Optimal Exercise of Executive Stock Options and Implications for Firm Cost*

Jennifer N. Carpenter   Richard Stanton   Nancy Wallace
New York University   U.C. Berkeley   U.C. Berkeley

January 7, 2010

Optimal Exercise of Executive Stock Options and Implications for Firm Cost

Abstract

This paper conducts a comprehensive study of the optimal exercise policy for an executive stock option and its implications for option cost, average life, and alternative valuation concepts. The paper is the first to provide analytical results for an executive with general concave utility. Wealthier or less risk-averse executives exercise later and create greater option cost. However, option cost can decline with volatility. We show when there exists a single exercise boundary, yet demonstrate the possibility of a split continuation region. We also show that, for CRRA utility, the option cost does not converge to the Black-Scholes value as the correlation between the stock and the market portfolio converges to one. We compare our model's option cost with the modified Black-Scholes approximation typically used in practice, and show that the approximation error can be large or small, positive or negative, depending on firm characteristics.

1 Introduction

As options have become a major component of corporate compensation, investors have become increasingly concerned about their cost to firms. The difficulty is that this cost depends on the exercise policies of option holders who face hedging constraints, so standard option theory does not apply. Indeed, evidence indicates that both executives and other employees exercise options well before standard theory would predict.

This paper conducts a comprehensive study of the optimal exercise policy for an executive stock option and its implications for option cost, average life, and alternative valuation concepts. Our paper is the first to provide analytical results on option cost, average life, and the nature of the continuation region for an executive with general concave utility. Working in a continuous-time framework, we give conditions under which wealthier or less risk-averse executives exercise later and create greater option cost. We prove that the continuation region is smaller the larger the dividend, and that when the interest rate is zero, the continuation region shrinks as time elapses. However, exercise policy is not monotonic in stock return volatility, and is nonmonotonic in time when interest rates are nonzero. Option cost to shareholders can decline as volatility rises. We show analytically when the exercise policy is completely characterized by a single stock price boundary; we also demonstrate the possibility of a split continuation region, where the executive exercises only at intermediate stock prices, but not at high or low prices.

When unconstrained trading of outside wealth in the market is possible, our numerical examples with constant relative risk averse (CRRA) utility show how the exercise boundary, option cost, and average life vary with the correlation between the stock return and the market return. We also demonstrate that the option's exercise policy and cost to the firm do not converge to their value-maximizing (Black-Scholes) counterparts as the correlation goes to one; this result is in sharp contrast to the case with constant absolute risk averse (CARA) utility. The presence of short-sale costs dampens the effect of correlation.

Recent accounting regulation requiring firms to recognize option expense has intensified the demand for better valuation methods. Until better alternatives emerge, the Financial Accounting Standards Boards (FASB) accepts the use of the Black-Scholes-Merton formula with the option's contractual term replaced by its average life; this approximation is used by the vast majority of firms. We compare our model's option cost with this approximate cost and show how the approximation error varies with the stock beta, volatility, and dividend rate. We find that the error can be large or small, positive or

negative. Moreover, in some cases, the response of the FASB approximation to a change in parameters is actually in the wrong direction.

Finally, we examine the option's value from the viewpoint of the executive. We prove that the executive's subjective option value is decreasing in the dividend payout rate. Our examples show how the subjective value discount from option cost varies with firm characteristics. The magnitude of the discount suggests that the incentive benefits of option compensation must be large to offset its cost relative to cash compensation. Our results are robust to the inclusion of restricted stock in the executive's portfolio.

Overall, our analysis underscores the importance of accurately characterizing the exercise policy for option valuation. As more data on exercises become available, it will be possible to estimate an empirical option exercise and cancellation rate function, and thus deduce option cost empirically. The results of this paper provide guidance about how to specify and interpret models for estimating exercise rates and option cost, and yield testable predictions about option exercise behavior. For example, our results suggest that average option lives should be increasing in non-option wealth and in the dividend rate.

The paper proceeds as follows. Section 2 reviews the related literature. Section 3 presents the general model with optimal trading of outside wealth and analyzes the dividend effect. Section 4 analyzes the effects of risk aversion and wealth and characterizes the shape of the exercise policy for a general utility function in the special case when outside wealth is invested risklessly. Section 5 studies the general case with optimal dynamic trading of outside wealth assuming CRRA utility, shows how the correlation between the underlying stock and the market affects exercise policy and option cost, and explains why they do not converge to their value-maximizing counterparts as this correlation tends to one. Section 6 shows how option average life and the FASB approximation error vary with the stock beta, volatility, and dividend payout rate. Section 7 examines the option's subjective value to the executive and shows how the subjective discount varies with the stock price parameters. Section 8 considers the effect of holding restricted stock on option cost. Finally, Section 9 concludes.

2 Related literature

The intuition that the need for diversification can lead an executive to sacrifice some option value by exercising early is well understood in the literature, but explicit theory of the optimal exercise of ESOs is still developing. Early papers establish the conceptual

foundation for ESO valuation and explain the rationale for value-destroying early exercise. In particular, Huddart (1994) and Marcus and Kulatilaka (1994) present binomial models of the exercise decision and option cost with CRRA utility and outside wealth in the riskless asset. Marcus and Kulatilaka (1994) include a positive risk premium on the stock return, so the riskless investment of outside wealth is not optimal and this can create a distortion in the exercise policy. Carpenter (1998) assumes outside wealth is invested in the Merton (1971) stock-bond portfolio, but this is still not fully optimal in the presence of the option. Detemple and Sundaresan (1999) study the case of simultaneous optimal-exercise and portfolio-choice decisions and provide a characterization of the solution in terms of certainty equivalents. Some recent papers use specific utility functions to focus on the optimal partial exercise of options. These include Jain and Subramanian (2004), Grasselli and Henderson (2009), and Rogers and Scheinkman (2007). None of these papers contains either general analytical results on option cost or a comprehensive numerical examination of policy and cost with dynamic trading of outside wealth.

Other recent papers have solved versions of the problem we describe here for the case of CARA utility. Kadam, Lakner, and Srinivasan (2003) model the optimal exercise policy for an infinite-horizon option, but the model links the manager's consumption date to the option exercise date, which can distort the exercise decision, even in the absence of trading restrictions. Henderson (2007) also models the optimal exercise policy for an infinite-horizon real option and links the manager's consumption date to the option exercise date, but uses a specialized utility function so that this link does not distort the exercise policy. Leung and Sircar (2009) solve the finite-horizon problem, and include the risk of job termination and the possibility of partial option exercise. Miao and Wang (2007) analyze a similar problem in a real-options context with CARA utility and an underlying asset that follows arithmetic Brownian motion. These papers obtain analytical results under the assumption of CARA utility so that outside wealth is not a state variable. Our paper allows for a more general utility specification and the study of wealth effects.

A number of papers value options using exogenous specifications of the exercise policy. Jennergren and Näslund (1993), Carr and Linetsky (2000), and Cvitanić, Wiener, and Zapatero (2008) derive analytic formulas for option cost assuming exogenously specified exercise boundaries and forfeiture rates. Hull and White (2004) propose a binomial model in which exercise occurs when the stock price reaches an exogenously specified multiple of the stock price and forfeiture occurs at an exogenous rate. Rubinstein (1995) and Cuny and Jorion (1995) also compute option cost under exogenous assumptions about the

timing of exercise. Given a specific exercise policy, these models are more computationally tractable, but the accuracy of their approximation of option cost is as yet unknown and they provide little insight about how option exercise policy should vary with option holder and firm characteristics.

Finally, another literature focuses on option subjective value to the executive and its discount from firm cost. Lambert, Larcker, and Verrecchia (1991) analyzes the case of a European option. Hall and Murphy (2002) allow for early exercise but invest non-option wealth in an exogenously specified portfolio of stock and bonds. Cai and Vijh (2005) fix outside wealth in the optimal constant-proportion portfolio. Ingersoll (2006) develops an analytic subjective option valuation methodology assuming a CRRA executive with a fixed proportion of wealth in restricted stock, a marginal option position and an optimal constant-barrier policy. In addition to our contributions to the literature on optimal exercise and option cost, our paper augments this subjective value literature with a comprehensive study of firm parameter effects in a model with optimal dynamic trading.

3   General framework

Options have become a significant part of corporate compensation, now representing over 40% of compensation for top executives at large firms (see Frydman and Saks, 2007). While option compensation is widely believed to create valuable performance benefits, firms, investors, and regulators are also increasingly concerned about its cost, since a better understanding of option cost can help firms decide how to use options most efficiently. Options strengthen performance incentives mainly because option holders cannot sell or fully hedge their options. In particular, options are explicitly nontransferable and Section 16-c of the Securities Exchange Act prohibits corporate insiders from taking short positions in their company's stock. However, these constraints on hedging mean that standard American option valuation theory does not apply. Standard theory assumes the option holders can trade freely and thus exercise according to a value-maximizing policy (see, for example, Merton, 1973; Van Moerbeke, 1976; Roll, 1977; Geske, 1979; Whaley, 1981; Kim, 1990). By contrast, constrained option holders exercise options in potentially value-destroying ways in order to diversify away some stock-specific risks.

The use of zero-cost collars and equity swaps by corporate insiders documented by Bettis, Bizjak, and Lemmon (2001) suggests that insiders may have some scope for hedging their incentive compensation. However, evidence that the vast majority of options are

exercised well before expiration, even when no dividend is present, suggests that option holders still face significant hedging constraints (see Bettis, Bizjak, and Lemmon, 2005).

Although we refer to the option holder as an executive, our formulation of the problem may apply equally well to lower-ranked employees. Non-insiders are not subject to Section 16-c short sales restrictions, but they may still find shorting stock to be quite expensive. Evidence of pervasive early exercise across all ranks, documented in Huddart and Lang (1996) and Carpenter, Stanton, and Wallace (2009), suggests that lower-ranked employees also face significant hedging constraints. This section lays out a general model of the executive's optimal-exercise problem in the presence of hedging restrictions and defines the resulting option cost to shareholders.

3.1 Executive's option exercise and portfolio choice problem

The executive has n finite-lived nontransferable options with strike price K and expiration date T, and additional wealth W that he can invest subject to a prohibition on short sales of the stock. The investment set includes riskless bonds with constant rate r, the underlying stock with price S_t, and a market portfolio with price M_t. These prices satisfy

$$\frac{dS_t}{S_t} = (\lambda - \delta)\, dt + \sigma\, dB_t, \tag{1}$$

$$\frac{dM_t}{M_t} = \mu\, dt + \sigma_m\, dZ_t, \tag{2}$$

where B and Z are standard Brownian motions with instantaneous correlation ρ. The stock return volatility, σ, the stock dividend rate, δ, and the mean and volatility of the market return, μ and σ_m, are constant. The mean stock return, λ, is equal to the normal return for the stock given its correlation with the market,

$$\lambda = r + \beta(\mu - r), \tag{3}$$

where $\beta = \rho\sigma/\sigma_m$. In particular, in the absence of the option, an optimal portfolio would contain no stock position beyond what is implicitly included in the market portfolio.

The executive simultaneously chooses an option exercise time τ and an outside wealth investment strategy in the market and the stock, $\pi_t \equiv (\pi_t^m, \pi_t^s)$. His goal is to maximize

the expected utility of time T wealth, or equivalently:

$$\max_{\{t_v \leq \tau \leq T, \pi^m, \pi^s \geq 0\}} \mathrm{E}\{V(W_\tau^\pi + n(S_\tau - K)^+, \tau)\}, \tag{4}$$

where t_v is the option vesting date, W^π denotes the outside wealth process under trading strategy π given by

$$dW_t^\pi = rW_t^\pi \, dt + \pi_t^m((\mu - r) \, dt + \sigma_m \, dZ_t) + \pi_t^s((\lambda - r) \, dt + \sigma \, dB_t). \tag{5}$$

V is the indirect utility of freely investable wealth,

$$V(W_t, t) \equiv \max_{\pi^m} \mathrm{E}_t\{U(W_T)\} \text{ s.t. } dW_u = rW_u \, du + \pi_u^m((\mu - r) \, du + \sigma_m \, dZ_u), \tag{6}$$

and the utility function U is strictly increasing, strictly concave, and twice continuously differentiable.

This formulation entails a number of simplifications. The executive's portfolio does not include a position in restricted shares of stock here, though Section 8 develops that extension (see also Kahl, Liu, and Longstaff, 2003; Ingersoll, 2006, for models of portfolio choice with restricted stock). It allows only for a single block exercise of the option, although the executive would probably prefer partial exercise. The model also considers only a single grant of options when in practice, executives are granted new options every year and typically build up large inventories of options with different strikes and expiration dates. It would be useful to understand which options are most attractive to exercise first and how the anticipation of future grants of options and other forms of compensation affects current exercise decisions. In addition, the model does not account for any control the executive has over the underlying stock price process through the exertion of effort and through project and leverage choices; these choices may interact with the exercise decision. Finally, the model as outlined here does not incorporate the possibility of cancellation, though we develop this extension in Section 6.2. Despite these simplifications, we believe this formulation captures the essence of the executive stock option problem.

3.2 Option cost to shareholders and executive's subjective value

The solution to the executive's optimal exercise problem, that is, the optimal exercise policy τ, defines the option payoff, $(S_\tau - K)^+$ that occurs at time τ. The cost of the

6

option to shareholders who can trade freely is the present value, or replication cost, of that payoff. This can be represented as the risk-neutral expectation of the risklessly discounted option payoff,

$$P = \mathrm{E}^*\{e^{-r\tau}(S_\tau - K)^+\}, \tag{7}$$

where E^* means expectation with respect to the probability measure under which the expected returns on both the market and the stock are equal to the riskless rate.

Standard theory for tradable options assumes the option holder chooses the exercise policy to maximize the option's present value, because when the option is tradable, maximizing present value is consistent with maximizing expected utility. When the option is nontransferable these objectives are different, and the utility-maximizing payoff typically has a lower present value.

In addition, when the option is nontransferable, its present value or cost to shareholders is different from its subjective value to the executive. We define the subjective option value as the amount of freely investable money that would give the executive the same utility as the option, i.e., the value of x such that

$$V(W + x, t) = f(W, S, t), \tag{8}$$

where V is the indirect-utility value of freely investable wealth as defined in Equation (6), and f is the value function for the executive's problem with the options, defined formally below. This definition is consistent with that in Kahl, Liu, and Longstaff (2003) for the subjective value of restricted stock. As Bergman and Jenter (2007) explain, in complete markets, the subjective value must be less than the option cost to shareholders. The discount in the executive's valuation of the option relative to its cost to shareholders is part of the price shareholders pay for improved performance benefits relative to cash compensation. Section 7 shows how this discount varies with firm characteristics

3.3 Exercise policy and the effect of dividends

The cost of the executive stock option depends on the executive's exercise policy. In the Markovian setting here, we can describe the exercise policy in terms of the so-called continuation region of the executive, the set of states in which he continues to hold the

option. Formally, the value function for the executive's problem is

$$f(W_t, S_t, t) \equiv \sup_{\{t \vee t_v \leq \tau \leq T, \pi^m, \pi^s \geq 0\}} E_t\{V(W_\tau^\pi + n(S_\tau - K)^+, \tau)\} \qquad (9)$$

and the executive's continuation region is the set

$$D \equiv \{(w, s, t) : t < t_v \text{ or } f(w, s, t) > V(w + n(s - K)^+, t)\}. \qquad (10)$$

The nature of the present-value-maximizing continuation region for an ordinary American option is well known (see, for example, Kim, 1990). There exists a critical stock price boundary above which the option holder exercises and below which he waits. The boundary is increasing in the stock return volatility and time to expiration and decreasing in the dividend rate. For an executive stock option, some of these results may fail to hold. However, the dividend effect is the essentially the same.

