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Market Selection and Welfare in a Multi-asset Economy*

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Abstract. We analyze the performance of irrational investors, who mistake expected returns of assets in a multi-asset economy. Mistakes by probabilistically unsophisticated investors that *a priori* seem small lead to severe underperformance compared with rational investors, under general conditions. Our results contrast with previous studies of single-asset economies, which find modest underperformance by irrational investors. In a calibration, an irrational investor who mistakes expected returns by 20% loses almost 95% of his consumption and wealth in about 25 years. The welfare cost of this underperformance is significant, about 40% of the total wealth in the economy.

JEL Classification: G0, G11

1. Introduction

The recent financial crisis has been blamed on the innovation of complex derivatives and other financial instruments that allowed unsophisticated investors to take on large risks they did not understand.[1] This view stands in contrast to the rational neoclassical view that such financial innovation increases efficiency by completing markets.

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[1] An early warning was issued by Mr Warren Buffet, calling these instruments "financial weapons of mass destruction" in his 2003 newsletter to Berkshire Hathaway's shareholders. In the letter, Mr Buffet argues that the range of derivatives is only limited by the imagination of madmen and that investors' biased forecasts together with fraudulent accounting impose severe systemic risk on the economy.

That derivative markets can be hazardous to investors, given their complexity and the potential arbitrage opportunities they offer, may not be surprising, but suboptimal behavior by unsophisticated individual investors has also been documented in the stock market by several recent studies. For example, Barber *et al.* (2009) find an underperformance of 2.1% per year for individual investors relative to institutional investors—a 50% underperformance over a 30-year time horizon. As shown in Calvet, Campbell, and Sodini (2007, 2009), the welfare costs of suboptimal investments by unsophisticated investors are also significant. A solid theoretical understanding of the equilibrium effects on consumption and wealth of investor irrationality in stock markets is therefore important.

The equilibrium effects of investor irrationality when agents make probabilistic mistakes can be understood by using investor survival analysis, developed in Blume and Easley (1992, 2006) and Sandroni (2000, 2005). The starting point is the first-order condition that in a complete market equilibrium relates the ratios of agents' marginal utilities to the ratios of their likelihood processes. The first-order condition completely determines the wealth and consumption dynamics of investors in the market and can be used to define a survival index that provides an asymptotic result (for large time periods) for the market selection process, that is, for the rate at which irrational investors underperform rational ones; see Blume and Easley (2006). In Kogan *et al.* (2006, 2011), the framework is incorporated into a standard asset pricing setting and used to analyze the long-term price impact and survival of irrational investors.

Given that irrational investors eventually die out, an important quantitative question is how much they underperform in a standard asset pricing setting.[2] If the

[2] Of course, as shown in several studies, irrational investors do not always die out. For example, overoptimistic investors may invest a larger share of their wealth in risky assets and ultimately dominate the market when prices are set exogenously (DeLong *et al.*, 1991). Similarly, irrational investors with a lower consumption-to-savings ratio than rational investors may come to dominate the market. Moreover, even when rational investors eventually dominate the market measured by fraction of wealth, irrational investors may still have nonnegligible impact on prices (Kogan *et al.*, 2006). However, when rational and irrational investors have identical utilities, irrational investors will lose out compared with rational ones except under special circumstances. In general equilibrium with complete markets, Sandroni (2000) shows that rational investors will eventually dominate the market under general conditions if agents have identical intertemporal discount factors [Blume and Easley (2006) show that in incomplete markets, this result may not hold in general, although Sandroni (2005) shows that the result can be extended to incomplete markets in some cases]. Loewenstein and Willard (2006) point out that models of the type of DeLong *et al.*, (1990, 1991) implicitly allow for real transfers of production (between risk-less storage and risky technology), due to sentiment, and for changes in aggregate consumption. We study a general equilibrium in a complete market, so in line with Sandroni (2000) irrational investors will eventually lose out. The literature also relates to the original literature on market selection, see Alchian (1950) and Friedman (1953), Cootner (1964), and Fama (1965). Other recent contributions to the literature include Cvitanic and Malamud (2010, 2011).

market selection process takes many centuries, and irrationality in the stock market thereby is only "mildly" punished, it could be argued that the effect is not very important in practice. If, on the other hand, the punishment is severe and the selection process therefore occurs in a matter of a few years, this may have important policy implications.

Using the survival index approach, and building on the general equilibrium literature with heterogeneous investors (see Detemple and Murthy, 1994a; Basak, 2000; David, 2009), Yan (2008) calibrates a standard exchange economy with a representative firm and shows that it may take several hundred years before rational investors significantly outperform irrational investors. Similar results are derived in Dumas, Kurshev, and Uppal (2009), under slightly different assumptions, and used in Branger, Schlag, and Wu (2006).

One caveat with these theoretical studies that find only modest underperformance by irrational investors is that they are based on representative firm economies, that is, on economies with a single traded risky asset. Although this is a harmless assumption when all investors are identical, since aggregate consumption is sufficient for all purposes in this case, it is unclear whether the assumption is harmless when some investors are irrational. In fact, one may suspect a source of underperformance to be that irrational investors hold very different portfolios compared with rational ones. With only one firm to invest in, however, the heterogeneity in portfolio holdings is severely limited. For example, with only one risky asset, all investors with long positions hold portfolios with the same Sharpe ratio.

In this paper, we study a general exchange economy with many risky assets. Our main result is that in such a multi-asset economy, mistakes by an irrational investor that *a priori* seem very small lead to severe underperformance, in contrast to what is obtained in the representative firm setting, in terms of consumption, wealth, and welfare dynamics. For example, over a 25-year horizon, a moderately irrational investor is expected to lose about 93% of consumption and wealth to a rational investor. This result stands in stark contrast to the previous theoretical studies. In other words, multi-asset stock markets may be as "dangerous" as derivative markets for unsophisticated investors, even though no pure arbitrage opportunities exist. Somewhat surprising, the welfare costs of irrationality are also severe, even in this standard exchange economy setting in which irrational investors neither affect the total output nor generate negative externalities by creating systemic risk. Instead, the welfare costs arise because the consumption allocation between the rational and irrational investor is severely suboptimal. When the initial wealth of the rational and irrational investor is the same, the welfare cost as a fraction of total wealth is about 40% in a 25-year horizon, highlighting the importance of financial education in markets with a large fraction of unsophisticated investors and suggesting that it may be welfare increasing to restrict the asset span in such markets.

The underperformance by irrational investors is proportional to the number of risky assets in the market. For example, if it takes 1,500 years to reach a prescribed loss in a market with one risky asset, it takes 3 years in a market with the same aggregate dynamics but with 500 risky assets. Thus, although the model with one representative firm qualitatively gives the same result as the multi-firm model (the eventual extinction of irrational traders), the quantitative difference is striking.

The intuition behind these results is straightforward. The larger state space in the multi-asset framework allows the rational investor to take advantage of the irrational investor much more efficiently than in the representative firm economy. Specifically, the rational investor invests in assets that the irrational investor is bearish about, and which are therefore underpriced, and sells assets that the irrational is bullish about. This portfolio strategy effectively cancels out most idiosyncratic risk in the rational agent's portfolio, while allowing for high expected returns. In other words, the rational investor can "diversify" over the irrational investor's mistakes across stocks. As long as such diversification opportunities are present, the market selection process will be fast when there are many assets. Indeed, we show that there are two cases under which the efficiency of the market selection process is not improved as the number of assets increases: when there is no spread of investor sentiment across assets and when the assets' dividend processes are independent. In these two cases, there is no opportunity for the rational investor to diversify over the irrational investor's mistakes, and the same slow market selection process as in the one-asset model is obtained.

The irrational investors in our model consistently mistake the growth rates of firms. Such behavior could, for example, arise if investors receive noisy signals about growth rates and are overconfident about the quality of these signals. The assumption can more generally be viewed as a reduced-form representation of the behavior of investors who are unsophisticated in how they treat probabilities. We show that our results are robust to several extensions and generalizations. Specifically, they continue to hold for more general risk structures in the market, when all agents make mistakes but some agents' mistakes are "smaller" than others', when rational agents do not know the parameters of the economy but learn about these, and when agents also mistake covariances in addition to growth.

In our analysis, we do not consider frictions. Transaction costs, for example, may be another source of underperformance by unsophisticated investors, as shown by Barber and Odean (2000). A more realistic model with frictions, although clearly of interest, is outside the scope of this paper. Also, we focus on investors who are irrational in the way that they update their probabilistic beliefs. Irrationality in the form of deviations from the expected utility framework is also outside the scope of our analysis.

The paper is organized as follows. In the next section, we introduce the model. In Section 3, we provide the main results on the consumption, wealth, and welfare of rational and irrational investors, in the context of a simple textbook style example. In Section 4, we discuss robustness under variations and generalizations of the base model. Finally, in Section 5, we make some concluding remarks. Details and proofs are left to the Appendix.

2. Model

We closely follow the complete market approach developed in Blume and Easley (1992, 2006) and Sandroni (2000, 2005) and further adjusted to an asset pricing context in Kogan *et al.* (2006, 2011) and Yan (2008) in our analysis. The key relationship that determines the dynamics of a Walrasian exchange economy equilibrium with two agents and intermediate consumption is[3]

$$\frac{u_1'(c_{1,t})}{u_2'(c_{2,t})} = \lambda_t. \tag{1}$$

Equation (1) states that the ratios of the two agents' marginal utilities of consumption in equilibrium is proportional to the ratio of their probability measures (represented by the stochastic weight λ_t), at all points in time and in each state of the world. We use this equilibrium relationship and introduce a general risk structure that can be calibrated to standard multi-asset economies.

