the riskless interest rate is assumed to be 5% (continuously compounded), and the option strike price is reset to the prevailing value of the underlying asset once per year.

value, as described above in Section 3.1. While the level of volatility has a significant effect on the option's value, it represents a sizable fraction of the asset value in each case. For all three volatilities, the option value is zero at expiration but quickly increases as the time to expiration increases, with the rate of increase falling over time. Even for the lowest volatility, $\sigma = 10\%$, the option adds 10% to the underlying asset value with 13 years to expiration and 15% with 39 years to expiration. For a plan invested in assets with a relatively high volatility, $\sigma = 30\%$, the option adds over 50% to the underlying asset value with 20 years to expiration and almost 70% with 50 years to expiration. Its value actually exceeds 100% of the underlying asset value with (an admittedly hard to achieve) 230 years to expiration. The option has significant value even for plans that allow only a small amount of time following separation in which to exercise the option. With just one year to expiration and quarterly revaluation, the option adds over 8% to the value of the assets in the plan with a volatility of 20% and adds over 13% if the volatility is 30%.

Fig. 5 focuses on the impact of valuation frequency. It plots the value of the option as a fraction of the underlying asset value against the time to expiration for annual, quarterly, monthly and daily valuation frequencies. The continuously compounded riskless interest rate is assumed to be 5%, and σ is fixed at

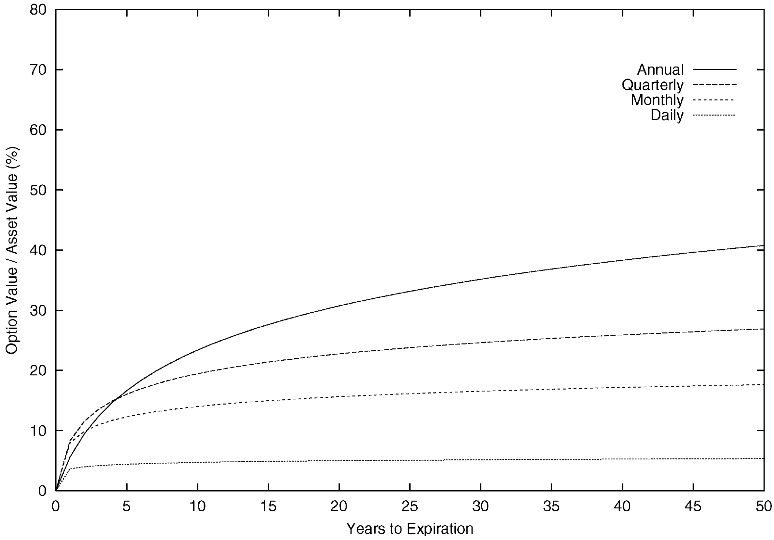

Fig. 5. Rollover put option value vs. reset frequency. Values are calculated via repeated applications of Eq. (10), using the Black–Scholes put option formula. For each revaluation frequency, the riskless interest rate is 5% (continuously compounded), and the volatility of the underlying asset is 20% annually.

20% annually. Valuation frequency has a significant impact on the value of the option. With 50 years to expiration, the option value is 40.7% of asset value with annual revaluation, 26.9% of asset value for quarterly valuation, 17.6% for monthly, and 5.3% for daily. It is interesting to note that, for some short expirations, the value is *nonmonotonic* in the revaluation frequency. For example, with one year to expiration, the option value is 5.57% of the asset value for annual valuation and 8.28% for quarterly valuation. From 5 years to maturity on, however, the value is decreasing in the revaluation frequency.

3.4. Optimal exercise policy

The investor optimally exercises the option if $S_\Delta < \underline{S}_i$, i.e., if the asset value has fallen since the last revaluation date by more than

$$\frac{K - \underline{S}_i}{K} = \frac{\mathscr{P}^{i-1}(1, 1, 0)}{1 + \mathscr{P}^{i-1}(1, 1, 0)}. \tag{19}$$

Fig. 6 plots this critical drop in value against the time to expiration for $\sigma = 10\%$, 20%, 30%. The continuously compounded riskless interest rate is assumed to be 5%, and asset revaluation occurs once per year. The critical drop rises sharply

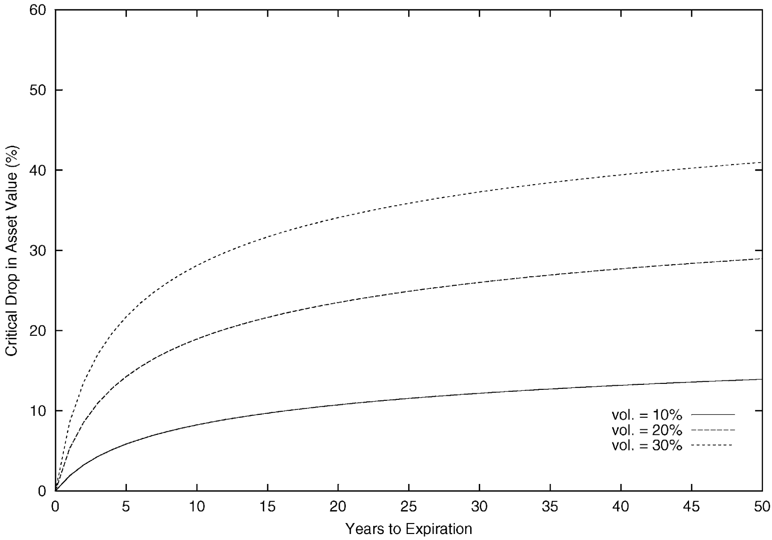

Fig. 6. Rollover put option optimal exercise boundary vs. volatility. The drop in asset value (since the previous valuation date) required for the exercise of the rollover option to be optimal, for different volatilities and times to expiration. In each case, the riskless interest rate is 5% (continuously compounded), and the option strike price is reset to the prevailing value of the underlying asset once per year.

from its initial value of zero, but once the expiration date reaches about 20 years the impact of changing volatility is much more significant than even a large change in expiration date. For example, with a volatility of 10% and time to expiration of 20 years, the optimal rule is to exercise the option (i.e., roll over the plan) if the assets have declined by at least 10.7% since the last valuation date. With a volatility of 30%, the corresponding critical drop is 34.1%.

4. Extensions

The analysis in Section 3 above considers the valuation of the option in a Black–Scholes or binomial world from the perspective of a single (type of) investor who has already left a firm, is trying to decide on the optimal withdrawal strategy, and does not have to think about the impact of others' behavior on this value. While this may be a reasonable approximation of the situation in practice, as very few participants in 401(k) and other retirement plans are even aware that the option described here exists, the analysis above can be extended in various directions. I here consider four such extensions. The first is to consider the impact of alternative assumptions about the dynamics of the underlying asset price, and in particular the effect of correlation between asset price movements and changes in volatility, on the value of the option. The second is to consider the possible dilution effects that occur for other investors in the plan when one investor decides to withdraw his or her funds. The third extension is to analyze the impact of transaction costs on the value and optimal exercise policy of the rollover option. Finally, I consider the decision of an employee who is still employed and thinking about leaving the firm to roll over his or her 401(k) plan.[3]

4.1. Alternative stock price dynamics

Instead of assuming a Black–Scholes world, we could assume different models for the evolution of stock prices, as in, for example, Cox and Ross (1976) or Merton (1976). Implementing such alternative pricing models would often be more numerically burdensome (due to the presence of additional state variables or the failure of the homogeneity assumption A1) and would almost certainly yield different numerical values for the option, but the basic intuition, that the option should be exercised if the stock price falls far enough, will continue to hold.