Proposition 1 *The executive's continuation region is larger the smaller the dividend rate on the stock.*

Proof Suppose a given state (w, s, t) with $t \geq t_v$ is in the continuation region when the dividend rate is δ_1 and let $\delta_2 < \delta_1$. Let $f(w, s, t; \delta)$ denote the value function and $S_t^{(\delta)}$ denote the stock price process when the dividend rate is δ. For every strategy π and τ,

$$V(W_\tau^\pi + n(S_\tau^{(\delta_2)} - K)^+, \tau) \geq V(W_\tau^\pi + n(S_\tau^{(\delta_1)} - K)^+, \tau), \qquad (11)$$

where W^π denotes the outside wealth process under trading strategy π. This implies

$$\sup_{\pi, \tau} E_t V(W_\tau^\pi + n(S_\tau^{(\delta_2)} - K)^+, \tau) \geq \sup_{\pi, \tau} E_t V(W_\tau^\pi + n(S_\tau^{(\delta_1)} - K)^+, \tau), \qquad (12)$$

so

$$f(w, s, t; \delta_2) \geq f(w, s, t; \delta_1) > V(w + n(s - K)^+, t). \qquad (13)$$

Therefore, (w, s, t) is in the continuation region for δ_2.

Note that this result holds for a general utility function, regardless of the shape of the continuation region or the existence of a critical stock boundary. Consistent with this, Leung and Sircar (2009) find the exercise boundary in the case of CARA utility declines with the dividend rate. Numerical examples described later show that, like the value of

ordinary options, executive option cost decreases in the dividend rate. The inequality $f(w, s, t; \delta_2) \geq f(w, s, t; \delta_1)$ above also implies the following:

Proposition 2 *The executive's subjective option value is greater the smaller the dividend rate on the stock.*

4 Special case with no portfolio choice

The presence of the outside portfolio-choice problem significantly complicates the analysis, and our remaining results for this general case are numerical. Before turning to these, we develop additional analytical results for the special case in which there is no market asset, and the executive simply invests outside wealth in riskless bonds. We also eliminate the risk premium on the stock because otherwise the executive could have a spurious incentive to hold the option in order to earn the risk premium. Thus, we consider the case in which the stock appreciates at the riskless rate,

$$\frac{dS_t}{S_t} = (r - \delta)\, dt + \sigma\, dB_t, \tag{14}$$

and there is no other risky asset available. After the executive exercises the options, his optimal portfolio contains only riskless bonds, so

$$V(W_t, t) = U(W_t e^{r(T-t)}). \tag{15}$$

The executive's problem at each time $t < T$ then becomes

$$f(S_t, t) \equiv \max_{\{t \vee t_v \leq \tau \leq T\}} \mathrm{E}_t\{U(n(S_\tau - K)^+ e^{r(T-\tau)} + W)\}, \tag{16}$$

where the constant W is outside wealth at time T, and $f : (0, \infty) \times [0, T] \to \mathcal{R}$ is a continuous function.

In the continuation region,

$$D = \{(s, t) \in (0, \infty) \times [0, T] : t < t_v \text{ or } f(s, t) > U(n(s - K)^+ e^{r(T-t)} + W)\}, \tag{17}$$

$f(S, t)$ satisfies $E(df) = 0$. If f is $C^{2,1}$ then, by Ito's Lemma, it satisfies the p.d.e.

$$f_t + \frac{1}{2}\sigma^2 S^2 f_{SS} + (r - \delta)S f_S = 0. \tag{18}$$

9

To calculate f numerically, simultaneously determining the optimal exercise policy, we solve Equation (18) backwards using an implicit finite-difference method, similar to valuing an ordinary American option (see Appendix A). The market value, or cost, of the option, $P(S, t)$, solves the usual Black-Scholes equation,

$$P_t + \frac{1}{2}\sigma^2 S^2 P_{SS} + (r - \delta)SP_S - rP = 0,$$

subject to the exercise policy determined in solving for f.

4.1   Existence of a single stock price boundary

This section explores whether a single stock price boundary $\bar{s}(t)$ separates the continuation region below from the exercise region above, as is the case for ordinary American calls. This is often assumed to be true in executive stock option models with exogenously specified exercise policies. However, we show that the utility-maximizing policy need not have this structure and provide conditions under which it does.

To formalize intuition about the various effects of waiting to exercise, let $g(s, t) \equiv U(n(s - K)^+ e^{r(T-t)} + W)$ denote the payoff function for the optimal stopping problem and note that on $(K, \infty) \times [0, T]$, g is $C^{2,1}$ and Itô's lemma implies that g has drift equal to $H(S_t, t)$ where

$$H(s, t) \equiv U'(h(s, t))(rK - \delta s)ne^{r(T-t)} + \frac{1}{2}U''(h(s, t))n^2 e^{2r(T-t)}\sigma^2 s^2 \tag{19}$$

and $h(s, t) \equiv n(s - K)e^{r(T-t)} + W$ is total time T wealth given exercise at time t and stock price s. This expression shows that when the option is in the money, the effects of waiting to exercise include the benefits of delaying payment of the strike price, the cost of losing dividends, and the cost of bearing stock price risk.

Proposition 3 *Suppose that $W > nKe^{rT}$ and that H is nonincreasing in the stock price s. For each time $t \in [t_v, T)$, if there is any stock price at which exercise is optimal, then there exists a critical stock price $\bar{s}(t)$ such that it is optimal to exercise the option if and only if $S_t \geq \bar{s}(t)$.*

Proof Fix $t \in [t_v, T)$. Suppose (s_1, t) is a continuation point. We show that if $s_2 < s_1$ then (s_2, t) is also a continuation point. First note that it must be optimal to continue holding the option if $S_t \leq K$. Stopping then would guarantee a reward of $U(W)$, which

is less than the expected utility of continuing, for example, until the first time the stock price rises to $K + c$, for some $c > 0$, or until expiration T.

So assume $s_1 > s_2 > K$. For $u \geq t$, let $S_u^{(i)}$ denote the stock price process starting from s_i at time t and note that $S_u^{(1)} > S_u^{(2)}$. Finally, let τ be the optimal stopping time given $S_t = s_1$. Since τ is a feasible strategy if $S_t = s_2$,

$$
\begin{aligned}
f(s_2, t) - f(s_1, t) &\geq \mathrm{E}_t\{U(n(S_\tau^{(2)} - K)^+ e^{r(T-\tau)} + W) - U(n(S_\tau^{(1)} - K)^+ e^{r(T-\tau)} + W)\} \\
&\geq \mathrm{E}_t\{U(n(S_\tau^{(2)} - K)e^{r(T-\tau)} + W) - U(n(S_\tau^{(1)} - K)e^{r(T-\tau)} + W)\} \\
&= g(s_2, t) - g(s_1, t) + \mathrm{E}_t \int_t^\tau (H(S_u^{(2)}, u) - H(S_u^{(1)}, u))du \\
&\geq g(s_2, t) - g(s_1, t). \tag{20}
\end{aligned}
$$

Therefore, $f(s_2, t) - g(s_2, t) \geq f(s_1, t) - g(s_1, t) > 0$.

The hypothesis is satisfied for CRRA utility functions with relative risk aversion less than or equal to one and sufficiently large wealth. Similarly, in the value-maximization problem for an ordinary option, the second-order term in H does not appear, and it follows that it is optimal to exercise if and only if the stock price is above a critical level. For executive stock options however, the risk aversion of the option holder gives rise to the second-order term, and the drift need no longer be monotonic in the stock price.

Example with a split continuation region Figure 1 shows the optimal exercise policy for utility function

$$
U(W) = \frac{W^{1-A}}{1 - A} + cW \tag{21}
$$

with $A = 10, c = 0.0001, S_0 = K = 1, t_v = 5, T = 10, r = 0.05, \sigma = 30\%$, and $\delta = 0$. As the figure shows, the executive exercises the option for intermediate stock prices, but does not exercise at either high or low stock prices. The intuition for this is that as the stock price grows large, the executive feels wealthier, the concave part of his utility becomes small relative to the risk neutral part, he starts to act like a cost maximizer, and refrains from exercising because the dividend rate is zero. With a positive dividend rate, there would also be another exercise region, above the upper continuation region, extending to infinity. In this example, if we ignore the presence of the upper continuation region, the option cost is 0.408 instead of the correct value of 0.432.

11

4.2 Risk aversion, wealth, and volatility effects

This section describes how exercise policy and option cost change with risk aversion, wealth, and stock return volatility.

4.2.1 Monotonicity with respect to risk aversion and wealth

Intuition suggests that less risk-averse managers are likely to exercise later, and consequently the cost of their options is greater. Similarly, one would expect that managers with decreasing absolute risk aversion will exercise later, implying greater option cost, if they have more non-option wealth. The following results verify this intuition and hold regardless of the actual shape of the continuation region:

Proposition 4 *An executive with less absolute risk aversion has a larger continuation region.*

Proof If U_1 and U_2 are utility functions and U_2 has everywhere less absolute risk aversion than U_1, then by Theorem 5 on page 40 of Ingersoll (1987),

$$U_2(W) = G(U_1(W)), \tag{22}$$

where the function G satisfies $G' > 0$ and $G'' > 0$. Now suppose a given state (s,t) with $t \geq t_v$ is in the continuation region for the problem with utility U_1. Let τ be the optimal stopping time for U_1. Let $f_i(s,t)$ and $g_i(s,t)$ denote the value and payoff functions for the problem with utility U_i, $i = 1, 2$. Since τ is feasible for the problem with U_2,

$$
\begin{aligned}
f_2(s,t) - g_2(s,t) &\geq \mathrm{E}_t\{U_2(n(S_\tau - K)^+ e^{r(T-\tau)} + W)\} - U_2(n(S_t - K)^+ e^{r(T-t)} + W) \\
&= \mathrm{E}_t\{G(U_1(n(S_\tau - K)^+ e^{r(T-\tau)} + W))\} - G(U_1(n(S_t - K)^+ e^{r(T-t)} + W)) \\
&\geq G(\mathrm{E}_t\{U_1(n(S_\tau - K)^+ e^{r(T-\tau)} + W)\}) - G(U_1(n(S_t - K)^+ e^{r(T-t)} + W)) \\
&= G(f_1(s,t)) - G(g_1(s,t)) > 0.
\end{aligned}
$$

Therefore, (s,t) is also in the continuation region for U_2.

Similarly, in the CARA models of Henderson (2007), Miao and Wang (2007), and Leung and Sircar (2009), the exercise boundary declines with the level of risk aversion.

Corollary 1 *If the executive has decreasing absolute risk aversion, then the continuation region is larger with greater wealth.*

Proof Let $W_2 > W_1$ and note that $U(w + W_2 - W_1) = G(U(w))$ for some function G satisfying $G' > 0$ and $G'' > 0$.

Note that when the stock price process is the same, but the continuation region is larger, the realized time to exercise or expiration is greater, and thus the option's expected life is greater. This leads to the following two corollaries:

Corollary 2 *The expected life of the option is greater if the executive has less absolute risk aversion.*

Corollary 3 *If the executive has decreasing absolute risk aversion, then the expected life of the option is greater if the executive is wealthier.*

This yields the testable prediction that the options of wealthier executives have longer average lives.

Proposition 5 *If the dividend is zero, option cost is greater if the executive has less absolute risk aversion.*

Proof Suppose U_1 and U_2 are utility functions and U_2 has everywhere less absolute risk aversion than U_1. For $i = 1, 2$, let τ_i be the optimal stopping time for the executive with utility U_i and let P_i be the resulting option cost. Finally, let

$$p(s, t) \equiv e^{-rt}(s - k). \tag{23}$$

By Proposition 4, $\tau_2 \geq \tau_1$, so

$$
\begin{aligned}
P_2 - P_1 &= \mathrm{E}\{p(S_{\tau_2}, \tau_2)^+ - p(S_{\tau_1}, \tau_1)^+\} & (24)\\
&= \mathrm{E}\{(p(S_{\tau_2}, \tau_2)^+ - p(S_{\tau_1}, \tau_1))1_{\{\tau_1 < T\}}\} & (25)\\
&\geq \mathrm{E}\{(p(S_{\tau_2}, \tau_2) - p(S_{\tau_1}, \tau_1))1_{\{\tau_1 < T\}}\} & (26)\\
&= \mathrm{E}\{\int_{\tau_1}^{\tau_2} e^{-rt}(rK\,dt + \sigma S_t\,dB_t)1_{\{\tau_1 < T\}}\} & (27)\\
&= \mathrm{E}\{\mathrm{E}_{\tau_1}\{\int_{\tau_1}^{\tau_2} e^{-rt}(rK\,dt + \sigma S_t\,dB_t)\}1_{\{\tau_1 < T\}}\} \geq 0. & (28)
\end{aligned}
$$

Corollary 4 *If the executive has decreasing absolute risk aversion and the dividend is zero, then option cost is greater with greater wealth.*

13

Proof From Corollary 1, the optimal stopping time for an executive with greater wealth is later. The rest follows like the proof of Proposition 5.

In numerical examples with CRRA utility, option cost decreases in risk aversion and increases in wealth with a positive dividend as well. All of the examples described in this section are generated using an implicit finite-difference method to solve the partial differential equations describing the executive value function and option cost. Even in examples in which the coefficient of relative risk aversion, A, is greater than one, or wealth is small, the continuation region is characterized by a single stock price boundary. In addition, the minimum wealth condition in Proposition 3, used only in the proof to ensure that the arguments of the utility function remain nonnegative, does not appear to be necessary in the numerical examples.

Figures 2 and 3 illustrate these effects with plots of exercise boundaries and option cost for various levels of risk aversion and wealth. On the left, the exercise boundary for each level of risk aversion is a plot of the critical stock price $\bar{s}(t)$ versus time t. On the right, the option cost, labeled "ESO cost," determined by the different boundaries are plotted against the level of risk aversion. Shown for comparison, the option value labeled "Max value" is the value of the option under the usual present-value-maximizing policy. In all of the figures, the options are at the money with ten years to expiration, the number of options and initial stock price are normalized to one, and the riskless rate is 5%. Other parameter values in Figures 2 and 3 are varied around a base case in which the executive has CRRA coefficient 2 and initial wealth 1.2, and the stock has zero risk premium, 3% dividend, and 50% volatility.

As indicated by Proposition 4 and Corollary 1 the exercise boundaries fall with risk aversion and rise with wealth. In addition, in the numerical examples, option cost falls with risk aversion and rises with wealth, even with a positive dividend, suggesting that though our proof of Proposition 5 uses the zero dividend condition, it is probably not necessary. The examples also suggest that as risk aversion grows large, or as outside wealth goes to zero, the boundary falls to $S = K$ and option cost falls to zero (or the value of a European option that expires on the vesting date). The intuition for this is that as risk aversion grows large, the risk premium required to trade a certain exercise value for a risky continuation value goes to infinity. On the other hand, as risk aversion goes to zero, or as wealth grows large, option cost converges to its maximized value.

4.2.2 Nonmonotonicity with respect to stock return volatility and time

A basic result in standard option pricing theory is that option value is increasing in volatility. This is also typically the case in executive stock option models with an exogenously specified exercise boundary (see, for example, Cvitanić, Wiener, and Zapatero, 2008). However, the utility-maximizing continuation region can shrink considerably with volatility and this can lead to option cost actually declining in volatility.

The risk-averse utility of the option payoff, as a function of the stock price, has both a convex region and a concave region, so in principle, an increase in volatility could either lead the executive to continue longer or exercise sooner. Figure 4 illustrates these effects with plots of the exercise boundaries and option cost for various levels of stock return volatility. The executive has CRRA coefficient 2 and initial wealth 1.2 and the stock has zero risk premium and zero dividend. As volatility rises from 10% to 200%, the exercise boundary tends to fall first, but can then rise slightly, especially at intermediate values of t. This is shown most clearly in Figure 4A, with risk aversion coefficient $A = 2$. The initial drop in the boundary as volatility increases is perhaps best understood by starting with the case $\sigma = 0$. In this case, if the dividend rate is less than the interest rate, it is never optimal to exercise early, that is, the boundary is infinite.[1] So starting at $\sigma = 0$, the boundary has to decline with volatility.