2.1 THE ECONOMY

We assume a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, $0 \le t \le T$, and N-dimensional $\mathcal{F}_t$-adapted standard Brownian motions $\mathbf{B}_t = (B_{1,t}, \ldots, B_{N,t})'$ (where $'$ denotes transpose) satisfying the usual assumptions. Here, T could be finite or infinite, although we mainly focus on the infinite horizon case. We use the notation $\mathbf{a} = [a_i]_i$ to "build" vectors from scalars and $a_i = (\mathbf{a})_i$ to extract scalars from vectors. Similarly, we define the matrix $\mathbf{A} = [a_{ij}]_{ij}$ and the scalars $a_{ij} = (\mathbf{A})_{ij}$.

There is an N-dimensional state vector, $\omega_t \in \mathbb{R}^N$, which evolves according to

$$d\omega_t = \mathbf{g}\, dt + \sigma_\omega\, d\mathbf{B}_t. \tag{2}$$

Here, $\mathbf{g} \in \mathbb{R}^N$ and $\sigma_\omega \in \mathbb{R}^{N\times N}$ are (smooth) functions of ω_t and t, so that ω_t is a general diffusion process. We sometimes suppress the time dependence of variables when this can be done without causing confusion, for example, writing ω instead of ω_t. The variance–covariance matrix of ω is $\Sigma = \sigma_\omega \sigma_\omega'$. We assume that Σ is invertible at all points in time. There is a dynamically complete competitive

[3] See Equation (6) in Kogan *et al.* (2011).

market of contingent claims such that claims on each realization of ω are traded, that is, Arrow–Debreu securities exist for each state of the world.

The first M elements (where $1 \leq M \leq N$) of the state vector are associated with firms that produce consumption goods. The latter $M - N$ elements are associated with claims on zero net supply assets. The latter elements could, for example, represent not only derivative markets on unspanned risk but also other claims on idiosyncratic risk, for example, insurance contracts. Specifically, firm i instantaneously produces $D_{i,t}\mathrm{d}t$ of a perishable consumption good, where

$$D_{i,t} = D_{i,0}\, \mathrm{e}^{\omega_{i,t}}, \quad D_{i,0} > 0, \tag{3}$$

and where $\omega_{i,t} = (\omega_t)_i$ is the ith element of the vector ω at time t. We define $\mathbf{D} = (D_{1,0}, D_{2,0}, \ldots, D_{N,0})'$, where $D_{M+1,0}, \ldots, D_{N,0} = 0$. The aggregate consumption is given by

$$C_t = \sum_{i=1}^{M} D_{i,t}. \tag{4}$$

Our results do not depend on the type of securities used to implement the complete market. A standard implementation, however, is to assume that there are N securities, where security i represents a claim to the consumption stream $\mathrm{e}^{\omega_{i,t}}$ and where the net supply of asset i is $D_{i,0}$ for $1 \leq i \leq M$ and 0 for $M+1 \leq i \leq N$, and that there is also a risk-free bond available in zero net supply.[4]

The setup is very general and contains several interesting subcases. If $M = N$ and $\mathbf{g}$ and σ_ω are constants, then the economy corresponds to a standard N-tree Lucas economy (see Cochrane, Longstaff, and Santa-Clara 2008; Martin, 2011; Parlour, Stanton, and Walden, 2011).[5] Also, the models in Anderson and Raimondo (2008), Santos and Veronesi (2006), Campbell and Cochrane (1999), and Bansal and Yaron (2004) fall within this setting, with $M = N$. The case when $M = 1$, that is, when there is one stock in positive net supply and many zero net supply claims, covers the models in Dumas, Kurshev, and Uppal (2009) (with $N = 2$) and Buraschi and Jiltsov (2006).[6] Further generalizations are also possible. For example, it is neither crucial that aggregate consumption is of the form of Equation (4) nor that

[4] Sufficient conditions that ensure that such assets dynamically span all state realizations are given in Anderson and Raimondo (2008), see also Hugonnier, Malamud, and Trubowitz. (2009).

[5] Though commonly referred to as a "Lucas" model, the first-order conditions for this economy and associated stochastic discount factor, $\rho\frac{U'(C_{t+1})}{U'(C_t)}$, were first derived by Rubinstein (1976) and later used by Lucas (1978).

[6] Another alternative is to have only zero net supply securities as the traded assets as in Duffie and Huang (1985), Huang (1987), Duffie and Zame (1989), and Karatzas, Lehoczky, and Shreve. (1990).

the underlying process ω is a diffusion process. For notational simplicity, we stick with the—already very general—setup.

Intuitively, it may be easiest to think of the situation when $M = N$ in which case each risky asset represents a stock, that is, the claim on the dividend process of a firm. Following this intuition, we will use the terminology N-stock or N-asset economy going forward. The important assumption needed for this analogy to be valid is that each stock contains idiosyncratic risk, so that a higher N corresponds to a higher dimensionality of traded risk in the economy.

2.2 AGENTS

There are two price-taking investors, $k \in \{1, 2\}$. Investor 1 is a rational von Neumann–Morgenstern expected utility optimizer, who has a complete understanding of parameters and dynamics in the economy. It has been repeatedly documented that many investors deviate from the rational expected utility framework. Broadly speaking, there are two types of deviations (see Barberis and Thaler, 2003). First, investor behavior is not consistent with agents having a (subjective) expected utility function. Second, investors do not update their beliefs in consistence with Bayes' rule. We will follow Yan (2008) (see also Kogan *et al.*, 2006, 2009; Cvitanic and Malamud, 2011; Chen *et al.*, 2011) and make a parsimonious assumption in line with the second type of deviation. Specifically, we assume that investor 2 mistakes the drift term for $\mathbf{g}^2 = (g_1^2, \ldots, g_N^2) = \mathbf{g} + \delta$, $\delta \neq \mathbf{0}$. We call δ the irrational investor's sentiment vector.

We motivate the irrationality of Agent 2 as a strong form of overconfidence (see Kahneman, Slovic, and Tversky, 1982). We could imagine a situation where Agent 2 at some point received a noisy signal about the true real growth rates of the firms in the economy and, because of overconfidence, trusted the signal to be infinitely precise. The signal was on average correct, but also contained an idiosyncratic random component for each firm. The idiosyncratic component could, for example, arise because the irrational agent wrongly believed that the age, accent, or fashion tastes of the CEO, the physical location of company headquarters, the sound of the company name, and so on were important for the future prospects of the firm.

Agent 1 may also have received such a noisy signal but correctly taken into account the signal's noisiness and arrived at a correct estimate by filtering out the idiosyncratic component or by incorporating other information. Investor 2 on the other hand, being strongly overconfident about the signal, does not update his belief.

This strong overconfidence assumption is obviously quite extreme but can be generalized in several more realistic directions, with similar results. For example, in Section 4.3, we extend the model to a Bayesian setting in which both investors face uncertainty about structural parameters in the economy. In this setting, the rational investor is also initially mistaken about parameters. However, the rational investor learns about the parameters over time. In contrast, the irrational investor at some point—because of overconfidence—decides that he has learned everything that needs to be learned.

In Section 4.4, we extend the model to explicitly take overconfidence into account, along the lines of Scheinkman and Xiong (2003) and Dumas, Kurshev, and Uppal (2009). In this setting, the growth rates of firms in the economy are time varying and unobservable to both agents. The irrational agent receives an uninformative idiosyncratic signal about the growth rate of each firm and chooses to trust the signal—again because of overconfidence.

As we shall see, the results for both these extensions are very similar to what we get in the base model in which the sentiment vector is constant over time. The intuition behind the results is easier to communicate under the assumptions of the base model however, and it leads to closed-form characterizations of several quantitative measures. We therefore mainly use the base model but show robustness to these (and several other) extensions in Section 4.

We further assume that investors $k \in \{1, 2\}$ have initial wealth W_k and Constant Relative Risk Aversion (CRRA) preferences with time discount factors ρ_k and common relative risk aversion parameter γ. For expositional reasons, we mainly focus on the case when $\gamma \neq 1$, although our results also hold under logarithmic utility. In Section 4.1, we generalize to agent-specific risk aversion coefficients. Thus, investor k optimizes

$$U_k = E_k\left[\int_0^T e^{-\rho_k t}\frac{c_{k,t}^{1-\gamma}}{1-\gamma}dt\right], \tag{5}$$

subject to his budget constraint, where $c_{k,t}$ is the instantaneous consumption at t of investor k. Here, since the two investors have different expectations, the k subscript of the expectation operator is motivated. The total initial wealth is $W = W_1 + W_2$.[7] The economic environment can be summarized by the quadruplet $\mathcal{E} = (\delta, \mathbf{g}, \Sigma, \mathbf{D})$, whereas the agents' preferences are summarized by the triplet (γ, ρ_1, ρ_2).

2.3 MEASURING WELFARE

The welfare measure we use is based on an expected *ex post* measure [see Harris (1978), Starr (1973), and also Hammond (1981)]. In our setting, this implies that

[7] A potential extension would be to endow the two agents with idiosyncratic endowment shocks, providing a motive for zero-net supply "insurance" assets (i.e., for $N - M > 0$).

welfare is derived from the objective expected utilities, that is, from the objective expectations of realized utility of consumption by the two agents,[8] as opposed to an *ex ante* measure that would be based on agents' subjective probabilities. Thus, although Agent 2 at $t = 0$ may believe that he is going to be very well off, the expected *ex post* utility he gets from consumption may be very low, with large resulting welfare costs.