4.1.1. Example – Stochastic volatility

Suppose we believe that volatility typically increases following a market crash and wish to calculate the impact that this correlation has on the value of the option. We cannot address this with the implementation in Section 3, in which volatility is always constant. Instead, assume that asset price movements are governed by the stochastic volatility model of Hull and White (1987). In this model, the joint (risk-neutral) dynamics of the asset value, S_t, and its instantaneous variance, v_t, are given by

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sqrt{v_t}S_t\,\mathrm{d}Z^1, \tag{20}$$

$$\mathrm{d}v_t = \mu v_t\,\mathrm{d}t + \xi v_t\,\mathrm{d}Z^2, \tag{21}$$

where the correlation between the innovations $\mathrm{d}Z^1$ and $\mathrm{d}Z^2$ is ρ. In this two-factor world, the price of a non-dividend-paying asset, $P(S, v, t)$, satisfies the partial differential equation

$$\frac{1}{2}\left[vS^2\frac{\partial^2 P}{\partial S^2} + \xi^2 v^2\frac{\partial^2 P}{\partial v^2} + 2\rho v^{3/2}\xi S\frac{\partial^2 P}{\partial S\partial v}\right] + rS\frac{\partial P}{\partial S} + \mu v\frac{\partial P}{\partial v} + \frac{\partial P}{\partial t} - rP = 0. \tag{22}$$

[3] I am grateful to the referee for suggesting these extensions.

Table 1
The impact of correlation between asset price and volatility on position value

The table shows the value of a 401(k) plan investor's position (per \$1 of nominal assets) for different asset volatilities (σ) and revaluation frequencies, under different assumptions about the correlation between changes in the asset price and changes in volatility (ρ). Asset price and volatility movements (risk-adjusted) are described by the Hull and White model,

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sqrt{v_t}S_t\,\mathrm{d}Z^1,$$

$$\mathrm{d}v_t = \mu v_t\,\mathrm{d}t + \xi v_t\,\mathrm{d}Z^2,$$

$$\mathrm{d}Z^1\,\mathrm{d}Z^2 = \rho\,\mathrm{d}t.$$

The investment horizon is assumed to be 5 years, the riskless interest rate is assumed to be 5% (continuously compounded), and the parameters of the volatility process are $\mu = 0$, $\xi = 1$ (matching those used in Hull and White, 1987). Values are calculated by solving Eq. (22) using the hopscotch algorithm of Gourlay and McKee (1977).

ρ		$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$
− 0.75	Annual	1.047	1.104	1.152
	Quarterly	1.046	1.099	1.145
− 0.30	Annual	1.053	1.114	1.162
	Quarterly	1.054	1.111	1.156
0.00	Annual	1.056	1.121	1.168
	Quarterly	1.060	1.118	1.164
0.30	Annual	1.059	1.127	1.175
	Quarterly	1.065	1.126	1.171
0.75	Annual	1.062	1.136	1.186
	Quarterly	1.071	1.138	1.182

The model described in Eqs. (20)–(21) satisfies the homogeneity property A1 (though not A2). The boundary conditions satisfied at each revaluation date are those given in Eq. (13), except that each price is now a function of both S and v; i.e.,

$$V^i(S_\Delta, v_\Delta, K, \Delta) = \max[S_\Delta V^{i-1}(1, v_\Delta, 1, 0), K]. \tag{23}$$

Table 1 shows the value of the investor's position just after a revaluation (so $K = S$) as a fraction of the value of the underlying assets for different assumptions about instantaneous volatility, valuation frequency, and correlation. The investment horizon in each case is five years, the continuously compounded riskless interest rate is 5%, and the parameters used for the volatility process are $\mu = 0$, $\xi = 1$ [matching those used in Hull and White (1987)]. Values are calculated by solving Eq. (22), subject to the boundary condition in Eq. (23), using the hopscotch algorithm of Gourlay and McKee (1977). Table 1 reveals that the option value is positively related to the level of volatility. The more negative the correlation (i.e., the more a crash is likely to be associated with an increase in volatility), the lower the value of the option. While this is consistent with the results obtained by Hull and White (1987), it is also important to note that the magnitude of this effect is relatively small. With an instantaneous volatility of 30% (where the effect is largest), changing the correlation from 0 to -0.3 only reduces the value of the position from \$1.168 to \$1.162 with annual revaluation. A change in correlation from $+0.75$ to -0.75 reduces the value from \$1.182 to \$1.145 with quarterly revaluation.

4.2. Multiple investors and dilution

Suppose there is more than one investor type, and one type decides to withdraw from the fund this period. For any other investor who does not also withdraw from the fund, there will be less remaining in the fund next period. This dilution effect will make exercise relatively more attractive and reduce the value of the option held by the investor.

To explore the significance of this effect, suppose there are two different investor types, with horizons differing by Δ_h years. Define:

$V^i(S, K, t)$ = Value of the short-horizon investor's position,

$V^j(S, K, t)$ = Value of the long-horizon investor's position with *no* short horizon investor,

$V^{j^*}(S, K, t)$ = Value of the long-horizon investor's position *with* a short – horizon investor,

where $j - i = \Delta_h/\Delta$. The boundary conditions on a revaluation date for the short-horizon investor (i) are identical to those described in Section 3 above, and are depicted by the dashed line in Fig. 7. The boundary conditions for the longer horizon investor (j) are a little more complicated; they are summarized by the solid line in Fig. 7.

1. $S_\Delta > \underline{S}_i$: In this case, the short-horizon investor chooses not to withdraw funds. This automatically means that the long-horizon investor will also optimally not withdraw, so the value of the long investor's position is

$$V^{j^*}(S_\Delta, K, \Delta) = V^{j-1^*}(S_\Delta, S_\Delta, 0) \tag{24}$$

$$= S_\Delta V^{j-1^*}(1, 1, 0), \tag{25}$$

the last equality following from homogeneity. The payoff from exercising today (per share) is the same for both investors, but the value of not exercising is higher for the long-horizon investor.