The subsequent slight rise in the boundary as volatility increases from a high level is in line with the theory for traded options. It may occur because when the option is in the money, low-volatility variation in the stock price remains in the concave portion of the utility payoff function, while high-volatility variation reaches across the convex portion, so higher volatility may be needed to tap the convexity effects of the option payoff from standard theory. In the real-option models of Henderson (2007) and Miao and Wang (2007) with CARA utility, the investment threshold is monotone in volatility, but the direction depends on the parameters. Empirically, Bettis, Bizjak, and Lemmon (2005) find that insider options are exercised earlier at higher-volatility firms.

Executive stock option cost can also be nonmonotonic in volatility. At lower levels of risk aversion, as shown in Figure 4A, option cost is generally increasing in volatility. However, at higher levels of risk aversion, as shown in Figures 4B and 4C, option cost is decreasing in volatility at low levels of σ. Here the negative effect on cost of the drop in the boundary offsets the positive effect of extreme stock prices becoming more likely. Other

---

[1]Indeed, Huddart and Lang (1996) find that at one of their sample firms with only 4% volatility, employees wait almost until expiration to exercise their options.

papers, such as Lambert, Larcker, and Verrecchia (1991), Ross (2004), Henderson (2007), and Miao and Wang (2007), have shown that the option's private value to the holder can decline with volatility. Ours is the first well-posed analysis to show that the option's cost to shareholders can also decline with volatility. In all cases, the gap between the ESO cost and the maximized option value widens as volatility increases. Consistent with this, Bettis, Bizjak, and Lemmon (2005) find empirically that the fraction of maximized option value that insiders realize at exercise declines with firm volatility.

Figure 4 also shows that in some cases, especially when volatility is high, the utility-maximizing exercise boundary may not decline monotonically in time (by contrast, the value-maximizing boundary always declines as expiration approaches). To understand this, consider first the following proposition, which indicates that when the interest rate is zero, the set of stock prices at which continuation is optimal shrinks as time t elapses; in particular, if there is a critical boundary, it declines monotonically in time.

Proposition 6 *Suppose $r = 0$. Fix stock price $s > 0$, let $t_v \leq t_1 < t_2$, and suppose it is optimal for the executive to continue with the option at (s, t_2). Then it is optimal to continue at (s, t_1).*

Proof For $t \geq t_v$, write the executive's value function as

$$f(s, t, T) \equiv \max_{\{t \leq \tau \leq T\}} \mathrm{E}_t\{U(n(se^{-(\delta+\sigma^2/2)(\tau-t)+\sigma(B(\tau)-B(t))} - K)^+ + W)\}, \qquad (29)$$

and note that

$$f(s, t, T) = f(s, t_v, T - (t - t_v)) . \qquad (30)$$

Next, note that if $T_1 < T_2$, then $f(s, t_v, T_1) \leq f(s, t_v, T_2)$ since any τ feasible for the problem with horizon T_1 is feasible for the problem with horizon T_2. Therefore,

$$
\begin{align}
f(s, t_1, T) &= f(s, t_v, T - (t_1 - t_v)) \tag{31}\\
&\geq f(s, t_v, T - (t_2 - t_v)) \tag{32}\\
&= f(s, t_2, T) \tag{33}\\
&> U(n(s - K)^+ + W) , \tag{34}
\end{align}
$$

so it is optimal to continue at (s, t_1).[2]

---

[2]Leung and Sircar (2009) have a similar result for the CARA case.

This result indicates that the nonmonotonicity of the boundary in time must stem from the presence of a positive interest rate. To gain some intuition for this, note that the value of continuing can be viewed as the in-the-money drift H of the utility payoff function (from equation (19)) plus the time value, or "put" value, of retaining the option not to exercise if the stock price falls below the strike price.[3] This put value of continuing, which is increasing in volatility and decreasing in time, is weighed against the cost of bearing more risk, which is summarized by the second-order "risk-aversion" term in H. That risk aversion term is increasing in volatility and may be decreasing in time when the interest rate is positive, the effect being greater at higher volatility. So the net effect of the passage of time may be to discourage exercise for high volatility and positive interest rates.

On the other hand, in the limit as time approaches the expiration date, the exercise decision merely hinges on the sign of H. In the CRRA case, at $t = T$ there is a single positive stock price, S^*, above which H is negative and below which H is positive. The limit of the exercise boundary as $t \to T$ is then $\max(S^*, K)$. When the dividend rate is zero this limit is decreasing in volatility, and for high volatility it is equal to the strike price, as Figure 4 shows.

5   General case with outside portfolio choice

This section examines the general problem with nontrivial outside-portfolio optimization, described in Section 3. Existing models of optimal exercise with an outside portfolio-choice problem, such as Cai and Vijh (2005), Henderson (2007), and Leung and Sircar (2009), either restrict the portfolio to constant proportions, or else assume CARA utility so that the exercise decision does not depend on wealth. By contrast, we assume CRRA utility for our numerical examples and allow for dynamic trading, so the value function f, the exercise decision, and the optimal outside-portfolio weights depend on both the stock price and outside wealth. In the continuation region, if f is sufficiently smooth and the portfolio weight in the market, π^m/W, lies in a compact set, then $f(W, S, t)$ satisfies the

---

[3]Formally, the evolution of the utility payoff $g(S_t, t) = U(n(S_t - k)^+ e^{r(T-t)} + W)$ is

$$dg(S_t, t) = 1_{\{S_t > K\}} H(S_t, t)dt + 1_{\{S_t > K\}} U'(h(S_t, t))ne^{r(T-t)}\sigma S_t \, dB_t + U'(W)ne^{r(T-t)} \, d\Lambda_t(K),$$

where $\Lambda_t(K)$ is the local time of the process S at the level K up to time t (see Karatzas and Shreve (1991) and Carr, Jarrow, and Myneni (1992)). This local time term captures the notion of the "put value" of continuing, described above.

Hamilton-Jacobi-Bellman equation,

$$\max_{\pi^m} f_t + f_W \left[rW + \pi^m(\mu - r) \right] + S f_S(\lambda - \delta) + \frac{1}{2} f_{WW}(\pi^m)^2 \sigma_m^2 \qquad (35)$$

$$+ \frac{1}{2} S^2 f_{SS} \sigma^2 + S f_{WS} \pi^m \rho \sigma \sigma_m = 0.$$

The option's market value, $P(W, S, t)$ satisfies the p.d.e.

$$P_t + WrP_W + SP_S(r - \delta) + \frac{1}{2} P_{WW}(\pi^m)^2 \sigma_m^2 \qquad (36)$$

$$+ \frac{1}{2} S^2 P_{SS} \sigma^2 + SP_{WS} \pi^m \rho \sigma \sigma_m - rP = 0,$$

subject to the exercise policy determined in solving for f. We solve Equations (35) and (36) simultaneously, using a locally one-dimensional implicit finite-difference method (see Appendix B).

Unreported results suggest that the wealth, risk aversion, and volatility effects from the previous section still hold in the presence of optimal trading in a market portfolio with a nonzero risk premium and correlation with the stock return. In particular, option cost is still increasing with executive wealth, decreasing with executive risk aversion, and nonmonotonic with respect to stock return volatility. In addition, the optimal exercise policy appears to be characterized by a critical stock price for each possible date and wealth level, above which it is optimal to exercise and below which it is optimal to continue. We also note that, when the market risk premium and the stock return correlation with the market are set to zero, the results are the same as those from the one-factor model of the last section. The remainder of this section focuses on the dependence of the exercise policy and option cost on the correlation between the stock return and the market return. Numerical examples in Figure 5 vary correlation around a base case in which the executive has CRRA coefficient 2 and initial wealth 1.2, the market has 20% volatility, and the stock has 50% volatility, zero dividend, and normal expected return given its correlation with the market. While some of the correlation effects are like those in the CARA case, others are very different.

5.1   Correlation effects with zero risk premium

The usual intuition from portfolio theory suggests that when the market risk premium $\mu - r$ is zero, the only reason to hold a market position in the outside portfolio is to

hedge the option position. Furthermore, all that should matter for the option exercise policy and cost is the magnitude of the correlation, ρ, not its sign, since that is what determines how much stock risk can be hedged away. Figure 5A confirms this intuition. Exercise boundaries and option cost for a given value of ρ are the same as for $-\rho$. To ease comparison with the previous section, the figures show exercise boundaries across time for wealth equal to its initial value. Also consistent with intuition, Figure 5A shows that option cost and exercise boundaries increase with $|\rho|$, as more and more of the option risk can be hedged away.

Limiting behavior as $|\rho| \to 1$: A surprising result from Figure 5A is that in the limit, as $|\rho|$ goes to one, the exercise boundaries and option cost remain strictly *below* their value-maximizing levels. With perfect correlation between the stock and the market, the executive can perfectly hedge the option, so there is no reason for a value-destroying early exercise, and the option cost is its maximized value (the Black-Scholes value in this case, indicated by the solid line in the right-hand plot of Figure 5A). However, this is *not* the limit of the option cost as $\rho \to \pm 1$. With anything less than perfect correlation, the basis risk in the hedge requires the executive to keep outside wealth positive, in order to avoid a possibly tiny, but still positive, probability that the option finishes out of the money while outside wealth remains nonpositive, which CRRA utility cannot tolerate. Therefore, in the limit as $|\rho| \to 1$, the executive is essentially solving the problem with perfect correlation, but also with a nonnegativity constraint on outside wealth. This bounds expected utility away from its unconstrained value and thus leaves open the possibility that value-destroying early exercise is optimal, as we now show by analytically solving the special case where correlation is perfect and the option is European.

If the executive is constrained to keep outside wealth nonnegative, then, using the martingale approach of Karatzas, Lehoczky, and Shreve (1987) and Cox and Huang (1989), his problem can be written in the form

$$\max_{W_T} EU(W_T + (S_T - K)^+) \text{ s.t. } E\{\zeta_T W_T\} \leq W_0 \text{ and } W_T \geq 0 \ , \tag{37}$$

where ζ_T is the stochastic discount factor

$$\zeta_T = e^{-rT - \xi B_T - \frac{1}{2}\xi^2 T} \ , \tag{38}$$

19

ξ is the Sharpe ratio

$$\xi = \frac{\mu - r}{\sigma_m} , \tag{39}$$

the terminal stock price is

$$S_T = S_0 e^{(\lambda - \delta)T + \sigma B_T - \frac{1}{2}\sigma^2 T} , \tag{40}$$

and $\lambda = r + \frac{\sigma}{\sigma_m}(\mu - r)$ is the normal mean stock return. Here the correlation between the stock and the market is plus or minus one, according to the sign of σ. The optimal outside-portfolio payoff is

$$W_T^* = [U'^{-1}(\gamma \zeta_T) - (S_T - K)^+]^+ , \tag{41}$$

where γ is the value of the Lagrange multiplier that makes the martingale budget constraint in problem (37) hold with equality, i.e.,

$$E\{\zeta_T[U'^{-1}(\gamma \zeta_T) - (S_T - K)^+]^+\} = W_0 . \tag{42}$$

The executive's optimal total terminal wealth with the option is $W_T^* + (S_T - K)^+$. For $U(W) = \frac{W^{1-A}}{1-A}$, $U'^{-1}(y) = y^{-1/A}$.

Note that the optimal total payoff in the unconstrained Merton problem without the option is of the form $(\gamma \zeta_T)^{-1/A}$, or equivalently $aW_0 S_T^{\theta^*}$, where a is a positive scalar and $\theta^* = \frac{\xi}{A\sigma}$ is the constant optimal proportion of wealth that would be invested in the stock if the investor could trade freely in the stock and riskless bonds. In other words, optimal terminal wealth is a power function of S_T. If the power $\theta^* \geq 1$ and the total present value of outside wealth and option wealth is sufficiently high, then outside wealth minus the option wealth, $aW_0 S_T^{\theta^*} - (S_T - K)^+$, will be positive for all values of S_T, in which case the nonnegativity constraint in problem (37) will be nonbinding and perfect hedging of the option will be possible. Otherwise, the constraint will bind for high values of S_T, and the executive will bear some unwanted option risk. For example, with zero risk premium, $\theta^* = 0$ and the optimal unconstrained payoff is a constant (generated by holding only riskless bonds). Perfect hedging would require negative outside wealth in high stock price states, so the nonnegativity constraint on outside wealth makes perfect hedging impossible. Similarly, in our examples with $\mu = 13\%$, $r = 5\%$, and $\sigma_m = 20\%$, the Sharpe ratio is $\xi = 0.4$, so with risk aversion coefficient $A = 2$ and stock return volatility 50%, the optimal portfolio weight in the stock in the unconstrained Merton problem is $\theta^* = 0.4$. Because this is less than 1, it does not generate a payoff large enough to absorb

the option payoff in high stock price states without violating the nonnegativity constraint.

When perfect hedging is not possible, there is the possibility that value-destroying early option exercise, which removes the hedging constraint, may be preferable to continuing with the option. We find numerically that, at a sufficiently high intermediate-date stock price, the expected utility of holding the European option to expiration (given the nonnegativity constraint) is less than the expected utility of exercising the option immediately and investing all wealth in the optimal unconstrained portfolio. For example, with a zero risk premium, $A = 2$, $K = 1$, $T = 10$, $r = 5\%$, $\sigma = 50\%$, and $\delta = 0$, at $t = 6$ (so there are 4 years until expiration), $S_t = 3$ and $W_t = 1.2$, the expected utility from the European option with the nonnegativity constraint above is -0.3022, while the expected utility from exercising immediately and investing all wealth in bonds is -0.2559.[4] Therefore, for an otherwise identical American option, holding to expiration is not optimal (even though the dividend is zero), and the cost of the option is strictly less than its maximized value. This result contrasts sharply with limiting behavior in CARA models such as Henderson (2007). There, the optimal exercise policy converges to the value-maximizing policy as $|\rho| \to 1$, and the option cost converges to its maximized value.

5.2   Correlation effects with a positive risk premium

When the market risk premium is positive, exercise boundaries and option cost are no longer necessarily symmetric in ρ, as Figure 5B illustrates. To understand this, recall that with less than perfect correlation between the stock and the market, the basis risk in hedging the option acts as a nonnegativity constraint on outside wealth. This means that not all of the market risk inherent in the option position can be hedged. Next, recall that the risk premium on the stock is equal to its normal level, i.e., the stock's market beta times the market risk premium. Therefore, when correlation is positive, the unhedged market risk in the option position carries a positive risk premium, while the unhedged market component of the option risk carries a negative risk premium when the correlation is negative. This means that for the same magnitude of correlation, holding the option is more attractive in the case of positive correlation than in the case of negative correlation.

Consistent with this explanation, we find that with a positive market risk premium,

---

[4]The continuation value is calculated by numerical integration, using Maple, i. to determine the value of the Lagrange multiplier, γ, from Equation (42); and then ii. to calculate the expected utility of total terminal wealth in problem (37). The expected utility obtained closely matches the expected utility calculated by our finite-difference solution to Equation (35) when $|\rho| = 0.99$ and the option is European.

expected utility in problem (37) above is higher when $\rho = 1$ than when $\rho = -1$. Since expected utility is lower when correlation is negative, the utility value of exercising immediately, which does not depend on the correlation, will exceed the expected utility of the European option in more states of the world. For example, with $A = 2$, $K = 1$, $T = 10$, $\mu = 13\%$, $r = 5\%$, $\sigma_m = 20\%$, $|\sigma| = 50\%$, and $\delta = 0$, at $t = 6$, $S_t = 1.75$ and $W_t = 1.2$, the expected utility from problem (37) is -0.3203 when $\rho = 1$ and -0.3995 when $\rho = -1$, while the expected utility from exercising immediately and investing all wealth in the optimal unconstrained Merton portfolio is -0.3578, regardless of the sign of ρ.[5] So when the option is American, it will be exercised sooner, and its cost will hence be lower, with negative correlation than with positive correlation.[6] By contrast, in the CARA model of Henderson (2007), the exercise policy depends only on the magnitude of the correlation, not the sign.