The expected realized utilities of consumption of the two agents are calculated using objective probabilities, that is, using the rational agent's probability estimates:

$$U_k^{\mathrm{OBJ}} = E_1\left[\int_0^T \mathrm{e}^{-\rho_k t}\frac{c_{k,t}^{1-\gamma}}{1-\gamma}\mathrm{d}t\right]. \tag{6}$$

We call U_1^{OBJ} and U_2^{OBJ} the objective expected utilities of the rational and irrational agents, respectively. The objective expected utility of the rational agent coincides with his "personal" expected utility. For the irrational agent, however, the personal and objective expected utilities differ because of his incorrect beliefs.

Measuring welfare under heterogeneous beliefs is of course not trivial, but within our model, the *ex post* approach is well motivated. One critique against the *ex post* measure (see, e.g., the discussion in Fleurbaey, 2010) is that, as a practical matter, choosing objective probabilities is not trivial. However, as we shall see, within our setting a social planner would be able to strongly reject the irrational investor's incorrect probability estimates. So, within the assumptions of our model, this would not be an issue.

A second critique against policies that are based on *ex post* welfare measures is that such policies, if they restrict the actions of agents, could be viewed as paternalistic and that being constrained may in itself dampen the well-being of agents in the economy (see, e.g., Harris and Olewiler, 1979; Fleurbaey, 2010). This may indeed suggest that policies that make Agent 2 better informed, so that he can himself correct his mistakes, are superior to policies that restrict his ability to trade based on his own incorrect beliefs (see also our discussion in Section 3.3). The argument does not mitigate the role of the *ex post* measure of welfare in our model, though. In fact, we shall see that the welfare costs in our setting may be so drastic as to be comparable with the analogue of "drinking a fatal poison in the mistaken belief that it was water," made in Hausman and McPherson (1994). The interpretation that the *ex post* welfare criterion measures the potential gains from making Agent 2 better informed—in line with the arguments for the *ex post* measure made in Nakata (2009) and Fleurbaey (2010)—is therefore well suited for our setting.

Finally, a third discussion, that is unrelated to our analysis, focuses on the merits of *ex post* versus *ex ante* welfare measures in allowing for inequality aversion, see,

[8] Within our exchange economy setting, expected *ex post* efficiency also implies so-called universal *ex post* efficiency, see Starr (1973) and Harris (1978).

for example, Diamond (1967), Hammond (1981), and Epstein and Segal (1992). Specifically, the classical (*ex ante*) social welfare measure introduced in Harsanyi (1955) does not allow a social planner to have a preference for "equal opportunity," (e.g., by randomizing who gets to be rich and who gets to be poor in an economy), and *ex post* measures, and other extensions, have been suggested to allow for such inequality aversion. The power and weaknesses of *ex post* welfare measures in allowing for inequality aversion are unrelated to our paper.

2.4 EQUILIBRIUM

It follows immediately that the perceived shocks to the state variable ω by agent $k = 1, 2$ are

$$d\omega_t - \mathbf{g}_t^k dt = \sigma_\omega \, d\mathbf{B}_t^k, \tag{7}$$

where $\mathbf{B}_t^k$ is a standard N-dimensional Brownian motion under agent k's probability measure. The relation between $\mathbf{B}_t^1 = \mathbf{B}_t$ and $\mathbf{B}_t^2$ is

$$d\mathbf{B}_t^2 = d\mathbf{B}_t - \Delta_t \, dt, \tag{8}$$

where $\Delta = \sigma_\omega^{-1}\delta$.

To solve for the Walrasian complete market equilibrium, we construct the social planner's problem with a representative agent (see Constantinidis, 1982; Dumas, 1989; Cuoco and He, 1994; Detemple and Murthy, 1994b; Wang, 1996; Basak, 2000; Gallmeyer and Hollifield, 2008) state by state and time by time from

$$\begin{aligned} u(C_t, \lambda_t, t) &= \max_{c_{1,t}, c_{2,t}} \left\{ e^{-\rho_1 t} \frac{c_{1,t}^{1-\gamma}}{1-\gamma} + \lambda_t \, e^{-\rho_2 t} \frac{c_{2,t}^{1-\gamma}}{1-\gamma} \right\}, \\ \text{s.t.} \\ c_{1,t} + c_{2,t} &= C_t. \end{aligned} \tag{9}$$

Here, $\lambda_t = \lambda \exp(-\frac{1}{2}\int_0^t \Delta_s' \Delta_s \, ds + \int_0^t \Delta_s' \, d\mathbf{B}_s)$ is proportional to the Radon–Nikodym derivative of the irrational agent's probability measure with respect to the rational agent's probability measure. It therefore measures how strongly the two agents disagree about the likelihood of events.[9]

Our objective is to study the consumption and wealth dynamics of the two agents. We therefore make the following definitions:

[9] Formally, $\lambda_t = \lambda_0 \eta_t$, where η_t is the Radon–Nikodym derivative of the irrational agent's probability measure with respect to the rational agent's probability measure, and $\lambda_0 \in \mathbb{R}_+$ determines Agent 2's weight in the social planner's representative agent problem.

Definition 1.

1. *The consumption share (of Agent 1) is* $f_t \overset{\text{def}}{=} \frac{c_{1,t}}{C_t}$.
2. *The wealth share (of Agent 1) is* $f_{W,t} \overset{\text{def}}{=} \frac{W_{1,t}}{W_t}$.
3. *The log-consumption ratio is* $h_t \overset{\text{def}}{=} \log[\frac{c_{1,t}}{c_{2,t}}] = \log[\frac{f_t}{1-f_t}]$.
4. *The log-wealth ratio is* $h_{W,t} \overset{\text{def}}{=} \log[\frac{W_{1,t}}{W_{2,t}}] = \log[\frac{f_{W,t}}{1-f_{W,t}}]$.

We note that the consumption share is obtained by a simple transformation of the log-consumption ratio, $f_t = \frac{e^{h_t}}{1+e^{h_t}}$. We will also work with the random stopping time

$$\tau_f = \inf_t \{t : f_t \geq f\}, \tag{10}$$

that is, τ_f defines the first point in time at which the consumption share of Agent 1 reaches f.

The following standard proposition, which follows directly from the agents' Euler conditions, summarizes the dynamics of consumption, wealth, and the stochastic discount factor in the complete market Walrasian equilibrium.

Proposition 1. *Agent 1's consumption share at time t is*

$$f_t = \frac{1}{1 + e^{(\rho_1-\rho_2)t/\gamma}\lambda_t^{\frac{1}{\gamma}}}, \tag{11}$$

which determines the agents' consumption:

$$\begin{aligned} c_{1,t} &= f_t C_t, \\ c_{2,t} &= (1-f_t)C_t. \end{aligned} \tag{12}$$

The stochastic discount factor at time t under the objective probability measure is

$$\xi_t \overset{\text{def}}{=} e^{-\rho_1 t}\left(\frac{c_{1,t}}{c_{1,0}}\right)^{-\gamma} = e^{-\rho_1 t}\left(\frac{f_t}{f_0}\right)^{-\gamma}\left(\frac{C_t}{C_0}\right)^{-\gamma}, \tag{13}$$

which determines the agents' wealth:

$$\begin{aligned} W_{1,t} &= E_t\left[\int_t^T \tfrac{\xi_s}{\xi_t} c_{1,s}\, ds\right] = E_t\left[\int_t^T \tfrac{\xi_s}{\xi_t} f_s C_s\, ds\right], \\ W_{2,t} &= E_t\left[\int_t^T \tfrac{\xi_s}{\xi_t} c_{2,s}\, ds\right] = E_t\left[\int_t^T \tfrac{\xi_s}{\xi_t}(1-f_s) C_s\, ds\right]. \end{aligned} \tag{14}$$

We see that the consumption share, f_t, together with the aggregate consumption, C_t, completely determine the dynamics of the economy.

3. Results

We study the dynamics of consumption and wealth of the two agents in equilibrium. The following proposition provides the key result for our analysis.

Proposition 2. *Define the instantaneous transfer index*

$$K = \delta' \Sigma^{-1} \delta. \tag{15}$$

Here, $\delta \in \mathbb{R}^N$ is the irrational agent's sentiment vector and Σ is the instantaneous variance–covariance matrix of ω. Then, the instantaneous dynamics of the log-consumption ratio, h_t, is

$$\mathrm{d}h_t = \left(\frac{1}{2\gamma}K + \frac{\rho_2 - \rho_1}{\gamma}\right)\mathrm{d}t + \frac{1}{\gamma}\sqrt{K}\,\mathrm{d}\tilde{B}_t, \tag{16}$$

where $\tilde{B}$ is a standardized Brownian motion.

Proposition 2 is valid under the most general assumptions of our model, with time- and state-dependent diffusion coefficients in which case K will also be time and state dependent. The transfer index is related to the survival index (see Blume and Easley, 2006; Yan, 2008), but it is preference independent, that is, it only depends on the economic environment. As we shall see, in multi-asset markets, the transfer index typically dominates preferences in determining if and how market selection takes place, that is, a large K typically implies a significant consumption and wealth transfer from Agent 2 to Agent 1.