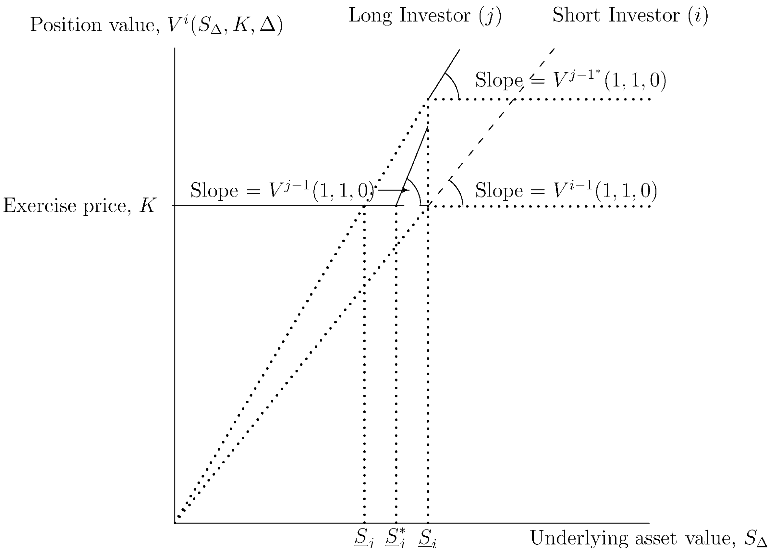

Fig. 7. Value of investment on a revaluation date – 2 investor types. The dashed line shows the value of the short-horizon investor's position on a revaluation date, with i revaluations left until expiration, the same as in Fig. 2. The solid line shows the value of the long-horizon investor's position on the same date, with j revaluations left until expiration, taking into account the dilution that occurs should the short-horizon investor withdraw from the fund. The slopes of the various line segments are given by

$V^{i-1}(1, 1, 0)$ = Value of the short-horizon investor's position (per share) should he or she not withdraw from the fund,

$V^{j-1}(1, 1, 0)$ = Value of the long-horizon investor's position if the short-horizon investor has already withdrawn,

$V^{j-1^*}(1, 1, 0)$ = Value of the long-horizon investor's position if the short-horizon investor has *not* withdrawn.

The different regions of the graph are determined by the value of S_Δ:

$0 \leqslant S_\Delta < \underline{S}_j$ – Both investors withdraw from the fund.

$\underline{S}_j \leqslant S_\Delta < \underline{S}_j^*$ – Both investors withdraw from the fund, but the long-horizon investor would not do so in the absence of the short-horizon investor.

$\underline{S}_j^* \leqslant S_\Delta < \underline{S}_i$ – The short-horizon investor withdraws from the fund; the long-horizon investor does not.

$\underline{S}_i \leqslant S_\Delta$ – Neither investor withdraws from the fund.

2. $S_\Delta < \underline{S}_i$: In this case, the short-horizon investor (i) optimally exercises the option and withdraws from the fund. Depending on the proportion of the fund owned by investor i and the difference between K and S_Δ, this withdrawal will

cause varying amounts of dilution for the long-horizon investor (j), should he or she decide *not* to withdraw from the fund.

Suppose at time Δ, before any withdrawals, the total number of shares in the fund is N_Δ, with type i owning a claim to a fraction θ_i of these. On withdrawal, type i takes out dollar amount $\theta_i N_\Delta K$, representing a fraction, $\theta_i N_\Delta K / N_\Delta S_\Delta = \theta_i (K/S_\Delta)$, of the total shares in the fund. The remaining investors used to own a fraction $1 - \theta_i$ of the fund. After investor i withdraws, if they do not also do so, they now own the fraction $1 - \theta_i(K/S_\Delta)$. This is smaller than the original fraction, since type i will only exercise if $K \geqslant S_\Delta$. The exact amount of dilution depends on the relative values of θ_i, K, and S_Δ. If investor i exercises at the money (i.e., at $S_\Delta = K$), or if $\theta_i = 0$, no dilution occurs at all. The larger θ_i, and the further in-the-money investor i exercises, the greater the dilution that occurs.

The long-horizon investor (j) thus faces a choice of withdrawing the full amount at the exercise price K or remaining in the fund and facing some dilution. If the investor does not withdraw from the fund, the position is worth (after investor i has withdrawn)

$$\frac{1 - \theta_i(K/S_\Delta)}{1 - \theta_i} V^{j-1}(S_\Delta, S_\Delta, 0) = \frac{S_\Delta - \theta_i K}{1 - \theta_i} V^{j-1}(1, 1, 0).$$

Note the V, rather than V^*, on the right-hand side. Since the short-horizon investor optimally withdraws at this point, the value of the long-horizon investor's position from now on needs to be calculated in the *absence* of the short investor. Alternatively, the long investor can withdraw from the fund, receiving K. The overall payoff is thus

$$V^{j*}(S_\Delta, K, \Delta) = \max\left(K, \frac{S_\Delta - \theta_i K}{1 - \theta_i} V^{j-1}(1, 1, 0) \right), \tag{26}$$

and the optimal strategy for the long investor is to withdraw if $S_\Delta < \underline{S}^*_j$, where $\underline{S}^*_j$ is given by

$$K = \frac{\underline{S}^*_j - \theta_i K}{1 - \theta_i} V^{j-1}(1, 1, 0), \tag{27}$$

i.e.,

$$\underline{S}^*_j = \frac{(1 - \theta_i)K}{V^{j-1}(1, 1, 0)} + \theta_i K. \tag{28}$$

Note that $V^{j-1}(1, 1, 0) \geqslant 1$, so

$$\begin{aligned} \underline{S}^*_j &= \frac{(1 - \theta_i)K}{V^{j-1}(1, 1, 0)} + \theta_i K \\ &\geqslant \frac{(1 - \theta_i)K}{V^{j-1}(1, 1, 0)} + \frac{\theta_i K}{V^{j-1}(1, 1, 0)} \end{aligned}$$

$$= \frac{K}{V^{j-1}(1, 1, 0)},$$

$$= \underline{S}_j. \tag{29}$$

As expected, the dilution effect caused by the presence of the short investor makes the long investor more likely to withdraw.

The long-horizon investor's boundary conditions are summarized by the solid line in Fig. 7. They apply at every revaluation date until the final expiration date of the short-horizon investor, at which point the boundary conditions for the long-horizon investor become the same as in Eq. (7) above. Note the relative slopes of the different line segments. First,

$$V^{j-1}(S, K, t) > V^{i-1}(S, K, t). \tag{30}$$

The value of the position to the long-horizon investor is greater than the value to the short-horizon investor because the long-horizon investor has longer to exercise his or her option. Second,

$$V^{j-1}(S, K, t) > V^{j-1^*}(S, K, t), \tag{31}$$

since the presence of the short-horizon investor lowers the value of the position held by the long-horizon investor.