Our results of i. nonconvergence to value-maximization as correlation goes to one, and ii. asymmetry in the sign of correlation, go beyond CRRA and are likely to hold more generally if there is a nonnegativity constraint on outside wealth. In particular, regardless of preferences and no matter how high the correlation between the option and tradable assets, if the option holder cannot use the option as collateral against which to take short positions, then perfect hedging may not be possible and the option cost might be bounded away from its maximized value. In other words, even without a prohibition on short selling stock (as in the case of non-insider employees), a lower bound on outside wealth may reduce option cost.

5.3 Correlation effects with costly short sales

The presence of costly short selling can produce an offsetting asymmetric correlation effect. That is, short sales costs dampen the increase in option cost with respect to correlation when correlation is positive. Suppose the executive incurs a proportional cost ϕ per unit time whenever he shorts the market. Then the Hamilton-Jacobi-Bellman equation

---

[5]Again, the continuation values are calculated by numerical integration, and again the expected utilities obtained closely match those calculated by our finite-difference solution when $\rho = 0.99$ or -0.99 and the option is European.

[6]Leung and Sircar (2009) find the opposite asymmetry, that is, for the same magnitude, the exercise boundary is higher with negative correlation than with positive correlation. This is because they hold the mean return on the stock fixed as they vary correlation, so the stock has an abnormal return with respect to the hedging instrument, which is larger the smaller, or more negative, the correlation. This means that for correlation of given magnitude, the unhedged option risk in their model is more attractive when correlation is negative than when it is positive.

describing the executive's value function becomes

$$\max_{\pi^m} f_t + f_W \left[rW + \pi^m(\mu - r + \phi 1_{\pi^m < 0}) \right] + S f_S(\lambda - \delta) + \frac{1}{2} f_{WW}(\pi^m)^2 \sigma_m^2 \qquad (43)$$

$$+ \frac{1}{2} S^2 f_{SS} \sigma^2 + S f_{WS} \pi^m \rho \sigma \sigma_m = 0.$$

The short sale cost impedes hedging only when the correlation is positive and large enough to adjust the natural long market position so far downward that the option holder would like to hold an overall negative position. Figure 5C illustrates the effect of costly shorting for the case with low outside wealth and risk aversion, and either zero market risk premium (on the left) or positive market risk premium (on the right). In both cases, the effect is to reduce the executive's willingness to hold the option, and consequently to reduce option cost, whenever the correlation is sufficiently high. The effect is weaker with a positive market risk premium than with zero market risk premium because higher correlation is accompanied by a higher expected stock return, which creates a larger natural long position to absorb the hedge. Indeed, with zero risk premium, option cost remains constant at its zero correlation level once the short cost becomes high enough that hedging ceases to take place. With a positive risk premium, the effect of costly shorting is very slight. This is consistent with Jenter (2002), who finds that short-sale constraints are not relevant at reasonable wealth levels in a model of the incentive effects of stock-based compensation which also allows the manager to trade a market asset.

6   The FASB approximation

In December 2004, the FASB issued SFAS 123(R), which requires firms to recognize option cost in earnings. The new standard requires firms to estimate option cost according to "established principles of financial economic theory" when market prices are unavailable. However, recognizing that full-blown estimation methods are still only beginning to develop, the FASB illustrates a variety of acceptable methods for approximating option cost, which include lattice methods and a modified Black-Scholes-Merton formula. The vast majority of firms use the latter method, which entails first estimating the option's expected term, conditional on vesting, and then valuing the option at its Black-Scholes-Merton value using the option's expected term in place of its contractual term. This amount is then multiplied by the probability that the option vests, and later updated to reflect the actual number of pre-vesting forfeitures. Equilar (2007) finds that 88% of

Fortune 1000 firms used this method in 2006.

To gain a better understanding of the properties of the FASB approximation, consider two polar cases, one in which the approximation error is always negative, and one in which it is always positive. In the first case, there is a positive dividend rate, and the option is exercised according to the value-maximizing policy of standard theory. Then the true option cost must be greater than the option cost approximated using any deterministic expiration date, so the FASB approximation will understate cost. For example, a 10-year at-the-money option on a stock with a 3% dividend rate and a 30% volatility will cost 0.34 per dollar of underlying stock, assuming a 5% riskless rate. If the expected stock return is 13%, then the expected life of the option is 7.9 years, so the FASB approximation, that is, the value of a European at-the-money option on the same underlying stock with expiration in 7.9 years, is only 0.30, understating the true cost.

In the second polar case, the exercise or cancellation time is random, and independent of the stock price (or any other priced risk). In this case, the true option cost is the average of Black-Scholes values over the distribution of possible stopping times. Since the Black-Scholes value for at-the-money call options is generally concave in time to expiration, the true cost will be less than the Black-Scholes value of the average expiration date by Jensen's inequality. For example, consider an option on the same stock as above, but suppose it is only ever stopped through exercise or cancellation (depending on whether it is in or out of the money) at an exogenous termination rate of 12% per year. Then the true option cost will be 0.24, the expected life will be 5.8 years, and the FASB approximation will be 0.27, overstating the true cost. Based on this reasoning, Huddart and Lang (1996) argue that the FASB's methodology overstates option cost, but this is not always true.

In practice, executives voluntarily exercise options in ways that may depart more or less from value-maximization, and also experience employment termination, much of which occurs exogenously, so the approximation error could be either positive or negative. To see how the FASB approximation compares with true ESO cost in our setting, we compute the expected option term implied by the exercise policy of the executive in our model, and then calculate the FASB approximation.[7] Since, in our model, the executive neither maximizes value nor exercises and cancels purely randomly, it is not clear ex ante whether we should expect the FASB errors to be positive or negative, and, indeed, we find that the approximation error can go either way.

---

[7]The expectation of the option's term is under the true probability measure (as would be estimated using historical data on realized option lives, according to FASB guidelines).

6.1 Baseline model with no employment termination

Table 1 examines the performance of the FASB approximation using the pure utility-maximizing model developed so far. The table presents expected terms, ESO cost, FASB approximations, and approximation errors, for a variety of parametrizations of the model. We start with two different base-case parametrizations. The first is for a firm with high volatility, high beta, and low dividend rate, which might be typical of a young firm or a technology firm. The second is for a lower volatility, lower beta, higher dividend firm, such as a more seasoned firm in the industrial sector. Then we alternately vary the beta, the volatility, and the dividend rate of the base-case firms, to show different effects in the cross-section. Throughout the examples, the riskless rate is 5%, the expected return on the market portfolio is 13%, and the volatility of the market return is 20%. The option vests at year 2 and expires at year 10. The executive has CRRA coefficient 4 and outside wealth 0.6 times the grant date value of shares under option. This gives the options expected terms of about 5 years in the base cases, which is the average expected term used by Fortune 1000 firms from 2004 to 2006, according to Equilar (2007).

In the first base case, the true option cost is 0.45 and the FASB approximation is 0.50, overstating by 10%. In the second base case, the true cost is 0.29, while the approximation is 0.26, understating by 9%. The utility-maximizing policy is closer to maximizing value at the higher dividend rate and lower volatility of the second base case,[8] which helps to explain why the second base case has a negative approximation error, while that in the first base case is actually positive. Indeed, the maximized value of the option in the second base case is 0.34, only 20% higher than the ESO cost. In the first base case, the maximized value is 0.67, 49% higher than the ESO cost.

Panel A of Table 1 shows the effect of increasing the stock's beta. This is equivalent to increasing its correlation with the market, since volatility is held constant in this panel. As explained in Section 5, increasing this correlation improves the executive's ability to hedge option risk, which raises the exercise boundary and increases the ESO cost, as Panel A shows. However, there are conflicting effects on the option's expected term. On the one hand, raising the boundary should increase the expected term, for a given expected stock return. On the other hand, when beta increases, the expected stock return rises commensurately, which reduces the time to reach a given boundary. For the first base

---

[8]This makes intuitive sense. With dividends, the value-maximizing exercise policy has the same form as the utility-maximizing policy: exercise when the stock price hits some boundary. With no dividend, the value-maximizing policy is always to wait until expiration, a completely different form.

case, the latter effect dominates, and option expected life and the FASB approximation *decline* as beta rises, moving in the opposite direction to the true ESO cost. In the second case, the expected term is U-shaped in beta, but the approximation error still decreases in algebraic value monotonically.

Panels B and C of Table 1 show the effects of changing stock return volatility. Panel B shows the effect with beta held constant, in which case only the idiosyncratic component of the stock risk is varying. Panel C shows the effect with correlation held constant, in which case the idiosyncratic and hedgeable components of risk are held in constant proportion, such as in the case of an increase in leverage. Like the examples from Section 4, the ESO cost is either increasing or U-shaped in volatility, as is the FASB approximation. However, the FASB approximation rises faster with volatility, so the approximation error tends to increase in algebraic value. Given the intuition from the polar cases described above, this may partly be understood by noting that ESO cost falls away from its maximized value as volatility increases, as shown in Figure 4.

The effect is less pronounced in Panel C for two reasons. First, in Panel C, some of the increase in risk is an increase in hedgeable risk, which has a positive effect on the exercise boundary and ESO cost because of the convexity of the option payoff, without the corresponding negative effect of increased net risk exposure. Second, in the FASB approximation in Panel C, there is the negative effect of increasing beta which increases the stock's expected return and reduces the option's expected term. Thus the approximation error increases more slowly in Panel C than in Panel B.

Panel D of Table 1 shows dividend effects. Both ESO cost and FASB approximations decline as the dividend increases. However, expected option term actually increases with a higher dividend rate because this reduces the appreciation rate of the stock, which therefore takes longer to reach the exercise region. In the low end of the dividend range, the FASB value declines faster than the ESO cost, which actually grows as a proportion of its maximized value, so the approximation error decreases. As the dividend grows large, all values converge to zero.

Finally, Panel E illustrates wealth effects. Both ESO cost and FASB approximations increase with executive wealth, but the ESO cost, which approaches its maximized value, increases faster. Therefore, the approximation error decreases with wealth and even becomes negative. This is especially apparent in the second base case, with a positive dividend. The ESO cost converges to the maximized value of 0.34, which is greater than the value assuming any deterministic stopping date, such as the expected term. By con-

trast, the corresponding FASB approximation is only 0.30. The effects of decreasing risk aversion, not reported, are qualitatively similar.

6.2 Model with employment termination

In the baseline model, early exercise results only from motives of diversification and dividend capture. In practice, employees can also be forced to exercise an in-the-money vested option, forfeit an unvested option, or cancel an out-of-the-money vested option, if employment terminates. We therefore here study the effect of adding exogenous termination to the utility-maximizing exercise model. In this case, the Hamilton-Jacobi-Bellman equation describing the executive's value function becomes

$$
\max_{\pi^m} f_t + f_W \left[rW + \pi^m(\mu - r) \right] + S f_S(\lambda - \delta) + \frac{1}{2} f_{WW}(\pi^m)^2 \sigma_m^2 + \frac{1}{2} S^2 f_{SS} \sigma^2 \tag{44}
$$
$$
+ S f_{WS} \pi^m \rho \sigma \sigma_m + \alpha \left(V \left(W + n(S - K)^+, t \right) - f \right) = 0,
$$

where α is the hazard rate governing exogenous termination. As a check on our calculations, we reproduce the option values and expected option terms in Table 1 of Carr and Linetsky (2000), which assumes exercise and cancellation at an exogenous rate. Other papers that incorporate employment termination risk in option valuation include Jennergren and Näslund (1993), Carpenter (1998), Cvitanić, Wiener, and Zapatero (2008), and Leung and Sircar (2009).

In Section 6.1 above, which assumed no employment termination, we needed to set the executive's outside wealth very low, 0.6 times the value of shares under option, to bring the option's expected term down to the empirically observed average of 5 years. However, if we introduce employment termination at a hazard rate of 12% per year (consistent with Carpenter, Stanton, and Wallace (2009)), we obtain an expected term of 5 years in the base cases with a more realistic wealth level of 4 times the value of shares under option.

Table 2 illustrates termination rate effects, as well as other parameter effects in the presence of a 12% termination rate. Note that, for the same expected term of 5 years, option cost is lower with the positive termination rate, even though wealth and the voluntary exercise boundary are higher. In particular, option cost is now 0.38 in base case 1 and 0.22 in base case 2, compared with 0.45 in base case 1 and 0.29 in base case 2 in the last section with only voluntary, utility-maximizing exercise.

To understand the termination rate effects in Panel A of Table 2, note that, in a model

with only exogenous termination and no voluntary early exercise, the FASB approximation error will be positive because of the Jensen's inequality effect described above. It will also be hump-shaped with respect to the termination rate, because the variance of the stopping time goes to zero as either i. the termination rate goes to zero and so the stopping time converges to the option expiration date, or ii. the termination rate goes to infinity, in which case the stopping time converges to the vesting date. The variance of the stopping time, and thus the Jensen's inequality effect, is highest for intermediate termination rates. In Table 2, there is both exogenous termination and voluntary utility-maximizing early exercise. In Panel A, we see the hump-shaped termination rate effect in the FASB approximation error in base case 1. However, in base case 2, where the base case FASB error is negative (because the utility-maximizing policy is closer to value-maximizing), the effect of increasing dispersion and the effect of the stopping time converging to the vesting date, which pushes the approximation error to zero, go in the same direction, and the approximation is monotonically increasing to zero.

The beta and dividend rate effects on the approximation error in Panels B and E are qualitatively the same as in the baseline case in Table 1, though somewhat dampened by the presence of the exogenous termination. The approximation error is still generally monotonically increasing in volatility in base case 2, as it was in Table 1, but is a U-shaped function of volatility in base case 1, and close to zero.

In Table 1, FASB errors were monotonically increasing in volatility. This is also true in Table 2, at least for volatilities 30% and higher. However, the FASB errors *decrease* in Panel C (and, to a lesser extent, D) when volatility increases from 25% to 30%. This is related to the accompanying sharp drop in the option's expected term. When volatility is 25%, there is much more time for exogenous termination to have an effect, so this case looks more like the purely random polar case studied above, where the Jensen's inequality effect pushed the FASB approximation above the true value. As volatility increases from 25% to 30%, the expected term drops sharply, so this effect is reduced, causing the FASB error to fall. As volatility continues to rise, it causes much smaller changes in expected term, and the relation with volatility looks more like that in Table 1.

The effects shown here assume the executive follows the exercise policy of our model. Analyzing the FASB approximation errors that occur in practice requires knowing the actual exercise policies of executives, from which both correct option cost and expected option terms could be estimated. Overall, Tables 1 and 2 suggest that FASB approximation errors can be small or large, positive or negative, depending on the firm profile.

7 The subjective option value discount

The focus of the paper is on the cost of the option to the firm, that is, the present value of the option payoff from the viewpoint of market participants who can trade freely. However, we can also use our framework to study the subjective value of the option from the viewpoint of the executive who cannot trade the option. The subjective option value is the amount of freely investable money that would give the executive the same utility as the option, as defined in Equation (8).