From Equation (16), it follows that $E[\mathrm{d}h_t] = (\frac{1}{2\gamma}K + \frac{\rho_2-\rho_1}{\gamma})\mathrm{d}t$, which, if K is constant, immediately implies that

$$E[h_t - h_0] = \left(\frac{1}{2\gamma}K + \frac{\rho_2 - \rho_1}{\gamma}\right)t. \tag{17}$$

Therefore, if $\rho_2 = \rho_1$, since Equation (15) implies that $K > 0$ (as the covariance matrix, Σ, is positive definite), the log-consumption ratio is expected to increase at each point in time. More generally, if the agents have different time preference parameters, the log-consumption ratio is expected to increase if $K > 2(\rho_1 - \rho_2)$.

If $\rho_1 = \rho_2$, it is straightforward to show that the consumption share is also expected to increase: $E_t[\mathrm{d}f_t] = \frac{\mathrm{e}^{h_t}(\gamma+1+\mathrm{e}^{h_t}(\gamma-1))}{2(1+\mathrm{e}^{h_t})^3\gamma^2}K\,\mathrm{d}t$. Thus, regardless of the sentiment of the irrational investor, he is expected to underperform (in consumption growth terms) the rational investor at each point in time and the larger the transfer index is, the more severe the underperformance.

To quantify the underperformance of the irrational investor, we focus on the case when model parameters are constant, so that K is constant. In this case, we get closed-form expressions for most variables of interest.[10] We study the expected stopping time, $E[\tau_f]$, that is, the time it is expected to take for the rational investor to reach a consumption share of f. We have the following proposition.

Proposition 3. *Assume that the irrational agent's sentiment, δ, and the covariance matrix, Σ, are constants and that the rational agent's (Agent 1's) initial consumption share is f_0. The expected time for the rational agent to reach the consumption share f, where $f > f_0$, is*

$$E(\tau_f) = \frac{2\gamma v}{K + 2(\rho_2 - \rho_1)}, \tag{18}$$

and the variance is

$$\mathrm{Var}(\tau_f) = \frac{8\gamma K v}{(K + 2(\rho_2 - \rho_1))^3}, \tag{19}$$

where $v = \log(\frac{f}{1-f}) - \log(\frac{f_0}{1-f_0})$.

We note that if K is large, any difference in the investors' discount factors will be swamped by K. We will show that K is indeed large, so going forward, we simply assume that the investors' discount factors are the same, $\rho_1 = \rho_2$.

3.1 AN EXAMPLE

We calibrate the model to an example. We consider a textbook style economy in which all risky assets are affected by a market-wide shock and also by independent idiosyncratic shocks. Specifically, we assume that $[\mathbf{g}]_i = g$, $i = 1, \ldots, N$, and that

[10] The theory of differential inequalities can be used to derive theoretical bounds on the underperformance when coefficients are state and/or time dependent, although the analysis becomes more complex. Intuitively, a lower bound on the transfer index leads to a lower bound on the underperformance through Equation (16) in this case.

$[\Sigma]_{i,i} = 2k_N\sigma^2$, $i = 1, \ldots, N$, $[\Sigma]_{i,j\neq i} = k_N\sigma^2$, $i, j = 1, \ldots, N$, where $k_N = \frac{N}{N+1}$ is a scaling factor introduced to make the total risk independent of N (without it, the aggregate consumption volatility would be higher for small N since the benefit of diversification is lower for small N).[11] For the time being, we allow M to be any number between 1 and N. The economy thus has one systematic risk factor and, for large N, one half of the risk is idiosyncratic, whereas the other half is systematic. We note that $\frac{g_i}{\sqrt{2k_N\sigma^2}} \sim \frac{g}{\sqrt{2}\sigma}$ for large N, so the growth normalized by uncertainty for each stock is basically independent of the number of stocks in the economy. A similar argument will hold for the sentiment, δ_i, normalized by uncertainty in each stock.

We will discuss several generalizations of this base example. For example, it follows from the discussion in Section 4.7 that the results continue to hold with random factor loadings and with multiple risk factors.

We choose a growth rate of $g = 0.02$, in line with standard practice, and a growth volatility of $\sigma = 0.03218$, as found in Campbell (2003). From Equations (16) and (18), we see that a high risk aversion coefficient will lead to a lower degree of underperformance by the irrational investor. We therefore choose a somewhat high risk aversion coefficient, $\gamma = 6$, to show that our results are robust. We use the common personal discount rate, $\rho_1 = \rho_2 = 1\%$. Finally, we assume that the irrational investor is slightly bullish about the first half of the stocks, $[\delta]_i = q \times g$, $i = 1, \ldots, N/2$, and slightly bearish about the other half, $[\delta]_i = -q \times g$, $i = N/2 + 1, \ldots, N$, where we for simplicity assume that N is even. We assume that $q = 0.2$. Thus, the irrational investor overestimates the real dividend growth rate by 20% for half of the stocks (believing that it is 0.024) and underestimates it by the same amount for the other half (believing that it is 0.016). Although not necessary, it is natural to think of the case when $M = N$, and the N risky assets represent the real growth processes of N firms.

It follows immediately from Equation (15) that the transfer index in this economy is

$$K = \frac{g^2}{\sigma^2} q^2 (N + 1), \quad N \text{ even.} \tag{20}$$

We see that K is increasing in the real growth rate and decreasing in the real growth volatility. Moreover, not surprisingly, it is increasing in the irrational investor's sentiment. What is crucial for our analysis, however, is that K is increasing in N. Thus, all else equal, the more risky assets there are in the economy, the more severe is the underperformance of the irrational investor. The intuition behind this

[11] We note that σ is a scalar that will be used in this and other symmetric examples to define the uncertainty of each risk, whereas the σ_ω matrix defines the variance–covariance matrix in the general model via the relation $\Sigma = \sigma_\omega \sigma_\omega'$.

result is that it is difficult for the rational investor to take advantage of the irrational one in a market with a restricted state space. For example, when there is only one stock, the only difference between the rational and irrational investors' portfolios is the amount they invest in the market, which does not allow much separation. When there are many stocks, more portfolio separation is possible, the rational investor can better take advantage of the irrational one, and the irrational investors' underperformance therefore becomes more severe.

Equation (20) holds for even N. It is easy to check that for odd N, the formula becomes

$$K = \frac{g^2}{\sigma^2} q^2 \left(N + 1 - \frac{1}{N} \right), \quad N \text{ odd.} \tag{21}$$

Therefore, K triples when the number of stocks increases from one to two. This drastic increase in the transfer index occurs because with two stocks, the irrational agent will push down the price of the stock he is pessimistic about. Similarly, he will push the price of the other stock up. The rational agent can then take on a relatively large position in the stock with the deflated price, financed with a relatively smaller position in the stock with an inflated price. This, in turn, leads to a higher Sharpe ratio for the rational agent than for the irrational agent. In contrast, when there is only one stock, differences in the two agent's portfolio holdings are financed by different holdings of the risk-free asset and the agents therefore hold portfolios with the same Sharpe ratio, leading to less underperformance by the irrational agent. Thus, the representative firm setting (i.e., the setting with $N = 1$) severely limits the underperformance by the irrational agent, and even with a modest number of assets, his underperformance may be much more severe. Such a situation could, for example, arise if the irrational agent invests in a small number of mutual funds.

In Figure 1, we show the expected time to reach different consumption shares for $N = 1$, 10, 50, and 100 stocks, when the rational and irrational investors have the same initial consumption share ($f_0 = 1/2$). We see that the difference between the representative firm ($N = 1$) and the multi-firm ($N \gg 1$) settings is indeed drastic. With one stock, it is expected to take 1,706 years for the rational investor's consumption share to reach 90%, whereas it takes 569 years with 2 stocks, 17 years with 100 stocks, and only 3.4 years with 500 stocks. With $N = 100$, the consumption share of the irrational agent is expected to be 3.7% after 25 years, so in this case the irrational agent is expected to lose (0.5 − 0.037)/0.5 = 93% of his initial consumption share of 50% to the rational agent.

The distribution of τ_f is thin-tailed, so the time it takes to reach f will, with high probability, be close to $E[\tau_f]$. In fact, as shown in the proof of Proposition 3, τ_f has the first passage time distribution of a Brownian motion with drift $\frac{K}{2\gamma}$ and

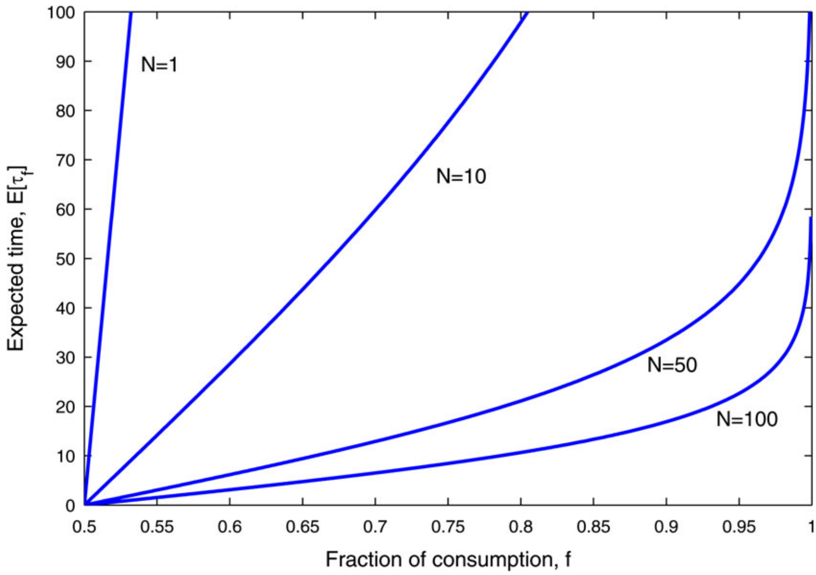

Figure 1. Expected time in years, $E[\tau_f]$, for the rational investor to reach the consumption share, *f*, for $N = 1, N = 10, N = 50$, and $N = 100$, when the irrational investor's sentiment is $q = 20\%$ and the initial consumption shares are the same.

variance $\frac{K}{\gamma^2}$ per unit time, with a probability density function that decreases faster than that of a normal distribution for any fixed *f*. In Figure 2, we show the distribution of the time it takes to reach $f = 90\%$, with $f_0 = 0.5$, for $N = 1$, $N = 10$, $N = 50$, and $N = 100$ stocks. For example, for $N = 100$, in which case the expected time is 17 years, the probability that it takes more than twice the expected time (34 years) is negligible. The expected log-consumption ratio of the rational agent after 25 years with 100 stocks, from Equations (17) and (20) is $E[h_{25}] = \frac{0.02^2}{0.03218^2} \times 0.2^2 \times 101 \times \frac{25}{2\times 6} = 3.25$.