Fig. 7 shows that the long investor's payoffs can be replicated by buying, and holding until time Δ, a portfolio containing K bonds with payoff \$1 each at time Δ; $V^{j-1}(1, 1, 0)/(1-\theta_i)$ call options on the assets, with expiration date Δ and exercise price $\underline{S}_j^*$; a *short* position in $[V^{j-1}(1, 1, 0)/(1-\theta_i) - V^{j-1^*}(1, 1, 0)]$ call options on the assets, with expiration date Δ and exercise price $\underline{S}_i = K/V^{i-1}(1, 1, 0)$; and $\underline{S}_i V^{j-1^*}(1, 1, 0) - ((\underline{S}_i - \theta_i K)/(1-\theta_i))V^{j-1}(1, 1, 0)]$ *binary 'cash-or-nothing' call options*, each with payout 1 if $S_\Delta > \underline{S}_i$ and 0 otherwise. By arbitrage, the value of the position must equal the value of this portfolio, so we obtain the formula

$$\begin{aligned} &V^j(S, K, t) \\ &= K e^{-r(\Delta - t)} + \frac{V^{j-1}(1, 1, 0)}{1-\theta_i} C(S, \underline{S}_j^*, \Delta - t) \\ &\quad - \left[\frac{V^{j-1}(1, 1, 0)}{1-\theta_i} - V^{j-1^*}(1, 1, 0)\right] C(S, \underline{S}_i, \Delta - t) \\ &\quad + \left[\underline{S}_i V^{j-1^*}(1, 1, 0) - \left(\frac{\underline{S}_i - \theta_i K}{1-\theta_i}\right) V^{j-1}(1, 1, 0)\right] BC(S, \underline{S}_i, \Delta - t), \end{aligned} \tag{32}$$

where $BC(S, K, t)$ is the value of a binary call option that pays out \$1 at time t if an asset with current price S is worth more than K, and zero otherwise.

Table 2
The impact of a short-term investor on position value

The table shows the value of a 401(k) plan investor's position (per $1 of nominal assets) for different horizons, asset volatilities (σ), and revaluation frequencies, in the presence of another investor with horizon 20 years shorter, who represents 20% of the fund. V is the value in the absence of the second investor; V^* is the value in the presence of the shorter horizon investor. Values are calculated using a binomial tree with 100 time steps per valuation period.

Horizon		$\sigma = 0.1$		$\sigma = 0.2$		$\sigma = 0.3$	
		Annual	Quarterly	Annual	Quarterly	Annual	Quarterly
20 years	V	1.120	1.102	1.307	1.227	1.517	1.364
	V^*	1.118	1.094	1.297	1.208	1.498	1.331
30 years	V	1.139	1.110	1.351	1.245	1.594	1.395
	V^*	1.127	1.098	1.317	1.216	1.530	1.345
40 years	V	1.152	1.116	1.383	1.258	1.650	1.416
	V^*	1.141	1.107	1.351	1.238	1.592	1.381

As in Section 3, if we make assumptions about the process driving movements in the underlying asset price, we can use this relation to price the long investor's position. For example, if asset prices were lognormal, we could use the Black–Scholes call option value in Eq. (32), along with the result that, in this economy,

$$BC(S, K, t) = \mathrm{e}^{-rt}\Phi(z - \sigma\sqrt{t}), \tag{33}$$

where $z = [\log(S/K) + (r + \sigma^2/2)t]/\sigma\sqrt{t}$. Alternatively, we can value the position using a binomial algorithm, as described in Section 3.2 above. This works just like the single investor considered there, except that three values now need to be calculated at each revaluation date, $V^{i-1}(1, 1, 0)$, $V^{j-1}(1, 1, 0)$ and $V^{j-1*}(1, 1, 0)$.

I implemented this binomial valuation algorithm for two investors with a wide range of different horizons, asset volatilities, times to maturity, and proportions of the short investor in the overall fund. In almost every case, the presence of the shorter horizon investor made little difference to the value of the position held by the longer horizon investor. As an example, Table 2 shows representative results when $\theta_i = 0.2$ (i.e., the short-horizon investor owns 20% of the assets in the fund) and the difference in horizons between long and short investors is 20 years. All values are generated using binomial trees with 100 time steps per valuation period. Although the presence of the shorter horizon investor does lower the value of the longer investor's position, the difference is small.

As the horizon and volatility vary, there are factors that suggest that the difference should be large in some cases, but for each such factor there is another, countervailing factor, which in practice seems to be more important. For example,

- At short horizons (e.g., 20 years in Table 2, when the corresponding horizon for the short investor is 0), there is a relatively wide difference in the optimal exercise policy of the two investors, and hence a range of possible asset values at which the short investor withdraws, and dilution may be an issue. Yet dilution only matters when there is a significant difference between the asset value and the exercise price. Close to maturity, the short-horizon investor exercises the option when it is only just in-the-money, so, although dilution occurs with significant probability, the amount of dilution that actually occurs is negligible.
- At longer horizons, the short investor only exercises the option when it is well in-the-money, leading to the possibility of significant dilution for the longer term investor. At long horizons, however, since the optimal exercise boundary is relatively flat (see Fig. 6), the optimal exercise policies of the long and short investors are almost the same, and so dilution, while significant in principle, is very unlikely actually to occur.
- High volatility makes it more likely that the short-horizon investor will exercise deep in-the-money, so more significant dilution may occur. At the same time, high volatility makes the optimal exercise boundary flatter at a wider range of horizons (see Fig. 6), so again the likelihood of this dilution actually occurring goes down as volatility increases.

4.2.1. Dilution with only a single investor type

For the rollover option to have value to one investor, there must be another investor giving up that value. So far, I have been valuing the option from the perspective of a single investor who represents a negligible fraction of the total plan. The investor's option (or, in Section 4.2, the long- and short-horizon investors' options) has been financed by the other investors in the plan, who were implicitly assumed either to be unaware of the existence of the option or else unable to exercise it (for example, because they were still employed, and the costs of changing jobs were too high). I here analyze the 'zero-sum' nature of the option in more detail, taking into account the fact that those withdrawing cannot take out more than the total current value of the assets in the plan.

Consider the position held by an investor of type i at time Δ, where all of the type i investors together own a claim to a fraction θ of the total shares in the fund. Suppose all investors of type i will act the same, and that the remaining fraction $1-\theta$ will definitely not withdraw from the plan.

Per share owned by the type i investors, there are $1/\theta$ shares in the fund altogether, so, if the type i investors all withdraw, they cannot take out more