As Bergman and Jenter (2007) note, as long as the executive is free to buy the option in the open market, either explicitly or synthetically, the option's subjective value cannot exceed its present value. If the executive were given an amount of cash equal to the option's present value, the executive could always buy the option in the open market, so the executive could get at least as much utility as with the option itself. Therefore, the executive would need no more than the present value of the option, in the form of freely investable cash, to be as well off as with the option. Violations of this inequality can arise if the executive is prevented from taking long stock or option positions in his outside portfolio, as Cai and Vijh (2005) show, but this inequality will certainly hold in the complete markets setting here.

In Table 3 we use our model to quantify subjective option values and their discounts from present value for the same parametrizations shown in Table 1. The discount is defined as one minus the ratio of subjective value to present value. Panel A shows that, like present values, subjective option values increase with beta or correlation, and the discount declines. Intuitively, better hedging opportunities narrow the gap as the executive can effectively monetize more and more of the option's value. Henderson (2007) and Leung and Sircar (2009) find similar effects in CARA models and Kahl, Liu, and Longstaff (2003) find a similar effect in the subjective value of restricted stock.

In Panel B, subjective option values are monotonically decreasing, and the discount is increasing, as the executive is exposed to more and more idiosyncratic risk. This is similar to the idiosyncratic risk effects in Kahl, Liu, and Longstaff (2003) and Henderson (2005). In Panel C, where correlation is held constant, subjective option value does not vary monotonically with volatility in the second base case. In this panel, in addition to the apparently negative effect of the increase in idiosyncratic risk, there is also the positive effect of an increase in market risk, which essentially increases the value of the tradable component of the option payoff. Henderson (2005) finds a similar effect with a

European option. In the second base case, this positive effect dominates at low volatility. The positive effect also operates on the exercise boundary and the option cost, however, and the subjective value discount remains increasing in volatility.

Panel D illustrates dividend effects. Proposition 2 established that increasing the dividend rate reduces the subjective option value. Panel D shows that in addition, the subjective discount is roughly constant at the low end of the dividend range. However, as the dividend grows large, both option cost and subjective value go to zero, and the discount appears to go to zero as the option becomes a trivial component of wealth.

The difference between the cost of the option to the firm and its subjective value to the employee is part of the cost of extracting better performance. It is the agency cost of the inefficient risk allocation necessary to elicit unobservable or noncontractible effort. In addition to this, there is the cost of compensating the executive for the extra effort, which must be paid even if the effort is contractible and the extra compensation can be paid in cash. The results in Table 3 suggest that the total cost of eliciting better performance may be quite high. On the other hand, evidence of positive stock price reactions to announcements of the adoption of an option plan in Brickley, Bhagat, and Lease (1985), DeFusco, Johnson, and Zorn (1990), Kato, Lemmon, Luo, and Schallheim (2005) and Langmann (2007) suggests that the market perceives the benefits to outweigh the costs.

8   The case with restricted stock

Since 2004, when the FASB issued a new standard requiring firms to recognize option cost in earnings, firms have increased their use of restricted stock as compensation. Chi and Johnson (2007) find that the share of compensation paid to executives in the form of restricted stock at S&P 500 firms rose monotonically from 4% in 1992 to 15% in 2005. Over the same period, the share of executive compensation paid in the form of options, as measured by Black-Scholes value, rose from 31% to 34%, with a peak of 65% in 2001.

To investigate the effects of restricted stock on option cost, we consider the case in which the executive's outside portfolio contains a fixed number of shares that vest on the option expiration date, in addition to wealth that the executive can trade dynamically in the market and riskless bonds. This does not change the Bellman equation for the problem, but it alters the boundary condition upon option exercise. As a check on our calculations, we reproduce the subjective option values in Table 1 of Lambert, Larcker, and Verrecchia (1991) by setting the correlation between the stock and the market to

zero, so that outside wealth is invested risklessly.

As Lambert, Larcker, and Verrecchia (1991) note, adding restricted stock can, in principle, have conflicting effects: a wealth effect that could increase option cost, and the effect of increasing the executive's exposure to stock risk, which could reduce option cost. We find that the second effect dominates, that is, adding restricted stock reduces option cost and average option life. Table 4 reports option cost and expected terms in the presence of restricted stock, using as a base case 0.3 shares of restricted stock and with outside wealth increased from 0.6 times the value of shares under option in the base case without restricted stock, to 0.95, in order to restore average option life to five years. The remaining base case parameter values match those of base case 1 in Tables 1 and 2.

Panel A of Table 4 shows that in the base case, option cost is 47% of the value of the underlying shares with no restricted stock, and falls to 39% as the restricted stock position grows large. The FASB approximation declines as the option average life falls with restricted stock, but not as fast as the option cost, so the approximation error increases. Option subjective value decreases with the size of the restricted stock holding and the subjective discount increases dramatically.

We find the remaining parameter effects are both qualitatively and quantitatively the same as in the case with no restricted stock. In fact, the numbers in Panels B through E of Table 4 are remarkably similar to those in Panels A through D of Tables 1 and 3, which assume no restricted stock but less outside wealth. We conclude that our previous results are robust to the inclusion of restricted stock.

In addition to holding restricted stock, executives typically have considerable investments of human capital in their firm. This firm-specific human capital represents another nondiversifiable component of the executive's total portfolio, which may be highly correlated with the firm's stock price. As such, like restricted stock, it should increase the executive's desire for diversification and thus promote earlier option exercise and lower option cost. This would further explain why our calibrated outside wealth level is so low.

9 Summary and conclusions

This paper advances the theory of executive stock option valuation with an in-depth study of the optimal exercise policy of a risk-averse executive and its implications for option cost. Many recent valuation models for executive stock options set the exercise policy exogenously, assuming a single stock price boundary. This paper provides a simple

example showing that the optimal exercise policy need not be of that form. However, we provide a sufficient condition for the existence of a single boundary when riskless bonds are the only investment available and the stock underlying the option appreciates at the riskless rate. This condition is satisfied by CRRA utility functions with risk aversion coefficient less than or equal to one, and we find no counterexamples among our numerical results for CRRA utility functions with risk aversion coefficient greater than one.

We also prove that, for general concave utility, the executive exercises later and option cost is greater when he has less absolute risk aversion, or more wealth combined with decreasing absolute risk aversion. The exercise region is also larger, and subjective option value to the executive is greater, the lower the dividend rate on the stock. Finally, if the interest rate is zero, the continuation region shrinks as time elapses. All these monotonicity results hold regardless of the exact shape of the continuation region.

Numerical examples with CRRA utility show how the exercise boundary and option cost vary with volatility. In contrast to results from standard option theory, or from executive stock option valuation models with a fixed exercise boundary, executive stock option cost can decline in stock return volatility when increases in volatility cause the optimal exercise boundary to drop sufficiently.

Next, we show numerically how exercise boundaries, option cost, and option average life vary with stock beta, volatility, correlation with the market, and dividend rates when trading outside wealth in the market is possible. The exercise boundary, option cost, and average life all increase with the magnitude of the correlation between the stock return and the market return. However, for CRRA utility, the option's exercise policy and cost to the firm do *not* converge to their maximized (Black-Scholes) values as the correlation between the stock and the market tends to one. When the market risk premium and stock beta are both nonnegative, increasing the stock beta increases option cost and average life. Option cost declines with the dividend payout rate, but average life actually increases because of the lower stock appreciation rate.

Finally, we examine the widely used approximation to option cost that is accepted by the FASB and find the approximation error can be small or large, positive or negative, depending on the firm's beta, volatility, and dividend rate. In addition, the subjective option value from the executive's viewpoint can be no greater than the option cost to shareholders, and we show how the subjective discount varies with firm profile. Our results suggest that the cost of providing performance incentives is high, and we conclude that the performance benefits must be quite significant.

A Numerical methods: one-factor model (Section 4)

In Section 4, the value function, $f(S, t)$ satisfies the one-factor partial differential equation

$$f_t + \frac{1}{2}\sigma^2 S^2 f_{SS} + (r - \delta)S f_S = 0. \tag{18}$$

To solve for f numerically, we first take a log transform of Equation (18).[9] Defining $s \equiv \log(S)$ and $g(s, t) \equiv f(S, t)$, the derivatives of f can be written as

$$
\begin{aligned}
f_t &= g_t, \\
f_S &= \frac{g_s}{S}, \\
f_{SS} &= \frac{g_{ss} - g_s}{S^2}.
\end{aligned}
$$

Substituting into Equation (18), we obtain

$$g_t + \frac{1}{2}\sigma^2 g_{ss} + \left(r - \delta - \frac{1}{2}\sigma^2\right) g_s = 0. \tag{45}$$

which we can write in the form

$$\frac{\partial g}{\partial t} + \mathcal{L}_s g = 0, \tag{46}$$

where the operator $\mathcal{L}_s$ is defined by

$$\mathcal{L}_s \equiv \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial s^2} + \left(r - \delta - \frac{1}{2}\sigma^2\right)\frac{\partial}{\partial s}.$$

We solve this p.d.e. numerically on a grid of s and t values using a finite-difference method, where the derivatives in Equation (45) are approximated using differences between neighboring values of g on the grid.[10] First, define the grid values as

$$g_{j,k} \equiv g(s_j, t_k) \equiv g(\underline{s} + j\,\Delta s, k\,\Delta t),$$

---

[9]This transformation simplifies numerical analysis by making the coefficients in the equation constant (see, for example, Brennan and Schwartz (1978), Geske and Shastri (1985), or Hull and White (1990)).

[10]Good general overviews of finite-difference methods include Ames (1992) and Lapidus and Pinder (1982); overviews focusing on asset-pricing applications include Tavella and Randall (2000) and Duffy (2006).

where $\underline{s}$ is the smallest value of s in the grid, and Δs and Δt are the grid spacings in the s and t directions respectively. Now approximate the time derivative, g_t, as

$$g_t \approx \frac{1}{\Delta t} \left(g_{j,k+1} - g_{j,k} \right), \tag{47}$$

and the s derivatives (at period k) via

$$g_{ss} \approx \frac{1}{\Delta s^2} \left(g_{j+1,k} - 2g_{j,k} + g_{j-1,k} \right), \tag{48}$$

$$g_s \approx \frac{1}{2\,\Delta s} \left(g_{j+1,k} - g_{j-1,k} \right), \tag{49}$$

and define the operator L_s, the finite-difference approximation to $\mathcal{L}_s$, as

$$L_s\, g_{j,k} \equiv \frac{\sigma^2}{2\,\Delta s^2} \left(g_{j+1,k} - 2g_{j,k} + g_{j-1,k} \right) + \frac{\left(r - \delta - \frac{1}{2}\sigma^2 \right)}{2\,\Delta s} \left(g_{j+1,k} - g_{j-1,k} \right). \tag{50}$$

Approximating $\mathcal{L}_s$ in Equation (46) using the (known) values at date $k+1$ leads to the *explicit* finite-difference approximation,

$$g_{j,k} = \left(1 + \Delta t\, L_s \right) g_{j,k+1}. \tag{51}$$

Alternatively, approximating $\mathcal{L}_s$ using the (as yet unknown) values at date k leads to the *implicit* finite-difference approximation,

$$\left(1 - \Delta t\, L_s \right) g_{j,k} = g_{j,k+1}. \tag{52}$$

As in Kahl et al. (2003), we use the implicit method, which has significant stability advantages (see, for example, Brennan and Schwartz (1978)). The "computational molecule" for this method is shown in Figure 6. This system of equations can be written (after imposing upper and lower boundary conditions at $j = 0$ and $j = n_j$) in matrix form as

$$\mathbf{V} g_k = g_{k+1}, \tag{53}$$

where $\mathbf{V}$ is tridiagonal. Such equations can be solved very efficiently (see, for example, Ames (1992)). Starting with the terminal values,

$$g_{j,n_k} = U \left(n(S_j - K)^+ + W \right),$$

we now calculate the value function for every s and t value, simultaneously determining the optimal exercise policy, much as we value an American option. For each date in turn, $k = n_k - 1, n_k - 2, \ldots, 0$,

1. Solve Equation (53) to calculate the value of g at date k, assuming no exercise.
2. For each S_j, also calculate the value function conditional on exercising,

$$U\left(n(S_j - K)^+ e^{r(T-t_k)} + W\right).$$

3. Replace g with the exercise value of the option whenever this is higher.

B   Numerical methods: two-factor model (Section 5)

In Section 5, the value function, $f(W, S, t)$, satisfies the Hamilton-Jacobi-Bellman equation,

$$\max_{\pi^m} \left[f_t + f_W \left[rW + \pi^m(\mu - r)\right] + S f_S(\lambda - \delta) + \frac{1}{2} f_{WW}(\pi^m)^2 \sigma_m^2 \right. \tag{35}$$
$$\left. + \frac{1}{2} S^2 f_{SS} \sigma^2 + S f_{WS} \pi^m \rho \sigma \sigma_m \right] = 0.$$

As with the one-factor model above, we solve this problem using a finite-difference method, similar to Brennan, Schwartz, and Lagnado (1997), and especially Kahl, Liu, and Longstaff (2003). Again, we start by taking a log transform. Letting $s \equiv \log(S), w \equiv \log(W), \theta \equiv \pi_m/W$, and $g(w, s, t) \equiv f(W, S, t)$, Equation (35) becomes[11]

$$\max_{\theta} \left[g_t + g_w \left[r + \theta(\mu - r) - 1/2\,\theta^2 \sigma_m^2\right] + g_s(\lambda - \delta - \frac{1}{2}\sigma^2) \right. \tag{54}$$
$$\left. + \frac{1}{2} g_{ww} \theta^2 \sigma_m^2 + \frac{1}{2} g_{ss} \sigma^2 + g_{ws} \theta \rho \sigma \sigma_m \right] = 0.$$

---

[11]Note that the investment *fraction*, θ, is well-defined because an investor with CRRA utility will never allow outside wealth to go to zero.