As mentioned, the large number of stocks allows the two agents to invest in portfolios with very different Sharpe ratios. This provides another metric for measuring the irrational investor's underperformance. In our example with $N = 100$ stocks, the Sharpe ratio of the rational investor at $t = 0$ is 0.65, whereas the irrational investor's Sharpe ratio at $t = 0$ is -0.62. Since individual investors' Sharpe ratios are not observable in the market, it is unclear whether these numbers are realistic. The (observable) Sharpe ratio of the market portfolio is 0.43, so the rational agent's Sharpe ratio is thus about 50% higher than the market's Sharpe ratio, whereas the irrational agent's Sharpe ratio is substantially lower.

The severe underperformance by the irrational investor is driven by the transfer index, *K*. In this example, we assumed that the personal discount rates of the two agents were the same. If the rational agent has a higher discount rate than the

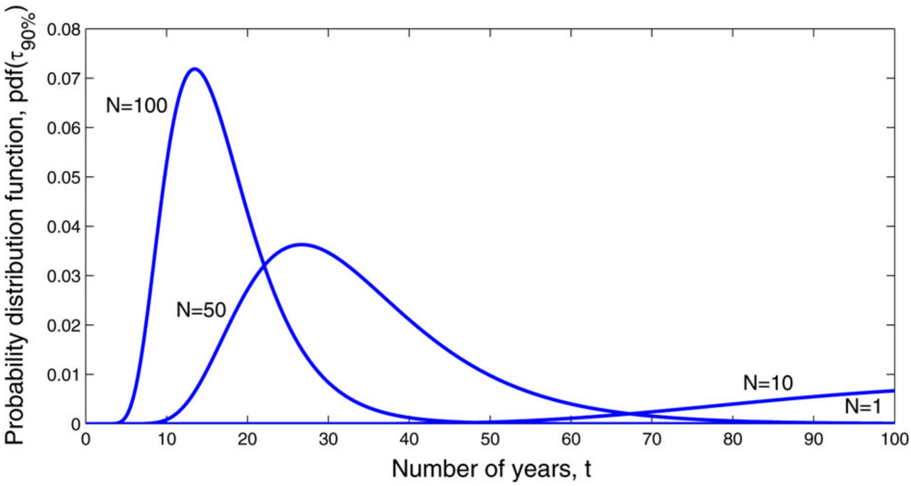

Figure 2. Probability distribution of the time it takes for rational investors to reach 90% of the consumption share for $N = 1$, $N = 10$, $N = 50$, and $N = 100$, when the irrational investor's sentiment is $q = 20\%$ and the initial consumption shares are the same.

irrational agent, the irrational agent's underperformance will decrease since the rational agent consumes at a higher rate. However, from Equation (18) it follows that the differences in discount rates have to be very large to offset the underperformance generated by the transfer index. For example, in the previous example with $N = 100$, the rational agent's discount rate has to be 39% per year to increase the expected time for the rational agent to reach a 90% consumption share by a factor of two, from 17 to 34 years.

We also note that the stopping time statistic may not provide the complete story about the underperformance of the irrational investor. For example, if the consumption share dynamics are "wild" (very volatile), it could well be that the expected time for the rational investor to reach a specific consumption share is short but so is the time for the irrational investor to reach the same consumption share. It turns out that such issues are not present in our model and that the first passage time provides a good summary statistic of the underperformance of the irrational investor. One way of seeing this is to compare the first passage distributions of the rational and irrational agents. Obviously, the first time the irrational agent reaches the consumption share f is the first time the rational agent reaches $1 - f$, so we can express the relationship using only the rational agent's stopping times. From the expressions for the stopping time distribution, it is straightforward to derive the following relationship when the two agents start with the same consumption share, $f_0 = 1/2$,

$$\mathrm{pdf}(\tau_{1-f}) = Z_f \times \mathrm{pdf}(\tau_f), \quad \text{where } Z = \left(\frac{1-f}{f}\right)^{\gamma}. \tag{22}$$

Thus, the distribution of the first time the irrational agent reaches the consumption share $f > \frac{1}{2}$ is a constant times the distribution of the time it takes for the rational agent to reach f, where interestingly the constant only depends on the agents' risk aversion coefficient but not on real variables (μ and σ).[12]

Since $Z_f < 1$, the total probability mass of the irrational agent is less than 1, reflecting the fact that there is a chance that he will never reach a consumption share of f; the expected time to reach f for the irrational agent is therefore infinite. Now, not only is $Z_f < 1$ but also it is typically a very small number. In the example we have studied so far, $Z_{90\%} = 1.88 \times 10^{-6}$, so the chance that the irrational agent will ever reach a consumption share of 90% is negligible. In fact, the chance that he will ever reach $f = 60\%$ is only $Z_{60\%} = 0.088$. Thus, we conclude that the consumption share dynamics are not "wild," and the rational agent's stopping time provides a good representation of his overperformance. To summarize, in consumption terms, the irrational agent is severely punished in the multi-asset economy. We next study the effects on wealth and welfare.

3.2 WEALTH AND WELFARE

It can be shown that the dynamics of the wealth share, $f_{W,t}$, are very similar to those of the consumption share, f_t. For some special cases, we have closed-form solutions, and for other cases we verify numerically that the dynamics of the consumption share and the wealth share are very similar. For simplicity, we restrict ourselves to the case when the agents have the same personal discount rates ($\rho_1 = \rho_2$) and the relative risk aversion coefficient is an integer; generalizations are straightforward.[13]

There are two special cases in which closed-form solutions for the wealth share are obtainable. First, when the investors have log-utility, $\gamma = 1$, it immediately follows that the dynamics of the consumption and wealth shares are identical since both investors consume constant fractions of their wealth at all times.

Proposition 4. *When the investors have logarithmic utility, $\gamma = 1$, $f_t \equiv f_{W,t}$ for all t, and the results for the consumption share therefore also hold for the wealth share.*

Second, for general γ, in the case when aggregate consumption follows a constant coefficient geometric Brownian motion (e.g., when $M = 1$ in the model in the previous section) we have the following proposition.

[12] In the case where the initial consumption share $f_0 \neq \frac{1}{2}$, similar results arise although the expression is slightly less clean. We have $\mathrm{pdf}(\tau_{\hat{f}}) = Z_f \times \mathrm{pdf}(\tau_f)$, where $Z = \left(\frac{f_0/(1-f_0)}{f/(1-f)}\right)^{\gamma}$, $\widehat{f} = (1-f)/(1 + (((1-f_0)/f_0)^2 - 1)f)$, and the relationship $\widehat{f} < f_0 < f$ holds.

[13] The proof in the Appendix covers the case with different personal discount rates and follows the approach in Yan (2008) extended to a multi-asset case. The approach in Bhamra and Uppal (2009) can be used to further extend the results to noninteger γ's.

Proposition 5. *When $M = 1$, γ is a positive integer, and* $\mathbf{g}$ *and* Σ *are constants, then the wealth dynamics of Agents 1 and 2 are*

$$W_{1,t} = \frac{C_t}{\left(1 + \lambda_t^{\frac{1}{\gamma}}\right)^{\gamma}} \sum_{k=0}^{\gamma-1} \binom{\gamma-1}{k} \lambda_t^{\frac{k}{\gamma}} \frac{1}{\alpha_k}\left(1 - \mathrm{e}^{-\alpha_k (T-t)}\right) \tag{23}$$

$$W_{2,t} = \frac{C_t}{\left(1 + \lambda_t^{\frac{1}{\gamma}}\right)^{\gamma}} \sum_{k=0}^{\gamma-1} \binom{\gamma-1}{k} \lambda_t^{\frac{k+1}{\gamma}} \frac{1}{\alpha_{k+1}}\left(1 - e^{-\alpha_{k+1} (T-t)}\right), \tag{24}$$

where $\alpha_k = \rho + \frac{k}{2\gamma}(1 - \frac{k}{\gamma})\Delta'\Delta + (\gamma - 1)(g_1 + \frac{1-\gamma}{2}\sigma'_{\omega_1}\sigma_{\omega_1} + \frac{k}{\gamma}\sigma'_{\omega_1}\Delta)$ *and where we have defined the vector* $\sigma_{\omega_1} = ((\sigma_\omega)_{11}, \ldots, (\sigma_\omega)_{1N})'$, *that is,* σ_{ω_1} *is the transpose of the first row of* σ_ω.