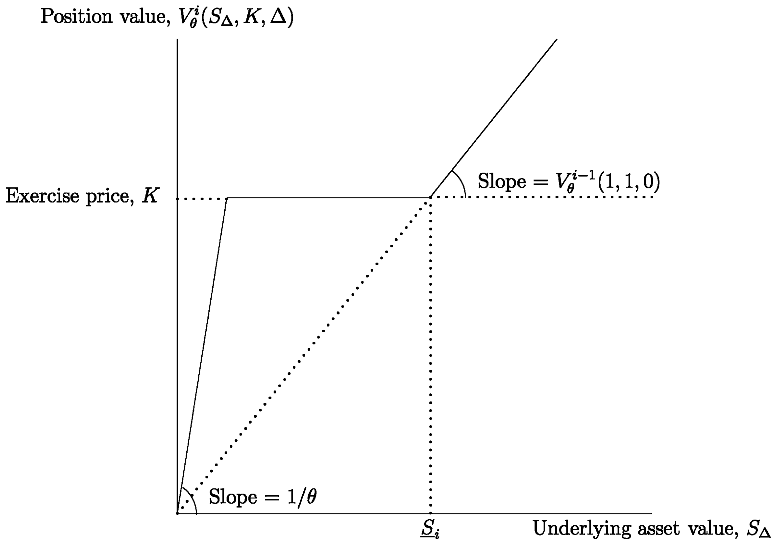

Fig. 8. Value of an investor's position on a revaluation date. The solid line shows the value of the investor's position on a revaluation date, with i revaluations left until expiration. Investors of type i are assumed to own a fraction θ of the assets in the plan, and all other investors will definitely not withdraw this period. If the underlying asset is worth more than $\underline{S}_i$, it is optimal to keep the money in the plan, now with $i-1$ revaluation periods until expiration. If the underlying asset value falls below the critical value $\underline{S}_i$, it is optimal to exercise the option and withdraw \$$K$. If the asset value falls far enough, there will not be enough assets in the plan to withdraw the full \$$K$, and the investor can therefore only withdraw S_Δ/θ per share owned. The overall value of the position is

$$V_\theta^i(S_\Delta, K, \Delta) = \max[S_\Delta V_\theta^{i-1}(1, 1, 0), \min\{K, S_\Delta/\theta\}].$$

than S_Δ/θ. Adding this cap to Eq. (13), the payoff at date Δ becomes

$$V_\theta^i(S_\Delta, K, \Delta) = \max[S_\Delta V_\theta^{i-1}(1, 1, 0), \min\{K, S_\Delta/\theta\}]. \tag{34}$$

Fig. 8 shows this payoff. It is identical to Fig. 2, except that, for low enough values of S_Δ, the cap of S_Δ/θ becomes binding. The smaller the value of θ, the less significant this cap. In the limit, as $\theta \to 0$ (i.e., the investors considering withdrawal represent a negligible fraction of the overall plan), the cap becomes irrelevant, and Fig. 8 turns into Fig. 2. At the other extreme, suppose $\theta = 1$, as, for example, in a plan with only a single investor. At maturity, Eq. (34) becomes

$$\begin{aligned} V_\theta^0(S_\Delta, K, \Delta) &= \max[S_\Delta V_\theta^{-1}(1, 1, 0), \min\{K, S_\Delta/\theta\}] \\ &= \max[S_\Delta, \min\{K, S_\Delta\}] \\ &= S_\Delta. \end{aligned} \tag{35}$$

If the value of the position is S_Δ at time Δ, it must equal S_t at any time prior to Δ. Hence, in particular,

$$V_\theta^0(1,\ 1,\ 0) = 1. \tag{36}$$

Now consider the value at the prior revaluation period.

$$\begin{aligned} V_\theta^1(S_\Delta, K,\ \Delta) &= \max[S_\Delta V_\theta^0(1,\ 1,\ 0),\ \min\{K, S_\Delta/\theta\}] \\ &= \max[S_\Delta,\ \min\{K, S_\Delta\}] \\ &= S_\Delta. \end{aligned} \tag{37}$$

Comparing Eqs. (35) and (37), it can be seen that the same relation will continue to hold for all i. Thus, when $\theta = 1$, for any i, S, K, and t,

$$V_\theta^i(S, K, t) = S. \tag{38}$$

Thus, if all investors behave identically, the option adds *no* additional value at all.

Fig. 8 reveals that the same payoff can be achieved by buying, and holding until time Δ, a portfolio containing K bonds with payoff \$1 each at time Δ; $V_\theta^{i-1}(1,\ 1,\ 0)$ call options on the assets, with expiration date Δ and exercise price $\underline{S}_i = K/V_\theta^{i-1}(1,\ 1,\ 0)$; and a short position in $-1/\theta$ put options on the assets, with expiration date Δ and exercise price θK. By arbitrage, the value of the position must equal the value of this portfolio, so

$$\begin{aligned} V_\theta^i(S,\ K,\ t) = {} & K\mathrm{e}^{-r(\Delta - t)} + V_\theta^{i-1}(1,\ 1,\ 0) C(S, K/V_\theta^{i-1}(1,\ 1,\ 0),\ \Delta - t) \\ & - \frac{1}{\theta} P(S,\ \theta K,\ \Delta - t), \end{aligned} \tag{39}$$

for $i \geqslant 0$, where $C(S, K, t)$ is the value of a standard call option with current asset value S, exercise price K, and time to expiration t, $P(S, K, t)$ is the value of the corresponding standard put option, and

$$V_\theta^{-1}(1,\ 1,\ 0) \equiv 1. \tag{40}$$

Table 3 shows the value of the investor's position as a fraction of the value of the underlying assets for different assumptions about horizon, volatility, and the fraction θ. The continuously compounded riskless interest rate is assumed to be 5%, and asset revaluation occurs once per year. I calculate the values using the Black–Scholes put and call option values in Eq. (39). In the table, $\theta = 0$ corresponds to the situation analyzed in Section 3, in which there was no dilution effect. The other extreme, $\theta = 1$, is where everyone in the plan either exercises or does not exercise at the same time. As expected, the option adds no value in this case. What is most interesting is that the difference in value between $\theta = 0$ and $\theta = 0.5$ is negligible for all horizons and volatilities considered, and even at $\theta = 0.7$ the difference is only slightly noticeable at $\sigma = 0.3$. In almost any

Table 3
The impact of dilution on position value

The table shows the value of a 401(k) plan investor's position (per \$1 of nominal assets) for different horizons, asset volatilities (σ), and proportions of the plan owned by that investor type (θ). The continuously compounded riskless interest rate is assumed to be 5%, and asset revaluation occurs once per year. The values are calculated using the Black–Scholes put and call option values in Eq. (39).

Horizon		$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$
5 years	$\theta = 0.0$	1.062	1.166	1.277
	$\theta = 0.2$	1.062	1.166	1.277
	$\theta = 0.5$	1.062	1.166	1.274
	$\theta = 0.7$	1.062	1.160	1.229
	$\theta = 1.0$	1.000	1.000	1.000
10 years	$\theta = 0.0$	1.090	1.234	1.391
	$\theta = 0.2$	1.090	1.234	1.391
	$\theta = 0.5$	1.090	1.234	1.385
	$\theta = 0.7$	1.090	1.223	1.309
	$\theta = 1.0$	1.000	1.000	1.000
15 years	$\theta = 0.0$	1.107	1.276	1.464
	$\theta = 0.2$	1.107	1.276	1.464
	$\theta = 0.5$	1.107	1.276	1.456
	$\theta = 0.7$	1.107	1.261	1.351
	$\theta = 1.0$	1.000	1.000	1.000

realistic setting, this potential dilution will have almost no impact on the value of the option. For long enough horizons, the value of θ does eventually have an impact on the value of the position. For example, it can be shown that, in a Black–Scholes world, for given values of S, K, and t, and any value of $\theta \in [0,\ 1]$,

$$\lim_{i \to \infty} V_\theta^i(S, K, t) = \frac{S}{\theta}, \tag{41}$$

so, in particular, for $\theta = 0$,

$$\lim_{i \to \infty} V_\theta^i(S, K, t) = \infty, \tag{42}$$

but it can take horizons of hundreds or thousands of years for these differences to be noticeable.