Assuming $f_{WW} < 0$ (or, equivalently, that $g_{ww} - g_w < 0$), the optimal portfolio position θ is obtained by solving the first order condition for θ:[12]

$$\theta^* = \frac{-(\mu - r)f_W - S\rho\sigma\sigma_m f_{WS}}{W\sigma_m^2 f_{WW}} = \frac{-(\mu - r)g_w - \rho\sigma\sigma_m g_{ws}}{\sigma_m^2 \left(g_{ww} - g_w\right)}. \tag{55}$$

Substituting back into Equation (54), we obtain the (nonlinear) p.d.e.

$$g_t + g_w \left[r + \theta^*(\mu - r) - 1/2\,\theta^{*2}\sigma_m^2\right] + g_s(\lambda - \delta - \frac{1}{2}\sigma^2) \tag{56}$$

$$+\frac{1}{2}g_{ww}\theta^{*2}\sigma_m^2 + \frac{1}{2}g_{ss}\sigma^2 + g_{ws}\theta^*\rho\sigma\sigma_m \quad = \quad 0.$$

Rewrite this, analogous to Equation (46) above, in the form

$$\frac{\partial g}{\partial t} + (\mathcal{L}_w + \mathcal{L}_s + \mathcal{L}_{ws})\,g = 0, \tag{57}$$

where

$$\begin{aligned}
\mathcal{L}_w &\equiv \frac{1}{2}\theta^{*2}\sigma_m^2\frac{\partial^2}{\partial w^2} + \left[r + \theta^*(\mu - r) - 1/2\,\theta^{*2}\sigma_m^2\right]\frac{\partial}{\partial w}, \\
\mathcal{L}_s &\equiv \frac{1}{2}\sigma^2\frac{\partial^2}{\partial s^2} + (\lambda - \delta - \frac{1}{2}\sigma^2)\frac{\partial}{\partial s}, \\
\mathcal{L}_{ws} &\equiv \theta^*\rho\sigma\sigma_m\frac{\partial^2}{\partial w\,\partial s}.
\end{aligned}$$

To solve this equation using finite-difference methods, first define the values of g on a grid of w, s and t values, analogous to Appendix A above, as

$$g_{i,j,k} \equiv g(w_i, s_j, t_k) \equiv g(\underline{w} + i\,\Delta w, \underline{s} + j\,\Delta s, k\,\Delta t).$$

Following Kahl et al. (2003), we linearize Equation (57) by evaluating θ^* at each time t and state (w, s) using the estimated values of the function g and its derivatives at time $t + \Delta t$. As in the one-factor case above, we approximate $\mathcal{L}_w$ and $\mathcal{L}_s$ using finite-difference

---

[12]For some parameter values, at very small wealth levels the value function, f, is *convex* in W, a possibility noted in Henderson and Hobson (2008). In this case, the solution given by Equation (55) is a *minimum*, rather than a maximum; the true solution is unbounded (Henderson and Hobson (2008) show that it is actually optimal for the agent to take an instantaneous large gamble in such a state), so we set $\theta^* = \pm\theta_{max}$, for some (finite) value θ_{max}, the sign being the opposite of the right hand side of Equation (55). In practice, we obtain virtually the same values this way as when we use Equation (55) regardless of the sign of f_{WW}, suggesting that this potential nonconcavity is not playing a big role.

operators, L_w and L_s defined analogously to Equation (50) above, and approximating $\mathcal{L}_{ws}$ using the operator

$$L_{ws}\, g_{i,j,k} \equiv \frac{\theta^* \rho \sigma \sigma_m}{4\,\Delta w\,\Delta s} \left(g_{i+1,j+1,k} - g_{i+1,j-1,k} - g_{i-1,j+1,k} + g_{i-1,j-1,k} \right). \tag{58}$$

This leads to a finite-difference approximation to Equation (57), analogous to the one-factor Equation (52),

$$(1 - \Delta t\,[L_w + L_s])\, g_{i,j,k} = (1 + \Delta t\, L_{ws})\, g_{i,j,k+1}, \tag{59}$$

where we have used implicit approximations to $\mathcal{L}_w$ and $\mathcal{L}_s$ on the left, and an *explicit* approximation to $\mathcal{L}_{ws}$ on the right. Writing this as a matrix equation relating g_k to g_{k+1}, as we did in the one-factor case, we obtain an equation which is sparse, but no longer tridiagonal, so it cannot easily be solved directly. Brennan et al. (1997) use successive over-relaxation to solve their version of this equation iteratively (see Tavella and Randall (2000) for a description). However, this technique is known to converge slowly as the number of grid points becomes large. Instead, we use a splitting method to turn the two-dimensional problem into a sequence of one-dimensional problems, each of which can be solved as in the one-factor case above.[13] This method involves approximating the left hand side of Equation (59) as

$$(1 - \Delta t\,[L_w + L_s])\, g_{i,j,k} \approx (1 - \Delta t\, L_w)(1 - \Delta t\, L_s)\, g_{i,j,k}. \tag{60}$$

This approximation introduces an error of order Δt^2, the same as that of the finite-difference approximation to g_t in Equation (47), so the overall order of the error is unchanged when we substitute into Equation (59) to obtain

$$(1 - \Delta t\, L_w)(1 - \Delta t\, L_s)\, g_{i,j,k} = (1 + \Delta t\, L_{ws})\, g_{i,j,k+1}. \tag{61}$$

This can be solved in two one-dimensional steps. Defining the fictitious "intermediate"

---

[13]Splitting methods, also known as "locally one dimensional" (LOD) methods, or the "method of fractional steps," were primarily developed in Russia in the 1960s (see, in particular, Yanenko (1971), Marchuk (1990), and, for a discussion of their use in financial applications, Duffy (2006)). They are closely related to Alternating Direction Implicit (ADI) methods (see, for example, Douglas and Rachford (1956) and Peaceman and Rachford (1955)). Note that handling the mixed derivative explicitly does not interfere with stability, unlike with classical ADI (see Yanenko (1971)).

value,

$$g_{i,j,k+\frac{1}{2}} \equiv (1 - \Delta t\, L_s)\, g_{i,j,k},$$

first solve, for each $j = 0, 1, \ldots, n_j$, the (tridiagonal) system of equations

$$(1 - \Delta t\, L_w)\, g_{i,j,k+\frac{1}{2}} = (1 + \Delta t\, L_{ws})\, g_{i,j,k+1}, \tag{62}$$

to obtain the intermediate values, $g_{i,j,k+\frac{1}{2}}$. Next, for each $k = 0, 1, \ldots, n_k$, solve the tridiagonal system of equations,

$$(1 - \Delta t\, L_s)\, g_{i,j,k} = g_{i,j,k+\frac{1}{2}}, \tag{63}$$

to calculate $g_{i,j,k}$. The mechanics of solving for g are now very similar to the one-factor case. Starting with the terminal values, $g_{i,j,n_k} = U\left(n(S_j - K)^+ + W_i\right)$, for each date in turn, $k = n_k - 1, n_k - 2, \ldots, 0$,

1. Solve Equations (62) and (63) to calculate the value of g at date k, assuming no exercise.

2. For each W_i and S_j, also calculate the value function conditional on exercising, which standard dynamic programming methods (see, for example, Merton (1969)) show to be given by[14]

$$
\begin{aligned}
V(W, t_k) &= \frac{e^{(1-A)\left(r + \frac{(\mu - r)^2}{2A\sigma_m^2}\right)(T - t_k)} W^{1-A}}{1 - A}, \qquad \text{where} \tag{64} \\
W &= n(S_j - K)^+ + W_i.
\end{aligned}
$$

3. Replace g with the exercise value of the option whenever this is higher.

Robustness Although our solution technique is similar to methods already in the literature, including Brennan et al. (1997) and Kahl et al. (2003), we performed various robustness checks to assure ourselves that the results obtained were reasonable. In particular: (1) We also calculated values using an explicit finite-difference algorithm, which does not require the linearization step because all derivatives with respect to w and s are evaluated at (known) date $t + \Delta t$. The results are virtually identical to those obtained using the implicit method; (2) In the case where the number of options equals zero ($n = 0$),

---

[14]In practice, we solve for V numerically using the same grid as for g, to ensure that any biases introduced by our numerical procedure affect both values roughly equally.

our numerical method gives results for both the value function and the optimal investment amount that are very close to their true, closed-form, values (see Equation (64)); (3) Using the same method to value a European call option, the value converges nicely to the Black and Scholes (1973) value; (4) We experimented with different numbers of w, s and t values in the grid, with little effect on our results; (5) We tried various boundary conditions, including imposing linearity at the boundary (see Tavella and Randall (2000)), and imposing a quadratic functional form at the boundary, with little effect on the solution. Intuitively (see Tavella (2002, page 237)), as long as the boundaries are a long way from the region in the middle of the grid that we are interested in, the exact choice of boundary condition is not very important.

References

Ames, W., 1992. Numerical Methods for Partial Differential Equations. Academic Press, Boston, MA, third ed.

Bergman, N., Jenter, D., 2007. Employee sentiment and stock option compensation. Journal of Financial Economics 84, 667–712.

Bettis, J., Bizjak, J., Lemmon, M., 2001. Managerial ownership, incentive contracting, and the use of zero-cost collars and equity swaps by corporate insiders. Journal of Financial and Quantitative Analysis 36, 345–370.

Bettis, J., Bizjak, J., Lemmon, M., 2005. Exercise behavior, valuation, and the incentive effects of employee stock options. Journal of Financial Economics 76, 445–470.

Black, F., Scholes, M., 1973. The pricing of options and corporate liabilities. Journal of Political Economy 81, 637–659.

Brennan, M., Schwartz, E., 1978. Finite difference methods and jump processes arising in the pricing of contingent claims: A synthesis. Journal of Financial and Quantitative Analysis 13, 461–474.

Brennan, M., Schwartz, E., Lagnado, R., 1997. Strategic asset allocation. Journal of Economic Dynamics and Control 21, 1377–1403.

Brickley, J., Bhagat, S., Lease, R., 1985. The impact of long-range managerial compensation plans on shareholder wealth. Journal of Accounting and Economics 7, 115–129.

Cai, J., Vijh, A., 2005. Executive stock and option valuation in a two state-variable framework. Journal of Derivatives Spring, 9–27.

Carpenter, J., Stanton, R., Wallace, N., 2009. Estimation of employee stock option exercise rates and firm cost. Unpublished working paper. U.C. Berkeley.

Carpenter, J., 1998. The exercise and valuation of executive stock options. Journal of Financial Economics 48, 127–158.

Carr, P., Jarrow, R., Myneni, R., 1992. Alternative characterizations of American put options. Mathematical Finance 2, 87–106.

Carr, P., Linetsky, V., 2000. The valuation of executive stock options in an intensity based framework. European Financial Review 4, 211–230.

Chi, J., Johnson, S., 2007. Restricted stock versus other sources of managerial incentives. Unpublished working paper. Arizona State University.

Cox, J., Huang, C., 1989. Optimal consumption and portfolio policies when asset prices follow a diffusion process. Journal of Economic Theory 49, 33–83.

Cuny, C., Jorion, P., 1995. Valuing executive stock options with an endogenous departure decision. Journal of Accounting and Economics 20, 193–205.

Cvitanić, J., Wiener, Z., Zapatero, F., 2008. Analytic pricing of executive stock options. Review of Financial Studies 21, 683–724.

DeFusco, R., Johnson, R., Zorn, T., 1990. The effect of executive stock options on stockholders and bondholders. Journal of Finance 45, 617–627.

Detemple, J., Sundaresan, S., 1999. Nontraded asset valuation with portfolio constraints: A binomial approach. Review of Financial Studies 12, 835–872.

Douglas, Jr., J., Rachford, H., 1956. On the numerical solution of heat conduction problems in two and three space variables. Transactions of the American Mathematical Society 82, 421–439.

Duffy, D., 2006. Finite Difference Methods in Financial Engineering: A Partial Differential Equation Approach. John Wiley, Sussex, England.

Equilar, 2007. 2007 Equity report: An analysis of equity compensation practices at Fortune 1000 companies.

Financial Accounting Standards Board, 2004. Statement of Financial Accounting Standards no. 123 (revised 2004): Share based payment. Financial Accounting Series 263-C, 1–286.

Frydman, C., Saks, R., 2007. Executive compensation: A new view from a long-term perspective. Unpublished working paper. Massachusetts Institute of Technology. forthcoming *Review of Financial Studies.*

Geske, R., 1979. A note on an analytic valuation formula for unprotected American call options on stocks with known dividends. Journal of Financial Economics 7, 375–380.

Geske, R., Shastri, K., 1985. Valuation by approximation: A comparison of alternative option valuation approaches. Journal of Financial and Quantitative Analysis 20, 45–72.

Grasselli, M., Henderson, V., 2009. Risk aversion and block exercise of executive stock options. Journal of Economic Dynamics and Control 33, 109–127.

Hall, B., Murphy, K., 2002. Stock options for undiversified executives. Journal of Accounting and Economics 33, 3–42.

Henderson, V., 2005. The impact of the market portfolio on the valuation, incentives, and optimality of executive stock options. Quantitative Finance 5, 35–47.

Henderson, V., 2007. Valuing the option to invest in an incomplete market. Mathematics and Financial Economics 1, 103–128.

Henderson, V., Hobson, D., 2008. The curious incident of the agent and the gamble. Unpublished working paper. Oxford University.

Huddart, S., 1994. Employee stock options. Journal of Accounting and Economics 18, 207–231.

Huddart, S., Lang, M., 1996. Employee stock option exercises: An empirical analysis. Journal of Accounting and Economics 21, 5–43.

Hull, J., White, A., 1990. Valuing derivative securities using the explicit finite difference method. Journal of Financial and Quantitative Analysis 25, 87–100.

Hull, J., White, A., 2004. How to value employee stock options. Financial Analysts Journal 60, 114–119.

Ingersoll, J., 1987. Theory of Financial Decision Making. Rowman & Littlefield, Totowa, New Jersey.

Ingersoll, J., 2006. The subjective and objective evaluation of incentive stock options. Journal of Business 79, 453–487.

Jain, A., Subramanian, A., 2004. The intertemporal exercise and valuation of employee options. The Accounting Review 79, 705–743.

Jennergren, L., Näslund, B., 1993. A comment on "Valuation of executive stock options and the FASB proposal". The Accounting Review 68, 179–183.

Jenter, D., 2002. Executive compensation, incentives, and risk. Unpublished working paper. Stanford University.

Kadam, A., Lakner, P., Srinivasan, A., 2003. Executive stock options: Value to the executive and cost to the firm. Unpublished working paper. New York University.

Kahl, M., Liu, J., Longstaff, F., 2003. Paper millionaires: How valuable is stock to a stockholder who is restricted from selling it? Journal of Financial Economics 67, 385–410.

Karatzas, I., Lehoczky, J., Shreve, S., 1987. Optimal portfolio and consumption decisions for a "small investor" on a finite horizon. SIAM Journal on Control and Optimization 25, 1557–1586.

Karatzas, I., Shreve, S., 1991. Brownian Motion and Stochastic Calculus. Springer-Verlag, New York, NY.

Kato, H., Lemmon, M., Luo, M., Schallheim, J., 2005. An empirical examination of the costs and benefits of executive stock options: Evidence from Japan. Journal of Financial Economics 78, 435–461.

Kim, I., 1990. The analytic valuation of American options. Review of Financial Studies 3, 547–572.

Lambert, R., Larcker, D., Verrecchia, R., 1991. Portfolio considerations in valuing executive compensation. Journal of Accounting Research 29, 129–149.

Langmann, C., 2007. Stock market reaction and stock option plans: Evidence from Germany. Schmalenbach Business Review 59, 85–106.

Lapidus, L., Pinder, G., 1982. Numerical Solution of Partial Differential Equations in Science and Engineering. John Wiley and Sons, New York, NY.

Leung, T., Sircar, R., 2009. Accounting for risk aversion, vesting, job termination risk and multiple exercises in valuation of employee stock options. Mathematical Finance 19, 99–128.

Marchuk, G., 1990. Splitting and alternating direction methods. In: Ciarlet, P., Lions, J. (eds.), *Handbook of Numerical Analysis*, North-Holland, Amsterdam, vol. 1, pp. 197–462.

Marcus, A., Kulatilaka, N., 1994. Valuing employee stock options. Financial Analysts Journal 50, 46–56.

Merton, R., 1969. Lifetime portfolio selection under uncertainty: The continuous-time case. The Review of Economics and Statistics 51, 247–257.

Merton, R., 1971. Optimum consumption and portfolio rules in a continuous-time model. The Review of Economics and Statistics 3, 373–413.

Merton, R., 1973. Theory of rational option pricing. Bell Journal of Economics and Management Science 4, 141–183.

Miao, J., Wang, N., 2007. Investment, consumption and hedging under incomplete markets. Journal of Financial Economics 86, 608–642.

Peaceman, D., Rachford, Jr., H., 1955. The numerical solution of parabolic and elliptic differential equations. Journal of the Society for Industrial and Applied Mathematics 3, 28–41.

Rogers, L., Scheinkman, J., 2007. Optimal exercise of executive stock options. Finance and Stochastics 11, 357–372.

Roll, R., 1977. An analytic valuation formula for unprotected American call options on stocks with known dividends. Journal of Financial Economics 5, 251–258.

Ross, S., 2004. Compensation, incentives, and the duality of risk aversion and riskiness. Journal of Finance 59, 207–225.

Rubinstein, M., 1995. On the accounting valuation of employee stock options. Journal of Derivatives 3, 8–24.

Tavella, D., Randall, C., 2000. Pricing Financial Instruments: The Finite Difference Method. John Wiley, New York, NY.

Tavella, D., 2002. Quantitative Methods in Derivatives Pricing: An Introduction to Computational Finance. John Wiley, New York, NY.

Van Moerbeke, P., 1976. On optimal stopping and free boundary problems. Archives of Rational Mechanical Analysis 60, 101–148.