It is straightforward to use Proposition 5 to see that when $M = 1$, the wealth and consumption shares are similar and that the severe underperformance of the irrational investor in terms of consumption therefore carries over to wealth. For example, in Figure 3, the wealth fraction divided by the consumption fraction, $Z = \frac{W_1/W_2}{c_1/c_2}$, is shown as a function of the consumption fraction, for $N = 1$, 10, 50, 100 assets and $M = 1$. The fraction is close to one, and it is greater than one when $c_1 > c_2$, implying that $\frac{W_1}{W_2} > \frac{c_1}{c_2}$ when $c_1 > c_2$. The dynamics of the wealth share is therefore very similar to that of the consumption share in this case, and since c_1 will quickly become larger than c_2 when N is large, the wealth share of the rational agent is typically even higher than the consumption share. We have also verified, using simulations, that the dynamics of the consumption share and wealth share when $M = N$ (i.e., for the N-tree Lucas economy) are almost identical to the case when $M = 1$. The results are reported in the Appendix. Continuing with the example of the previous section, with $N = 100$ trees, the wealth share of the irrational agent is expected to be 3.24% after 25 years, that is, it is even lower than the expected consumption share, which is 3.7%. The irrational agent is therefore expected to lose (0.5 − 0.0324)/0.5 = 93.5% of his wealth in 25 years.

It is *a priori* unclear whether the underperformance of the irrational investor is associated with large welfare costs. In fact, since we are studying an exchange economy, the actions of the irrational investor do not influence total output, so one might suspect that the main effect of the underperformance is a transfer from the irrational to the rational investor with no welfare effects. This intuition is incorrect, and the welfare costs of Agent 2's irrationality may actually be high.

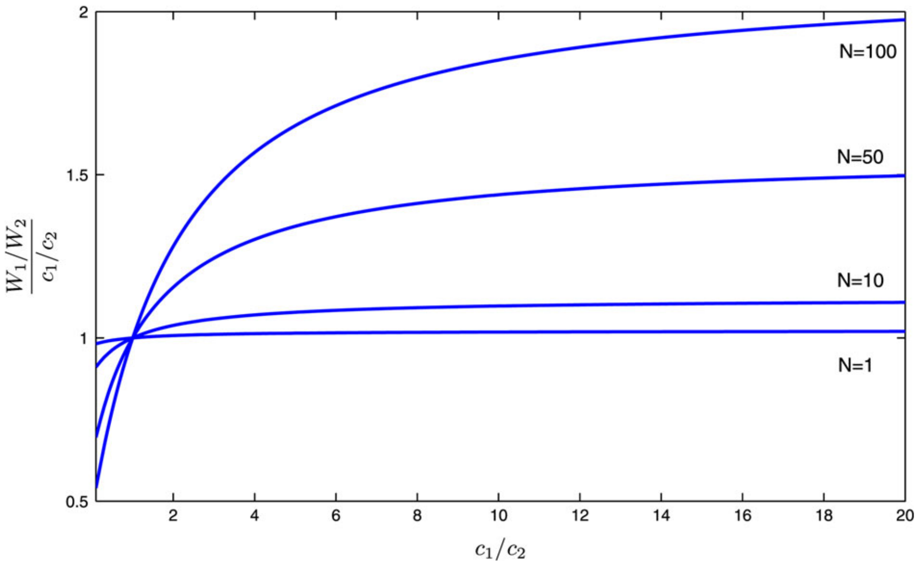

Figure 3. Wealth fraction divided by consumption fraction, $Z = \frac{W_1/W_2}{c_1/c_2}$, as a function of consumption fraction, for $N = 1, 10, 50, 100$, and $M = 1$. Z is close to one, implying that the wealth fraction is very similar to the consumption fraction. Parameters: $q = 20\%$, $\sigma = 0.0328$, $g = 0.02$, $\rho_1 = \rho_2 = 1\%$, $\gamma = 6$, $T = \infty$.

The reason is that he consumes in the wrong states of the world and at the wrong points in time. Especially, he tends to consume too early compared with what is objectively optimal.

In analyzing the welfare costs of Agent 2's irrationality, we use the objective expected utilities, as motivated in Section 2.3, that is, the expected realized utility of consumption of the two agents. We compare the objective expected utilities of Agents 1 and 2 with the expected utilities they could realize in an economy with the same aggregate consumption process and utility weights but with Pareto efficient consumption allocations in all states. We focus on the case when $T < \infty$ and $\gamma > 1$.[14] Formally, we define the utility weight, $y = \frac{U_1^{\mathrm{OBJ}}}{U_2^{\mathrm{OBJ}}}$. A Pareto efficient allocation with the same utility weights would be achieved by allowing Agent 1 to consume ζC_t in all states of the world at all times, where $\zeta = \frac{y^{\frac{1}{1-\gamma}}}{1+y^{\frac{1}{1-\gamma}}}$. Agent 2 would then consume $(1 - \zeta)C_t$. We define the expected utility of a rational representative investor,

[14] If $T = \infty$, the objective utility of the irrational agent may not even be defined since it may be negative infinity. The analysis of the case when $\gamma = 1$ is straightforward, although the formulas differ.

$$\widehat{U}^{\mathrm{OBJ}} = E_1\left[\int_0^T \mathrm{e}^{-\rho t}\frac{C_t^{1-\gamma}}{1-\gamma}\mathrm{d}t\right]. \tag{25}$$

The expected utilities of the two agents with this Pareto efficient allocation would then be $\widehat{U}_1^{\mathrm{OBJ}} = \zeta^{1-\gamma}\widehat{U}^{\mathrm{OBJ}}$ and $\widehat{U}_2^{\mathrm{OBJ}} = (1-\zeta)^{1-\gamma}\widehat{U}^{\mathrm{OBJ}}$. Therefore, the relative expected utility improvements of the two agents, under the efficient allocation, would have the same utility weights,

$$\zeta^{1-\gamma}\frac{\widehat{U}^{\mathrm{OBJ}}}{U_1^{\mathrm{OBJ}}} = \frac{y}{\left(1+y^{\frac{1}{1-\gamma}}\right)^{1-\gamma}}\frac{\widehat{U}^{\mathrm{OBJ}}}{yU_2^{\mathrm{OBJ}}} = \frac{1}{\left(1+y^{\frac{1}{1-\gamma}}\right)^{1-\gamma}}\frac{\widehat{U}^{\mathrm{OBJ}}}{U_2^{\mathrm{OBJ}}} = (1-\zeta)^{1-\gamma}\frac{\widehat{U}^{\mathrm{OBJ}}}{U_2^{\mathrm{OBJ}}}.$$

It follows almost immediately that the welfare cost of Agent 2's irrationality is $\theta = 1 - \zeta^{-1}\left(\frac{U_1^{\mathrm{OBJ}}}{\widehat{U}^{\mathrm{OBJ}}}\right)^{\frac{1}{1-\gamma}}$.[15] That is, given that the two agents have wealth W_1 and W_2 respectively in the original economy and objective expected utilities U_1^{OBJ} and U_2^{OBJ}, the objective expected utilities they would realize in the Pareto efficient implementation correspond to what they would realize with wealth $\frac{1}{1-\theta}W_1$ and $\frac{1}{1-\theta}W_2$ in the original economy. We rewrite and summarize this in the following proposition.

Proposition 6. *Given that the objective expected utilities of Agents 1 and 2 are U_1^{OBJ} and U_2^{OBJ} and that the objective expected utility of a representative agent is $\widehat{U}^{\mathrm{OBJ}}$, the welfare cost of Agent 2's irrationality is*

$$\theta = 1 - \frac{|U_1^{\mathrm{OBJ}}|^{\frac{1}{1-\gamma}} + |U_2^{\mathrm{OBJ}}|^{\frac{1}{1-\gamma}}}{|\widehat{U}^{\mathrm{OBJ}}|^{\frac{1}{1-\gamma}}}. \tag{26}$$

For special cases, we have closed-form solutions for the welfare cost, just as was the case for the wealth dynamics.

Proposition 7. *When $M = 1$, and $\mathbf{g}$ and Σ are constants, the relative welfare cost in an economy with horizon T is*

$$\theta = 1 - \frac{\left(\sum_{k=0}^{\gamma-1}\binom{\gamma-1}{k}\lambda_0^{\frac{k}{\gamma}}\left(\frac{1-\mathrm{e}^{-\alpha_k T}}{\alpha_k}\right)\right)^{\frac{1}{1-\gamma}} + \left(\sum_{k=0}^{\gamma-1}\binom{\gamma-1}{k}\lambda_0^{\frac{k+1-\gamma}{\gamma}}\left(\frac{1-\mathrm{e}^{-\alpha_{k+1-\gamma} T}}{\alpha_{k+1-\gamma}}\right)\right)^{\frac{1}{1-\gamma}}}{\left(\frac{1}{\sigma_0}(1-e^{-\alpha_0 T})\right)^{\frac{1}{1-\gamma}}}. \tag{27}$$

[15] Since expected utility is homogeneous of degree $1-\gamma$ in wealth for both agents, the corresponding relative wealth increase for both agents under a proportional sharing rule is $\beta = \zeta(\frac{\widehat{U}^{\mathrm{OBJ}}}{U_1^{\mathrm{OBJ}}})^{\frac{1}{1-\gamma}}$. The relative welfare of the economy with irrationality is therefore a fraction $\frac{1}{\beta}$ of the optimal economy, so the welfare cost is $1-\frac{1}{\beta}$, which takes the given form.