4.3. Transaction costs

While exercising the option to withdraw from a 401(k) plan is costless, there are costs associated with leaving money in the plan, in the form of management fees that are paid out of plan assets. These reduce the returns earned by investors in the plan and lower the value of the position held by an investor, compared with the corresponding value in the absence of such costs. I here consider the impact of these costs on an investor's optimal rollover policy and on the exact relation between the values with and without costs.

4.3.1. Rollover to another plan

Suppose the investor is considering rolling over his or her 401(k) plan into another plan that has the same costs as the first, and assume that the date of the rollover will have no impact on the date of the final withdrawal of assets from the second plan. In this case, the costs are incurred whether or not the option is exercised, though the amount of the costs will depend on the exercise strategy.

Proportional costs. Suppose that, in both plans, proportional costs equal to c times the current plan balance are charged each period, and consider the valuation of the investor's position at the final maturity date. Let \$K be the current exercise price and S_Δ the value of the assets in the plan. If the investor rolls over the account immediately prior to maturity, he or she receives $K(1-c)$. I assume that the \$K is rolled into the second plan, where it immediately incurs costs at rate c, and is then withdrawn. If the investor does *not* roll over the account, he or she withdraws the amount in the plan, less transaction costs, $S_\Delta(1-c)$. Writing the value of the overall position in the presence of costs as V_c (c for 'cost'), the overall payoff at maturity can thus be written as

$$\begin{aligned} V_c^0(S_\Delta, K, \Delta) &= \max[S_\Delta(1-c), K(1-c)] \\ &= (1-c)\max[S_\Delta, K] \\ &= (1-c)V^0(S_\Delta, K, \Delta), \end{aligned} \tag{43}$$

where V^0 is the value of the position in the *absence* of transaction costs. The last equality follows from Eq. (13). Given this relation between the payoffs at time Δ, it follows that, at any time t in the last period,

$$V_c^0(S, K, t) = (1-c)V^0(S, K, t), \tag{44}$$

and the optimal exercise strategy at the last date is exactly the same, regardless of whether there are transaction costs. This follows because the maximum is taken over the same two quantities in both cases.

Now consider the investor's decision one period earlier, at the last revaluation date prior to maturity. If the investor rolls over the account immediately prior to the revaluation, he or she receives \$K, which is invested into \$K worth of assets in the second plan, and withdrawn at maturity after incurring transaction costs *twice*. The present value of this is $K(1-c)^2$. If the investor does *not* roll over the account, costs c are paid out, and the investor now possesses

$$V_c^0(S_\Delta(1-c), S_\Delta(1-c), 0) = S_\Delta(1-c)V_c^0(1, 1, 0). \tag{45}$$

The boundary condition for the last revaluation date can thus be written as

$$\begin{aligned} V_c^1(S_\Delta, K, \Delta) &= \max[S_\Delta(1-c)V_c^0(1, 1, 0), K(1-c)^2] \\ &= \max[S_\Delta(1-c)^2 V^0(1, 1, 0), K(1-c)^2] \\ &= (1-c)^2 \max[S_\Delta V^0(1, 1, 0), K] \\ &= (1-c)^2 V^1(S_\Delta, K, \Delta), \end{aligned} \tag{46}$$

the last equality again following from Eq. (13). As above, given this relation between the payoffs at time Δ, it follows that, at any time t within the valuation period,

$$V_c^1(S, K, t) = (1-c)^2 V^1(S, K, t). \tag{47}$$

Again, the optimal exercise strategy is exactly the same, regardless of whether there are transaction costs.

Working back one period at a time, it is straightforward to show that the boundary condition i revaluation periods prior to maturity is

$$\begin{aligned} V_c^i(S_\Delta, K, \Delta) &= \max[S_\Delta(1-c)V_c^{i-1}(1, 1, 0), K(1-c)^{i+1}] \\ &= (1-c)^{i+1} \max[S_\Delta V^{i-1}(1, 1, 0), K] \\ &= (1-c)^{i+1} V^i(S_\Delta, K, \Delta). \end{aligned} \tag{48}$$

Hence, for any i and any time t within the valuation period,

$$V_c^i(S, K, t) = (1-c)^{i+1} V^i(S, K, t), \tag{49}$$

and the optimal exercise policy is completely unaffected by the presence of transaction costs. While they reduce the value of the overall position, they do so by a fixed fraction each period regardless of whether the option is exercised.

Fixed costs. The irrelevance of transaction costs to the optimal exercise policy above is due to the combination of proportional costs and the homogeneity of the option value in S and K. It is *not* a general property. For example, assume that transaction costs are a constant *dollar* amount, \$C, per period, instead of a constant proportion. Writing the value of the position in the presence of costs as V_C, the payoff at maturity can be written as

$$V_C^0(S_\Delta, K, \Delta) = \max[S_\Delta - C, K - C]$$

$$
\begin{aligned}
&= \max[S_\Delta, K] - C \\
&= V^0(S_\Delta, K, \Delta) - C, \qquad (50)
\end{aligned}
$$

the last equality following from Eq. (13). It follows that, at any time t in the last period,

$$V_C^0(S, K, t) = V^0(S, K, t) - \frac{C}{(1+r)^{\Delta - t}}, \qquad (51)$$

and, just as with proportional costs, the optimal exercise strategy at the last date is exactly the same, regardless of whether there are transaction costs.

Things change, however, if we go back one period. If the investor rolls over the account immediately prior to the last revaluation, he or she receives \$K, which is invested into \$K worth of assets in the second plan, and withdrawn at maturity after incurring transaction costs twice, once now, and once after Δ years. The present value of this is $K - C - C/(1+r)^\Delta$. If the investor does not roll over the account, cost C is paid out, and the investor now possesses $V_C^0(S_\Delta - C, S_\Delta - C, 0)$. The boundary condition at this revaluation date can thus be written as

$$V_C^1(S_\Delta, K, \Delta) = \max\left[V_C^0(S_\Delta - C, S_\Delta - C, 0), K - C - \frac{C}{(1+r)^\Delta} \right]. \qquad (52)$$

More generally, the boundary condition at any revaluation date can be written as

$$
\begin{aligned}
&V_C^i(S_\Delta, K, \Delta) \\
&\quad = \max\left[V_C^{i-1}(S_\Delta - C, S_\Delta - C, 0), K - C - \frac{C}{(1+r)^\Delta} - \cdots - \frac{C}{(1+r)^{i\Delta}} \right].
\end{aligned}
\qquad (53)
$$