Whaley, R., 1981. On the valuation of American call options on stocks with known dividends. Journal of Financial Economics 9, 207–211.

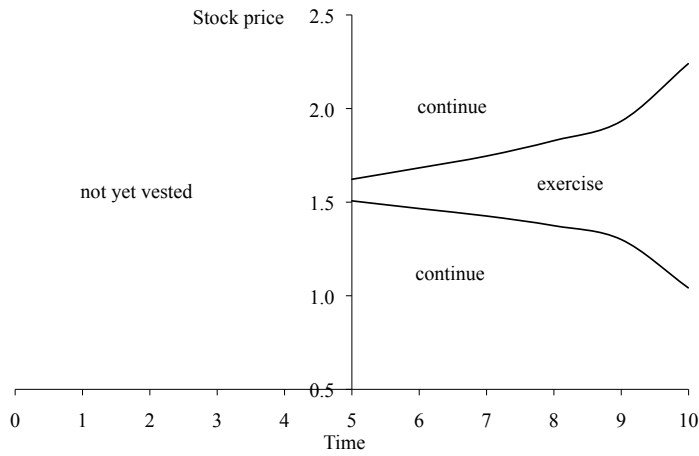Yanenko, N., 1971. The Method of Fractional Steps. Springer-Verlag, Berlin.

Fig. 1. Exercise policy with split continuation region. Executive has utility U(W)=W^(1-A)/(1-A)+cW with A=10 and c = 0.0001, and initial wealth 1.2. Executive follows optimal exercise policy and invests outside wealth risklessly. Option is at the money, vests in 5 years, and expires in 10 years. Riskless rate is 5%. Stock has zero risk premium, zero dividend rate, and 30% volatility. Initial stock price is 1.
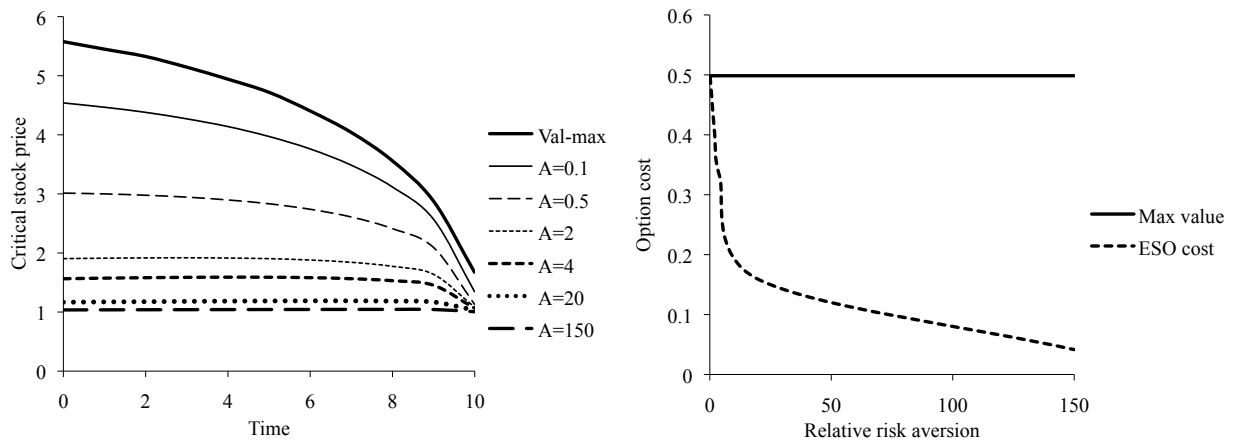
Fig. 2. Exercise boundaries and option cost for various levels of risk aversion. Executive has CRRA utility, initial wealth 1.2, follows optimal option exercise policy, and invests outside wealth risklessly. Option is at the money, vested, and expires in 10 years. Riskless rate is 5%. Stock has zero risk premium, 50% volatility, and 3% dividend rate. Initial stock price is 1.
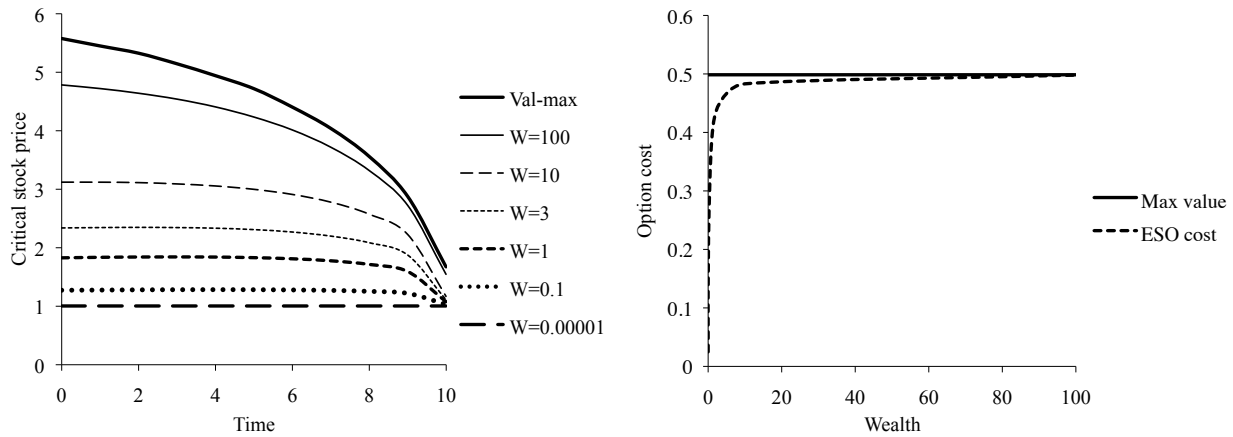
Fig. 3. Exercise boundaries and option cost for various levels of initial wealth. Executive has CRRA coefficient 2, follows optimal option exercise policy, and invests outside wealth risklessly. Option is at the money, vested, and expires in 10 years. Riskless rate is 5%. Stock has zero risk premium, 50% volatility, and 3% dividend rate. Initial stock price is 1.
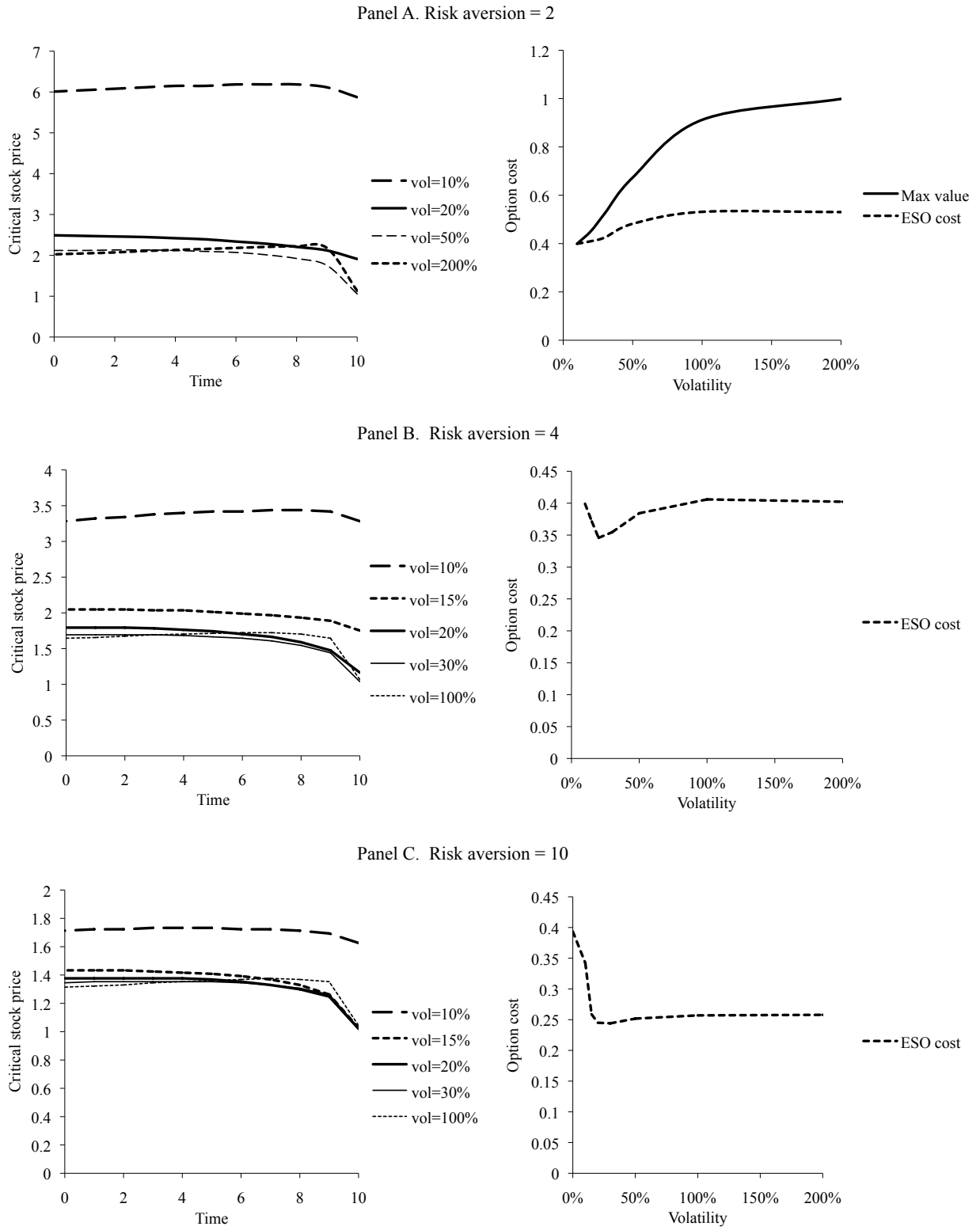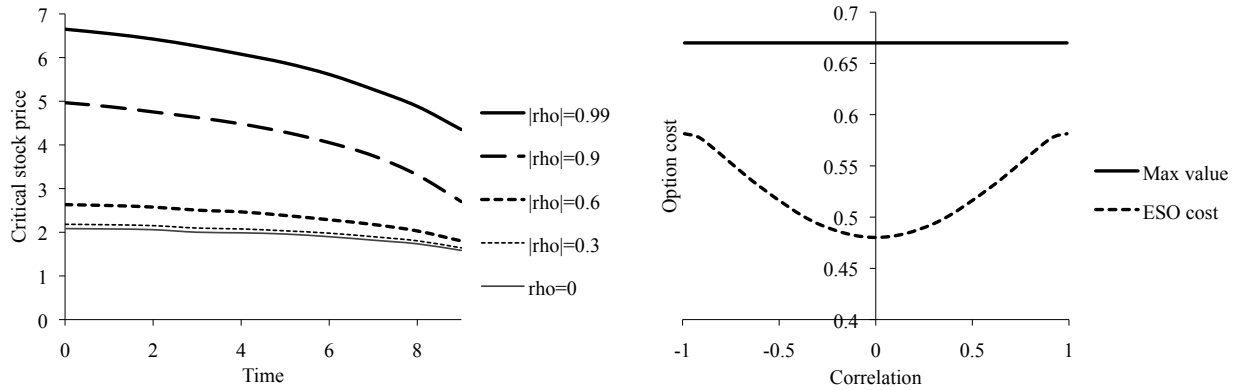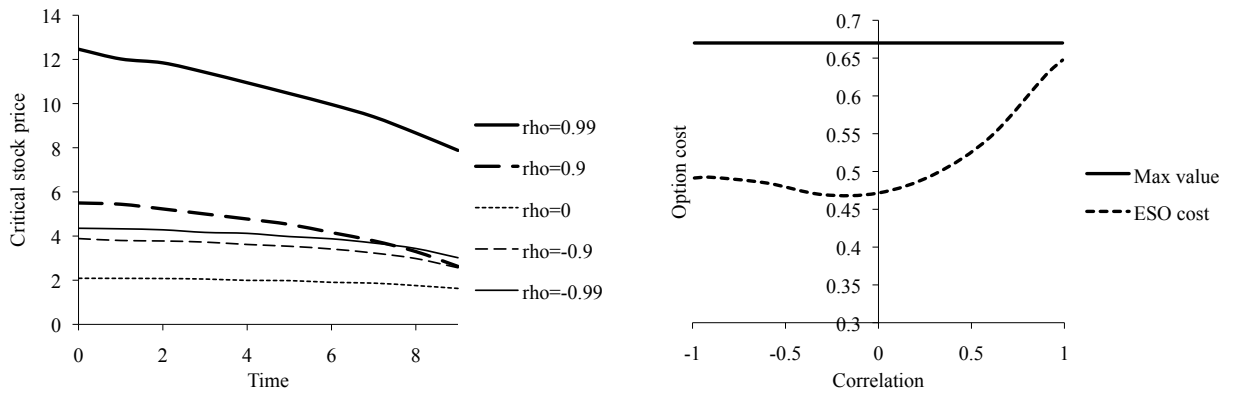
46

Fig. 4. Exercise boundaries and option cost for various levels of stock volatility. Executive has CRRA utility, initial wealth 1.2, follows optimal option exercise policy, and invests outside wealth risklessly. Option is at the money, vested, and expires in 10 years. Riskless rate is 5%. Stock has zero risk premium, 50% volatility, and 3% dividend rate. Initial stock price is 1.

Panel A. Market premium = 0

Panel B. Market risk premium = 8%

Panel C. With costly short sales

Market Risk Premium = 0
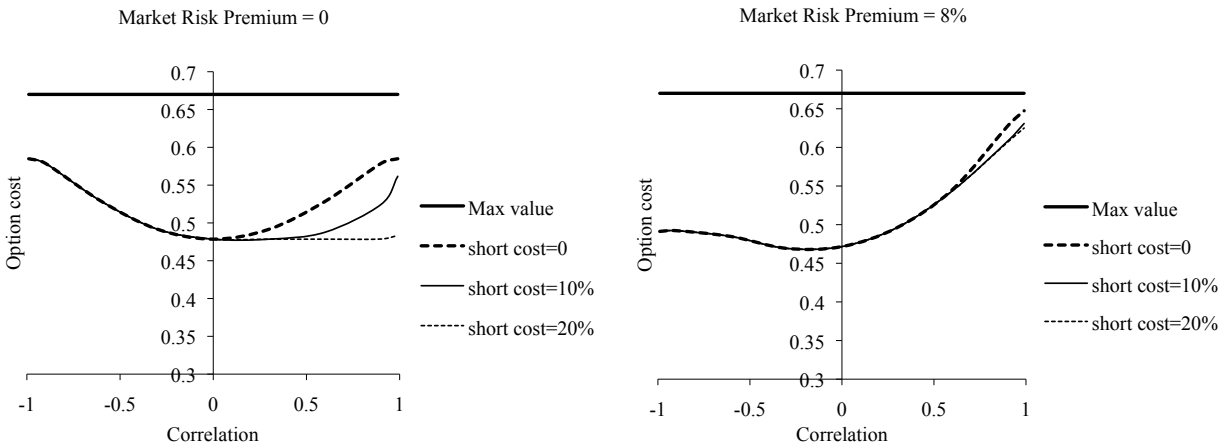
Market Risk Premium = 8%

Fig. 5. Exercise boundaries and option cost for various levels of correlation between stock return and market return. Executive has CRRA coefficient 2, initial wealth 1.2, follows optimal option exercise policy, and invests outside wealth optimally in market and riskless bond. Option is at the money, vested, and expires in ten years. Riskless rate is 5%. Stock has 50% volatility, zero dividend, and normal expected return given its correlation with market. Initial stock price is 1.
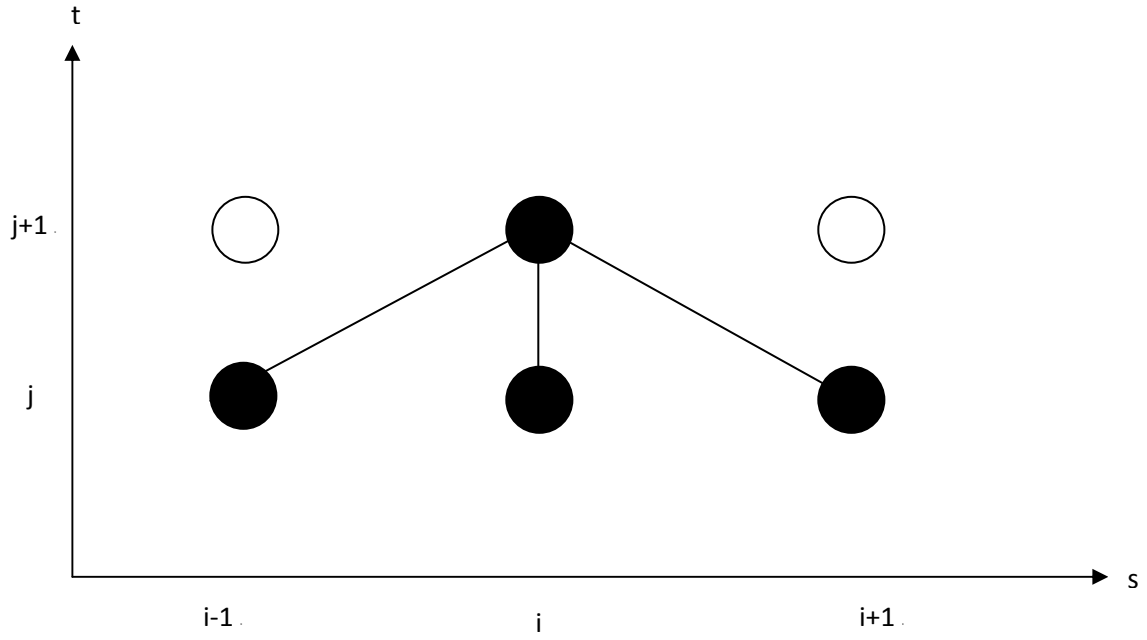
Figure 6: Implicit computational molecule. Solid circles show values used in calculation.