Here, $\alpha_k = \rho + \frac{k}{2\gamma}(1-\frac{k}{\gamma})\Delta'\Delta + (\gamma-1)(g_1 + \frac{1-\gamma}{2}\sigma'_{\omega_1}\sigma_{\omega_1} + \frac{k}{\gamma}\sigma'_{\omega_1}\Delta)$, $\sigma_{\omega_1} = ((\sigma_\omega)_{11}, \ldots, (\sigma_\omega)_{1N}))'$, *and* λ_0 *is the social planner's weight coefficient, defined in Section* 2.

In Figure 4, we show the welfare cost of Agent 2's irrationality for economies with $N = 1$, 10, 50, 100 risky assets and $M = 1$, as a function of the horizon of the economy, T, given that the two agents have the same initial wealth, $W_{1,0} = W_{2,0}$. It can indeed be high. For example, for the economy with $N = 100$ risky assets and a 25-year investment horizon, the welfare cost is about 40% of the total wealth in the economy. For $N = 50$, it is even higher in longer horizons, about 43%.[16]

Interestingly, the welfare cost is neither monotone in the number of risky assets, N, nor in the time horizon, T: a higher N may lead to a lower welfare cost, given T, as may a longer horizon, given N. The intuition is as follows. The model is calibrated such that the initial amount of wealth of the two agents is the same. For a low N, the underperformance of the irrational agent is not severe and the welfare costs

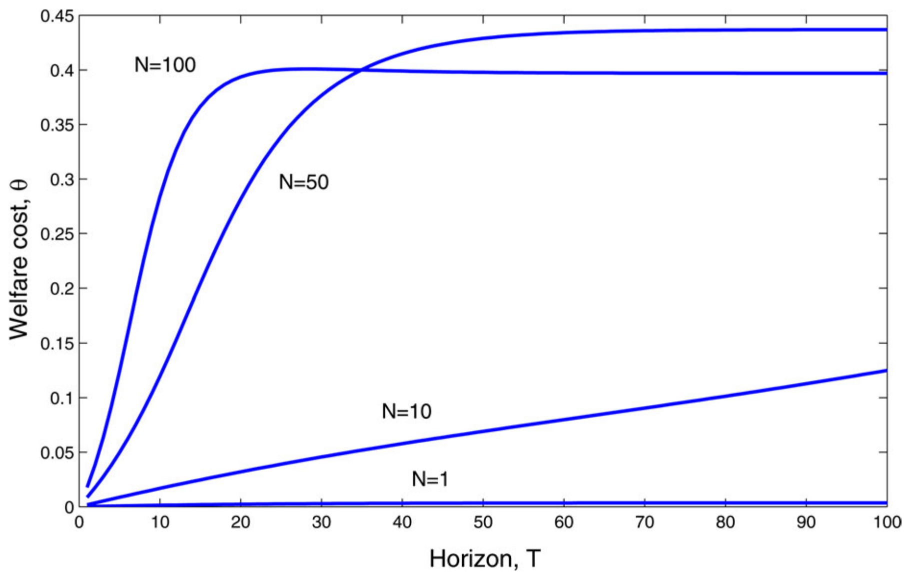

Figure 4. Welfare costs as a function of time for $N = 1$, $N = 10$, $N = 50$, and $N = 100$ risky assets and $M = 1$. The initial amount of wealth of the two agents is the same, $W_{1,0} = W_{2,0}$, in each calibration. Parameters: $q = 20\%$, $\sigma = 0.0328$, $g = 0.02$, $\rho_1 = \rho_2 = 1\%$, $\gamma = 6$.

[16] The closed-form solution is valid when $M = 1$. We have verified with simulations that the results are almost identical in the N-tree Lucas economy, that is, in the economy with $M = N$. The results are available from the authors upon request.

are therefore limited. As N grows, the severity of the irrational agent's mistakes increases and so do the welfare costs. For very large N, however, the rational agent very quickly captures the bulk of the consumption and it is therefore not possible to make him much better off than he already is. Thus, although the irrational agent can be made drastically better off by smoothing consumption over time and would be willing to pay almost all his wealth to the rational agent if illuminated about this, the extra value for the rational agent—in terms of increased consumption—is quite limited for large N. This, in turn, leads to limitations on the total potential welfare gains.

A similar argument can be made for why, for a fixed high N, the welfare cost may be nonmonotone as a function of the investment horizon, T: initially θ is increasing in T since the higher T is the easier it is to make Agent 2 better off. For larger T however, it is not only very easy to make Agent 2 better off, since his expected realized utility is so low, but also difficult to make Agent 1 much better off, since even if he consumes the whole of C_t, the consumption increase is so lumped toward early time periods that his utility does not increase that much. Of course, the larger T is, the more "lumped" is the consumption increase given to Agent 1. Therefore, the welfare cost may be decreasing in T for large T.

3.3 POTENTIAL POLICY IMPLICATIONS

As shown, in the exchange economy setting the welfare costs of agent irrationality can be severe. In a production economy, in which the irrational agents would also affect output, the costs may be even higher, as may also be the case if the irrational agents' actions create systemic risk, potentially leading to economy-wide negative externalities. Our model is, of course, too stylized to allow for any definite conclusions, but it is suggestive about the presence of significant welfare costs and potential policy responses.

The root of the welfare costs is the suboptimal behavior by the unsophisticated investors in our model. The most straightforward approach would therefore be to educate these investors. This approach also avoids the potential critique of an *ex post* welfare measure, for example, that constraining agents' behavior may in itself dampen their well-being. The importance of financial education has been emphasized in the household finance literature; see Campbell (2006). However, Campbell also stresses that financial education alone may not be enough and that regulation may also be important. He writes: "As a financial educator, I am tempted to call for an expansion of financial education. However, academic finance may have more to offer by influencing consumer regulation, disclosure rules and the provision of investment default options . . . ."

So what is the appropriate regulatory response to the type of irrational behavior studied in this paper? One potential policy would be to support delegated management. Another policy would be to restrict the asset span. In our model, it is the market completeness that allows the irrational agents to invest in severely suboptimal portfolios. If the asset span were restricted to include only claims on aggregate consumption, together with a risk-free asset, that is, if the economy would have $N = 1$ risky assets, the welfare costs would be negligible (see Figure 4).[17] Thus, when irrational agents are present, the neoclassical view that financial innovation allows efficient risk sharing by completing markets—and thereby is welfare increasing—may be misleading and a restricted asset span may lead to higher welfare.

In the absence of other sources of agent heterogeneity than differences in the degree of rationality, only claims on aggregate consumption are needed and a severely restricted asset span is optimal. In practice, additional sources of heterogeneity exist, for example, because of agents' hedging motives for idiosyncratic risk and heterogeneous preferences. When determining the optimal asset span, a regulator would therefore have to weigh the benefits of risk sharing against the costs of suboptimal investments, the general rule being that unsophisticated investors may be restricted from trading in financial instruments on (disaggregated) idiosyncratic risks that offer no or few hedging benefits. An example of such a policy may be found in US hedge fund regulations. Only accredited investors, that is, investors with a minimum net worth of USD 1 million are allowed to invest in hedge funds, keeping poorer (less sophisticated) investors away from hedge funds' volatile returns, dynamic trading strategies, and investments in nontraditional asset classes.

4. Extensions

In this section, we show that our results are robust to several generalizations and variations.

4.1 DIFFERENT RISK AVERSION COEFFICIENTS

Although the analysis becomes less tractable, the irrational agent also underperforms severely when the investors have different risk aversion coefficients, γ_1 and γ_2, respectively. For example, it is possible to derive the following bound on the expected change in the log-consumption ratio, $E[h_t - h_0]$.

Proposition 8. *Given that the agents have risk aversion coefficients* γ_1 *and* γ_2 *and personal discount rates* ρ_1 *and* ρ_2*, respectively, in the constant coefficient economy*

[17] In other settings, finding an efficient regulatory policy when irrational agents are present is often difficult since the agents' responses to a well-intended regulation may actually lead to a worse outcome; see Salanie and Treich (2009) for an example.

$\mathcal{E} = (\delta, \mathbf{g}, \Sigma, \mathbf{D})$, define $\bar{g} = \max_i(\mathbf{g})_i$ and $\underline{g} = \min_i(\mathbf{g})_i$. The following inequalities hold for the expectation of the log-consumption ratio, $h_t = \log(c_{1t}/c_{2t})$:

$$E[h_t - h_0] \geq \begin{cases} \left(\frac{K}{2\gamma_2} + \frac{\rho_2-\rho_1}{\gamma_2} + \frac{\gamma_2-\gamma_1}{\gamma_2}\bar{g}\right)t + \frac{\gamma_2-\gamma_1}{\gamma_2}\log(1 + e^{-h_0}), & \gamma_1 > \gamma_2, \\ \left(\frac{K}{2\gamma_2} + \frac{\rho_2-\rho_1}{\gamma_2} + \frac{\gamma_2-\gamma_1}{\gamma_2}\underline{g}\right)t - \frac{\gamma_2-\gamma_1}{\gamma_2}\log(2) - O(e^{-qKt}), & \gamma_2 > \gamma_1. \end{cases} \tag{28}$$

Here $z = O(f(t))$ means that, for large t, $|z| \leq af(t)$, for some positive constant, a. The term $O(e^{-qKt})$ thus quickly becomes small as t grows, for large K.