This boundary condition can be used to solve for V_C but, except at maturity, the relation between V_C and V is not as simple as in the case of proportional transaction costs. For example, from Eqs. (51) and (52),

$$
\begin{aligned}
V_C^1(S_\Delta, K, \Delta) &= \max\left[V_C^0(S_\Delta - C, S_\Delta - C, 0), K - C - \frac{C}{(1+r)^\Delta} \right] \\
&= \max\left[V^0(S_\Delta - C, S_\Delta - C, 0) - \frac{C}{(1+r)^\Delta}, K - C - \frac{C}{(1+r)^\Delta} \right] \\
&= \max[(S_\Delta - C)V^0(1, 1, 0), K - C] - \frac{C}{(1+r)^\Delta}
\end{aligned}
$$

$$= \max[(S_\Delta V^0(1,1,0) - C\{V^0(1,1,0)-1\}, K] - C - \frac{C}{(1+r)^\Delta}. \quad (54)$$

This would be equal to $V^1(S_\Delta, K, \Delta) - C - C/(1+r)^\Delta$, and the optimal exercise policy would be unaffected by the transaction costs, only if $V^0(1, 1, 0)$ were equal to 1. However, $V^0(1, 1, 0)$ is greater than 1. As a result, the optimal exercise decision here is not the same as in the absence of transaction costs. In fact, exercising the option is relatively *more* attractive in the presence of the fixed cost. This also means that there is no simple relation between the values with and without costs.

4.3.2. Final withdrawal

Suppose, instead of rolling over the assets into another plan, any withdrawal from the plan is final (e.g., it accompanies retirement). Then payment of transaction costs stops on withdrawal from the plan. The transaction costs in this case are very similar to a dividend payment. In the presence of proportional costs c, the boundary condition in Eq. (13) becomes

$$V^i(S_\Delta, K, \Delta) = \max[S_\Delta(1-c)V^{i-1}(1,1,0), K]. \quad (55)$$

This is similar to Eq. (13), but the position's value if the option is not exercised is reduced by a factor $(1 - c)$. Relative to the no-cost case, this makes the option less valuable and makes exercise more likely.

4.4. Exercise while still employed

A plan participant can only exercise the option after leaving the firm. In principle, however, someone currently employed could decide to leave the firm to exercise the option. Consider an employee who can leave the firm at any time (rolling over the 401(k) investment in the process), and that leaving the firm early would force the employee to incur a cost of $\$Z$. Assume also that, each revaluation period while employed, the employee contributes an additional $\$X$ to the fund. On a revaluation date, suppose the value of the employee's underlying assets in the plan is S_Δ, and write $\hat{V}^i(S, K, t)$ for the value of the investor's position (including the option). The analysis is similar to that considered in Section 4.3.1 in the case of fixed transaction costs.

At the final maturity date, assume that the investor would retire anyway, so the investor does not incur the additional cost. The payoff is the same as in Eq. (13),

$$\begin{aligned}\hat{V}^0(S_\Delta, K, \Delta) &= \max[S_\Delta, K] \\ &= V^0(S_\Delta, K, \Delta). \quad (56)\end{aligned}$$

Given this relation between the payoffs at time Δ, it follows that, at any time t in the last period,

$$\hat{V}^0(S,K,t) = V^0(S,K,t), \tag{57}$$

and the optimal exercise policy in the last period is the same as in Section 3.

Just as in the case of fixed transaction costs considered above, things change if we go back one period. If the employee leaves the firm to exercise the option immediately prior to the last revaluation date before maturity, the payoff of the option is the same as in Section 3, except that an additional cost of \$$Z$ is incurred. The payoff is thus $K - Z$. If the investor does *not* leave the firm, the next period's additional contribution of \$$X$ to the plan will increase his or her investment in the plan (reducing cash reserves by the same amount). The value of the position therefore becomes $\hat{V}^0(S_\Delta + X, S_\Delta + X, 0) - X$, and the overall boundary condition is

$$\hat{V}^1(S_\Delta, K, \Delta) = \max[\hat{V}^0(S_\Delta + X, S_\Delta + X, 0) - X, K - Z]. \tag{58}$$

More generally, the boundary condition at any revaluation date is

$$\hat{V}^i(S_\Delta, K, \Delta) = \max[\hat{V}^{i-1}(S_\Delta + X, S_\Delta + X, 0) - X, K - Z]. \tag{59}$$

This boundary condition can be used to solve for $\hat{V}$, but, except at maturity, there is no simple relation between $\hat{V}$ and V. For example, from Eqs. (56) and (58),

$$\begin{aligned}
\hat{V}^1(S_\Delta, K, \Delta) &= \max[\hat{V}^0(S_\Delta + X, S_\Delta + X, 0) - X, K - Z] \\
&= \max[V^0(S_\Delta + X, S_\Delta + X, 0) - X, K - Z] \\
&= \max[(S_\Delta + X)V^0(1, 1, 0) - X, K - Z] \\
&= \max[S_\Delta V^0(1, 1, 0) + X\{V^0(1, 1, 0) - 1\}, K - Z].
\end{aligned} \tag{60}$$

This is shown, for different values of S_Δ, in Fig. 9. The cost associated with moving jobs reduces the value should the employee exercise the option by \$$Z$ (the distance between the solid and dashed horizontal line segments in Fig. 9). On the other hand, the payoff of the option should the employee *not* exercise it is *higher* than in Section 3 by $X\{V^0(1, 1, 0) - 1\}$ (the distance between the solid and dashed sloping line segments in Fig. 9), since the employee's additional contribution to the plan increases holdings of the option in future.

Both of these effects make exercise relatively less attractive, but the effect on the option's value today, relative to that calculated in Section 3, is ambiguous. It depends on the relative values of X and Z, as well as the time to expiration. For example, if $Z = 0$ and $X > 0$, i.e., there is no cost to moving jobs, the option is unambiguously more valuable than in Section 3. The payoff if exercised is the same, but the value of not exercising is higher, as it reflects the value of future units of the option that will be added to the account. If $X = 0$ and $Z > 0$, i.e.,