Table 1

Option values and FASB approximations

Executive follows optimal option exercise policy and optimally invests outside wealth in market portfolio. Executive has CRRA coefficient 4 and initial wealth 0.6 times value of shares under option. Option is at the money, vests in 2 years, and expires in 10 years. Risk free rate is 5%. Market portfolio return has mean 13% and volatility 20%. Initial stock price is 1.

Changing Parameter	Base Case 1: Vol=50%, Div=0, Beta=1.2						Base Case 2: Vol=30%, Div=3%, Beta=0.9					
	Expected Term	ESO Cost	FASB Approx	Approx Error	Percent Error	Max Value	Expected Term	ESO Cost	FASB Approx	Approx Error	Percent Error	Max Value
Panel A: Beta Effects												
0.0	5.98	0.425	0.539	0.114	27%	0.665	6.12	0.263	0.278	0.014	5%	0.341
0.5	5.51	0.430	0.519	0.089	21%	0.665	5.39	0.271	0.266	-0.005	-2%	0.341
0.9	5.19	0.440	0.505	0.066	15%	0.665	5.05	0.287	0.260	-0.027	-9%	0.341
1.2	5.02	0.451	0.497	0.046	10%	0.665	5.19	0.309	0.262	-0.047	-15%	0.341
1.4	4.94	0.460	0.493	0.033	7%	0.665	5.82	0.332	0.273	-0.059	-18%	0.341
Panel B: Volatility Effects Holding Beta Constant												
25%	7.71	0.467	0.420	-0.047	-10%	0.488	5.14	0.265	0.227	-0.038	-14%	0.298
30%	5.38	0.426	0.375	-0.051	-12%	0.526	5.05	0.287	0.260	-0.027	-9%	0.341
40%	4.85	0.421	0.422	0.001	0%	0.602	5.21	0.333	0.330	-0.003	-1%	0.425
50%	5.02	0.450	0.497	0.047	11%	0.665	5.45	0.378	0.400	0.022	6%	0.502
60%	5.30	0.486	0.574	0.088	18%	0.737	5.71	0.422	0.466	0.044	10%	0.571
Panel C: Volatility Effects Holding Correlation Constant												
25%	5.12	0.339	0.330	-0.010	-3%	0.488	5.16	0.257	0.227	-0.030	-12%	0.298
30%	4.97	0.359	0.358	0.000	0%	0.526	5.05	0.287	0.260	-0.027	-9%	0.341
40%	4.93	0.404	0.426	0.022	5%	0.602	5.05	0.346	0.327	-0.019	-5%	0.425
50%	5.02	0.450	0.497	0.047	11%	0.665	5.12	0.398	0.392	-0.006	-1%	0.502
60%	5.18	0.494	0.568	0.075	15%	0.737	5.24	0.446	0.456	0.009	2%	0.571
Panel D: Dividend Rate Effects												
0%	5.02	0.451	0.497	0.046	10%	0.665	4.95	0.376	0.357	-0.019	-5%	0.524
3%	5.25	0.388	0.396	0.008	2%	0.503	5.05	0.287	0.260	-0.027	-9%	0.341
10%	5.85	0.271	0.209	-0.062	-23%	0.309	5.72	0.156	0.105	-0.051	-33%	0.165
50%	9.04	0.024	0.000	-0.024	-100%	0.024	9.78	0.002	0.000	-0.002	-100%	0.002
90%	9.93	0.001	0.000	-0.001	-100%	0.001	10.00	0.000	0.000	0.000		0.000
Panel E: Wealth Effects												
0.006	3.68	0.327	0.427	0.101	31%	0.665	3.29	0.188	0.218	0.031	16%	0.341
0.06	3.95	0.355	0.443	0.087	25%	0.665	3.68	0.217	0.229	0.012	6%	0.341
0.6	5.02	0.451	0.497	0.046	10%	0.665	5.05	0.287	0.260	-0.027	-9%	0.341
6	7.05	0.581	0.581	0.000	0%	0.665	6.95	0.334	0.289	-0.045	-13%	0.341
60	8.92	0.654	0.643	-0.011	-2%	0.665	7.76	0.341	0.299	-0.042	-12%	0.341
600	9.34	0.665	0.655	-0.010	-2%	0.665	7.88	0.341	0.300	-0.041	-12%	0.341

Table 2
Option values and FASB approximations with exogenous employment termination

Executive follows optimal option exercise policy in the presence of an exogenous employment termination rate and optimally invests outside wealth in market portfolio. Executive has CRRA coefficient 4 and initial wealth 4 times value of shares under option. Option is at the money, vests in 2 years, and expires in 10 years. Risk free rate is 5%. Market portfolio return has mean 13% and volatility 20%. Initial stock price is 1.

Changing Parameter	Base Case 1: Vol=50%, Div=0, Beta=1.2, Term. rate=12%						Base Case 2: Vol=30%, Div=3%, Beta=0.9, Term. rate=12%					
	Exp. Term Given Vest	ESO Cost	FASB Approx	Approx Error	Percent Error	Max Value	Exp. Term Given Vest	ESO Cost	FASB Approx	Approx Error	Percent Error	Max Value
Panel A: Termination Rate Effects												
0%	6.65	0.558	0.566	0.008	1%	0.665	6.66	0.330	0.286	-0.044	-13%	0.341
6%	5.70	0.459	0.468	0.009	2%	0.665	5.73	0.269	0.241	-0.028	-11%	0.341
12%	5.01	0.382	0.390	0.008	2%	0.665	5.05	0.223	0.204	-0.019	-9%	0.341
18%	4.49	0.323	0.329	0.006	2%	0.665	4.54	0.188	0.174	-0.014	-7%	0.341
50%	3.20	0.148	0.146	-0.002	-1%	0.665	3.22	0.084	0.080	-0.004	-5%	0.341
Panel B: Beta Effects												
0.0	5.54	0.371	0.409	0.039	10%	0.665	5.68	0.218	0.213	-0.005	-2%	0.341
0.5	5.28	0.373	0.400	0.027	7%	0.665	5.27	0.220	0.207	-0.012	-6%	0.341
0.9	5.11	0.378	0.394	0.016	4%	0.665	5.05	0.223	0.204	-0.019	-9%	0.341
1.2	5.01	0.382	0.390	0.008	2%	0.665	5.02	0.227	0.204	-0.023	-10%	0.341
1.4	4.96	0.386	0.388	0.002	1%	0.665	5.19	0.229	0.206	-0.023	-10%	0.341
Panel C: Volatility Effects Holding Beta Constant												
25%	7.06	0.307	0.314	0.007	2%	0.488	5.13	0.197	0.178	-0.019	-10%	0.298
30%	5.95	0.323	0.312	-0.011	-3%	0.526	5.05	0.223	0.204	-0.019	-9%	0.341
40%	5.12	0.349	0.341	-0.008	-2%	0.602	5.06	0.273	0.257	-0.016	-6%	0.425
50%	5.01	0.382	0.391	0.008	2%	0.665	5.12	0.318	0.309	-0.009	-3%	0.502
60%	5.05	0.416	0.442	0.026	6%	0.737	5.20	0.358	0.358	0.000	0%	0.571
Panel D. Volatility Effects Holding Correlation Constant												
25%	6.04	0.288	0.286	-0.001	0%	0.488	5.16	0.195	0.179	-0.017	-9%	0.298
30%	5.57	0.304	0.301	-0.003	-1%	0.526	5.05	0.223	0.204	-0.019	-9%	0.341
40%	5.15	0.343	0.343	0.000	0%	0.602	4.94	0.276	0.255	-0.021	-8%	0.425
50%	5.01	0.382	0.391	0.008	2%	0.665	4.91	0.324	0.305	-0.020	-6%	0.502
60%	4.98	0.420	0.440	0.020	5%	0.737	4.92	0.367	0.352	-0.015	-4%	0.571
Panel E: Dividend Rate Effects												
0%	5.01	0.382	0.390	0.008	2%	0.665	5.60	0.310	0.302	-0.008	-3%	0.524
3%	5.00	0.321	0.307	-0.015	-5%	0.503	5.05	0.223	0.204	-0.019	-9%	0.341
10%	5.11	0.216	0.168	-0.049	-22%	0.309	4.87	0.118	0.085	-0.033	-28%	0.165
50%	6.61	0.018	0.000	-0.018	-97%	0.024	7.00	0.001	0.000	-0.001	-100%	0.002
90%	7.10	0.001	0.000	-0.001	-100%	0.001	7.14	0.000	0.000	0.000	-100%	0.000

Table 3

Subjective option values and discount from firm cost

Executive follows optimal option exercise policy and optimally invests outside wealth in market portfolio. Executive has CRRA coefficient 4 and initial wealth 0.6 times value of shares under option. Option is at the money, vests in 2 years, and expires in 10 years. Risk free rate is 5%. Market portfolio return has mean 13% and volatility 20%. Initial stock price is 1.

Changing Parameter	Base Case 1: Vol=50%, Div=0, Beta=1.2				Base Case 2: Vol=30%, Div=3%, Beta=0.9			
	Subj. Value	ESO Cost	Max Value	Subj. Discount	Subj. Value	ESO Cost	Max Value	Subj. Discount
Panel A: Beta Effects								
0.0	0.158	0.425	0.665	63%	0.139	0.263	0.341	47%
0.5	0.166	0.430	0.665	61%	0.151	0.271	0.341	44%
0.9	0.180	0.440	0.665	59%	0.179	0.287	0.341	37%
1.2	0.197	0.451	0.665	56%	0.223	0.309	0.341	28%
1.4	0.210	0.460	0.665	54%	0.283	0.332	0.341	15%
Panel B: Volatility Effects Holding Beta Constant								
25%	0.411	0.467	0.488	12%	0.189	0.265	0.298	29%
30%	0.298	0.426	0.526	30%	0.179	0.287	0.341	37%
40%	0.229	0.421	0.602	46%	0.165	0.333	0.425	50%
50%	0.195	0.450	0.665	57%	0.153	0.378	0.502	60%
60%	0.170	0.486	0.737	65%	0.140	0.422	0.571	67%
Panel C. Volatility Effects Holding Correlation Constant								
25%	0.214	0.339	0.488	37%	0.171	0.257	0.298	34%
30%	0.212	0.359	0.526	41%	0.179	0.287	0.341	37%
40%	0.205	0.404	0.602	49%	0.188	0.346	0.425	46%
50%	0.195	0.450	0.665	57%	0.187	0.398	0.502	53%
60%	0.182	0.494	0.737	63%	0.180	0.446	0.571	60%
Panel D. Dividend Rate Effects								
0%	0.197	0.451	0.665	56%	0.233	0.376	0.524	38%
3%	0.168	0.388	0.503	57%	0.179	0.287	0.341	37%
10%	0.116	0.271	0.309	57%	0.097	0.156	0.165	38%
50%	0.011	0.024	0.024	52%	0.001	0.002	0.002	30%
90%	0.000	0.001	0.001	44%	0.000	0.000	0.000	

Table 4

Option values in the presence of restricted stock

   Executive has restricted stock and options, follows optimal option exercise policy, and optimally invests outside wealth in market portfolio. Executive has CRRA coefficient 4, initial outside wealth 0.95 times the value of shares under option, and 0.3 shares of restricted stock in the base case. Option is at the money, vests in 2 years, and expires in 10 years. Risk free rate is 5%. Market portfolio return has mean 13% and volatility 20%. In the base case, the stock has volatility 50%, beta 1.2, and zero dividend. Initial stock price is 1.

Changing Parameter	Expected Term	ESO Cost	FASB Approx	Approx Error	Percent Error	Subj. Value	Subj. Discount
Panel A: Restricted Stock Effects							
0	5.35	0.473	0.512	0.039	8%	0.246	48%
0.15	5.13	0.457	0.502	0.045	10%	0.203	56%
0.3	5.01	0.450	0.497	0.047	10%	0.186	59%
1	4.79	0.431	0.486	0.055	13%	0.144	66%
2	4.64	0.418	0.478	0.060	14%	0.118	72%
5	4.47	0.404	0.470	0.066	16%	0.087	79%
10	4.38	0.396	0.465	0.069	18%	0.072	82%
Panel B: Beta Effects							
0.0	6.00	0.425	0.540	0.115	27%	0.149	65%
0.5	5.53	0.429	0.520	0.091	21%	0.156	64%
0.9	5.21	0.436	0.506	0.069	16%	0.170	61%
1.2	5.01	0.450	0.497	0.047	10%	0.186	59%
1.4	4.95	0.456	0.494	0.038	8%	0.201	56%
Panel C: Volatility Effects Holding Beta Constant							
25%	7.48	0.463	0.413	-0.050	-11%	0.408	12%
30%	5.33	0.423	0.373	-0.050	-12%	0.285	33%
40%	4.81	0.417	0.420	0.003	1%	0.213	49%
50%	5.02	0.449	0.497	0.048	11%	0.183	59%
60%	5.34	0.488	0.576	0.087	18%	0.165	66%
Panel D. Volatility Effects Holding Correlation Constant							
25%	5.10	0.337	0.329	-0.008	-2%	0.198	41%
30%	4.94	0.356	0.357	0.001	0%	0.194	45%
40%	4.91	0.402	0.425	0.023	6%	0.189	53%
50%	5.02	0.449	0.497	0.048	11%	0.183	59%
60%	5.21	0.496	0.570	0.074	15%	0.177	64%
Panel E. Dividend Rate Effects							
0%	5.01	0.450	0.497	0.047	10%	0.186	59%
3%	5.28	0.390	0.396	0.007	2%	0.164	58%
10%	5.91	0.273	0.209	-0.064	-23%	0.122	55%
50%	9.08	0.024	0.000	-0.024	-100%	0.022	7%