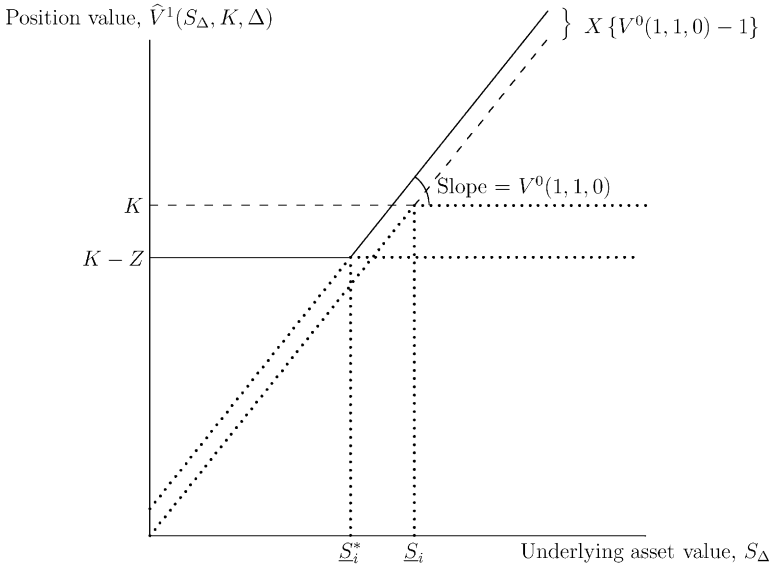

Fig. 9. Value of the position on the last revaluation date. The solid line shows the value of the position held by an investor who is currently employed, whose position was worth \$$K$ on the prior valuation date, who faces a cost of \$$Z$ associated with leaving the firm, and who will make an additional contribution of \$$X$ to the plan next period if he or she does *not* leave the firm. For comparison, the dashed line is the payoff shown in Fig. 2. Relative to that payoff, the cost associated with moving jobs reduces the payoff should the employee exercise the option by \$$Z$ (the distance between the solid and dashed horizontal line segments). The payoff of the option should the employee *not* exercise it is *higher* than in Fig. 2 by $X\{V^0(1, 1, 0) - 1\}$ (the distance between the solid and dashed sloping line segments), since the employee's additional contribution to the plan increases the holdings of the option in future.

moving is costly, and no further money will be added to the account, the value is unambiguously lower than in Section 3 due to the cost of moving. Over time, both of these effects will become less and less significant as the value of the assets currently in the plan increases relative to both the cost of leaving and the value of possible future contributions.

5. Summary

Under current regulations, participants in some 401(k) and other retirement plans have an option to time retirement or rollovers from these plans strategically. I analyze this option and find that, given reasonable assumptions about life expectancy and volatility, the option may be worth up to 40% of the value of the underlying assets, financed entirely by those still contributing to the plan. Even when a plan insists that rollovers must occur within one year of leaving the firm, the option can add over 13% to the value of the underlying assets. Recent events such as the accelerated retirements of American Airlines pilots in late 1998 show that this option can have very large payoffs in practice and causes significant changes in investors' behavior.

One way to prevent these wealth transfers is to value the accounts more often, e.g., daily rather than annually, but this may require substantial additional administrative overhead. An alternative way of achieving the same objective, without changing the valuation frequency, and without any extra administrative burden, is merely to require any request for a rollover or retirement payout to be carried out immediately following the *next* valuation. Since participants can no longer condition their withdrawal request on this (as yet unknown) value, they lose the option described here.

Appendix. Rollover rules

While 401(k) plans have a fair degree of flexibility in their valuation and distribution policies, there are a number of rules limiting this flexibility. Below is a summary of some of the main applicable rules and regulations. This section borrows from the much more extensive treatment in Franz et al. (1997), which should be consulted for more details (see especially Chapters 3, 4, and 6).

- Valuation of a 401(k) plan may be performed as often as daily, or as infrequently as annually. Traditional 401(k) plans are valued quarterly, though the trend is towards more frequent valuation (Franz et al., 1997, pp. 3–53).
- Distributions of funds in a 401(k) plan may always be made upon retirement, death, disability, or separation from service [see Treas. Reg. Sections 1.401–1(b)(1)(ii), 1.401(k)–1(d)(1)(i)].
- Employer contributions are, in general, more accessible at other times than elective (i.e., employee, as opposed to employer) contributions. Elective contributions may also be distributed under the following circumstances [see Treas. Reg. Sections 1.401(k)–1(d)(1)(ii–v)]:
  - Attainment of age 59 1/2.
  - Hardship
  - Termination of the plan
  - Sale or other disposition of at least 85% of the assets used by a corporation in its trade or business to an unrelated corporation
  - Sale or other disposition by a corporation of its interest in a subsidiary to an unrelated individual or entity.

- Any such distribution can be 'rolled over' without tax consequences[4] directly to another eligible plan, such as another employer's 401(k) plan (if it accepts rollovers) or a rollover IRA account. A plan loses its tax-qualified status if it does not allow such a direct rollover [see IRC Section 401(a)(31); Treas. Reg. Section 1.401(a)(31)–1T, Q&A 1–3].
- A rollover does not have to occur immediately on separation from the company. Although a person *may* roll over the account immediately on leaving a firm, the law provides that, as long as the person has more than $3,500 in the plan, he or she cannot be forced to rollover immediately [see Treas. Reg. Section 1.411(a)–11(c)(3), Section 1.411(a)–11(c)(4)].

References

Arak, M., Goodman, L., 1987. Treasury bond futures: valuing the delivery options. Journal of Futures Markets 7, 269–286.

Black, F., Scholes, M.S., 1973. The pricing of options and corporate liabilities. Journal of Political Economy 81, 637–659.

Chance, K., Hemler, M., 1993. The impact of delivery options on futures prices: a survey. Journal of Futures Markets 13, 127–155.

Cohen, H., 1995. Isolating the wild card option. Mathematical Finance 5, 155–165.

Cox, J.C., Ross, S.A., 1976. The valuation of options for alternative stochastic processes. Journal of Financial Economics 3, 145–166.

Cox, J.C., Ross, S.A., Rubinstein, M., 1979. Option pricing: a simplified approach. Journal of Financial Economics 7, 229–263.

Fleming, J., Whaley, R.E., 1994. The value of wildcard options. Journal of Finance 49, 215–236.

Franz, S., McDonagh, J., Maier, J., Kalke, W., 1997. 401(k) Answer Book. Panel Answer Book Series. Aspen Publishers, New York.

Gay, G., Manaster, S., 1986. Implicit delivery options and optimal delivery strategy for financial futures contracts. Journal of Financial Economics 16, 41–72.

Gourlay, A.R., McKee, S., 1977. The construction of hopscotch methods for parabolic and elliptic equations in two space dimensions with a mixed derivative. Journal of Computational and Applied Mathematics 3, 201–206.

Gray, S.F., Whaley, R.E., 1997. Valuing S&P 500 bear market warrants with a periodic reset. Journal of Derivatives 5, 99–106.

Hull, J., White, A., 1987. The pricing of options on assets with stochastic volatility. Journal of Finance 42, 281–300.

Kane, A., Marcus, A., 1986. Valuation and optimal exercise of the wild card option in the Treasury bond futures market. Journal of Finance 41, 195–207.

Merton, R.C., 1976. Option pricing when underlying stock returns are discontinuous. Journal of Financial Economics 3, 125–144.

[4] If a 401(k) plan contains employer stock, the net unrealized appreciation (the difference between the market value of the security at the time of distribution and the cost basis of the security) is not subject to tax until the securities are sold. Only the cost basis of the securities is included in taxable income at the time of the distribution [See IRC Section 402(e)(4)]. This tax advantage is lost if the securities are rolled over into an IRA.

Valerio, N., 1992. Valuation of cash-settlement call options containing a wild-card exercise feature. Working paper, Emory University.

Wall Street Journal, 1987. Lockheed workers protect nest eggs by retiring early. David J. Jefferson, November 2.

Wall Street Journal, 1998. American Air faces pilot shortage because of many early retirements. Scott McCartney, October 8.