Compared with Equation (17), which is valid when $\gamma_1 = \gamma_2$, when $\gamma_1 > \gamma_2$ there are two additional terms in this bound. The first additional term depends on the maximum growth rate, $\bar{g}$. This term appears because the difference in relative risk aversions of the two agents provides an additional motive for a transfer of consumption, beyond the heterogeneous probability measures. Specifically, since Agent 1's marginal utility decreases at a faster rate than Agent 2's when $\gamma_1 > \gamma_2$, a higher consumption is worth relatively more for Agent 2 than for Agent 1. Agent 2 is therefore willing to give up more of today's consumption for future consumption, so his current consumption is lower and his future consumption is higher compared with what they would be if his risk aversion coefficient was γ_1, that is, his consumption is expected to grow at a higher rate. This offsets the underperformance that is due to irrationality. The second additional term does not depend on t. In total, since the transfer index is large for large N, it will dominate the other terms and the irrational agent severely underperforms also when $\gamma_1 > \gamma_2$. A similar argument holds when $\gamma_2 > \gamma_1$. We note that Equation (28) generalizes Equation (18) in Yan (2008) in that it shows the rate at which market selection occurs and is also valid for multi-asset economies, $N > 1$.

Similar to what was shown for personal discount factors, when N is large, it follows immediately from Equation (28) that the differences in risk aversion need to be very large to offset the market selection generated by the transfer index. In our example in Section 3.1, with $N = 100$, Agent 1 needs a risk aversion coefficient of $\gamma_1 = 25.5$ to double the expected time it takes for him to reach 90% of the consumption share (from 17 to 34 years), when $\gamma_2 = 6$.

These results cannot be generalized to arbitrary utility functions outside of the CRRA class, as shown in Kogan *et al.* (2011). Specifically, the authors show that the irrational investor may survive in the long run when the investors have unbounded relative risk aversion. It is intuitively clear that the irrational investors may survive within our setting too, as seen in Equation (28): if the economy reaches a state where $\gamma_2 - \gamma_1 << 0$, then market selection may not occur even if K is large.

So, in an economy with varying risk aversion coefficients, in which K is large, we initially expect the irrational investor to severely underperform the rational one since K will initially dominate $\gamma_2 - \gamma_1$. However, if the utilities are such that eventually $\gamma_2 - \gamma_1 << 0$, then eventually the irrational investor's underperformance slows down and he survives—although with a very small consumption share.

4.2 SEVERAL INVESTORS

The generalization to economies with $L > 2$ investors is also straightforward. The following proposition generalizes Proposition 3 to economies with multiple investor groups.

Proposition 9. *In the economy with L investors, define the consumption share of investor i with respect to investor j,* $f_t \stackrel{\text{def}}{=} \frac{c_{it}}{c_{it}+c_{jt}}$, *the stopping time* $\tau_f \stackrel{\text{def}}{=} \inf_t\{t : f_t \ \geq f\}$, *and the constants* $K_i \stackrel{\text{def}}{=} \delta_i' \Sigma^{-1} \delta_i$, $K_j \stackrel{\text{def}}{=} \delta_j' \Sigma^{-1} \delta_j$, $K_{ij} \stackrel{\text{def}}{=} \delta_i' \Sigma^{-1} \delta_i$ c *If* $K_j > K_i$, *then*

$$E(\tau_f) = \frac{2\gamma v}{K_j - K_i + 2(\rho_j - \rho_i)} \tag{29}$$

and

$$\mathrm{Var}(\tau_f) = \frac{8\gamma v (K_i + K_j - 2K_{ij})}{(K_j - K_i + 2(\rho_j - \rho_i))^3}, \tag{30}$$

where $v = \log(\frac{f}{1-f}) - \log(\frac{f_0}{1-f_0})$.

It also immediately follows that the log-consumption ratio of investor i with respect to j, $h_t = \log(f_t/(1-f_t))$, is expected to grow as

$$E[h_t - h_0] = \left(\frac{1}{2\gamma}\left(K_j - K_i\right) + \frac{\rho_j - \rho_i}{\gamma}\right)t, \tag{31}$$

generalizing Equation (17) to the case with multiple investors who make different mistakes.

Thus, the severity of the underperformance of group j relative to i is decided by the difference of their two transfer indices, $K_j - K_i$, in the case with multiple investors.

4.3 BOTH AGENTS USE WRONG DRIFT TERMS—LEARNING

The assumption that Agent 1 knows the exact drift term is obviously very strong, but it is easy to show that similar results arise if we relax this assumption.

First, we use Proposition 9 to understand what happens if both investors are irrational (in the case with two investors) in that they both "stubbornly" mistake drift terms. As seen from Equation (29), in this case it is the difference between the two agents' transfer indexes that is important. For example, assume that Agent 1 makes similar mistakes as Agent 2 in the one-factor model of Section 3.1 but that his mistake in each stock is one half of Agent 2's, that is, $\frac{q}{2}$.

In this case, using a similar argument as the one leading to Equation (20), it follows that $K_2 - K_1 = \frac{g^2}{\sigma^2}(q^2 - \frac{q^2}{4})(N+1) = \frac{3}{4}K$, where K is the transfer index in the original example. Thus, the market selection is still fast in this case, although it is 25% slower than in the case when Agent 1 is exactly right about the drift term. The result can easily be extended to show that, under general conditions, fast market selection occurs when Agent 1 also makes mistakes about the drift terms, as long as these mistakes are "smaller" than the ones made by Agent 2.

It is also quite straightforward to extend the model to the case when Agent 1 is rational, but uncertain about drift terms, and learns about $\mathbf{g}$ by solving a Bayesian filtering problem.[18] For simplicity, we assume that Agent 1's beliefs about $\mathbf{g}$ at $t = 0$ are formed by observing ω_t for T years before trading begins (equivalently, we could assume that he observes, but does not participate in, the market between $-T \leq t \leq 0$). Specifically, his beliefs at $t = 0$ are given by solving a Bayesian filtering problem between $-T$ and 0, with a diffuse prior at $t = -T$.[19] At each point in time, Agent 1 then has a posterior belief about the true $\mathbf{g}$, as a normally distributed variable, with mean $\widehat{\mathbf{g}}_t = \mathbf{g} + \delta_t^1$, where $\delta_t^1 = \frac{1}{T}\sigma_\omega(\mathbf{B}_t - \mathbf{B}_{-T})$, and variance $\frac{1}{T+t}\Sigma$. We focus on the case when the agents have the same personal discount rate, $\rho_1 = \rho_2$.

In line with our previous analysis, we assume that Agent 2 does not update but stubbornly sticks to his initial estimate of the drift term. To focus on the effect of learning, we assume that Agent 2's initial estimate is the same as Agent 1's, $\delta_0^2 = \delta_0^1$. In practice, we would expect Agent 2 to have a larger error term since he would not be rational in forming his initial beliefs either. This would lead to even faster market selection.

As shown in the Appendix, the expected log-consumption ratio then develops in line with the following proposition.

[18] The detailed derivation of this case is provided in the Appendix.

[19] Similar results arise if we assume that his prior is not diffuse, but the formulas become more complicated.

Proposition 10. *The expected log-consumption ratio, when Agents* 1 *and* 2 *start with the same error from T periods of pre-learning,* $\delta_0^1 = \delta_0^2 = \frac{1}{T}\sigma_\omega(\mathbf{B}_0 - \mathbf{B}_{-T})$*, and where Agent 1 continues to update his beliefs, but Agent 2 does not, is*

$$E[h_t - h_0] = \frac{N}{2\gamma}\left(\frac{t}{T} - \log\left(1 + \frac{t}{T}\right)\right). \tag{32}$$

From Equation (32), it follows that again the market selection process scales linearly with N. For example, using our previous example from Section 3.1, with $N = 100$ stocks, $\gamma = 6$, and equal initial consumptions share of the two agents, and further assuming that the learning period is $T = 25$ years, it follows from Equation (32) that after 25 years, $E[h_{25}] = \frac{100}{2\times 6}(1 - \log(2)) = 2.56$, corresponding to a consumption share of 93%. Thus, the market selection process is almost as efficient in this case as in the case when Agent 1 knows the exact drift term in which case the rational agent's expected consumption ratio and consumption share are $E[h_{25}] = 3.25$ and 96%, respectively.

4.4 EXPLICITLY MODELING OVERCONFIDENCE

Our model for overconfidence so far has been in significantly reduced form. We have assumed that the irrational agent receives a signal that he is completely confident about, although in reality the signal is noisy. The rational agent has either received enough information to completely know the growth rate, as in the base model, or learns about it in a Bayesian fashion, as in the extension in the previous section. These assumptions have allowed for tractability in the analysis, but our results continue to hold when the analysis is extended to explicitly include dynamic updating of the beliefs of the irrational investor, using a standard approach to modeling overconfidence. We follow the assumptions made in Scheinkman and Xiong (2003) and further developed in Dumas, Kurshev, and Uppal (2009) closely.

Specifically, the state vector evolves according to Equation (2) as before, but we assume that expected growth is time varying and follows

$$\mathrm{d}\mathbf{g}_t = \alpha(\bar{\mathbf{g}} - \mathbf{g}_t)\mathrm{d}t + \sigma\,\mathrm{d}\mathbf{B}_t^g. \tag{33}$$

The agents in the economy observe ω_t but not $\mathbf{g}_t$. There is also an observable N-dimensional signal process, $\mathbf{s}_t$, that follows

$$\mathrm{d}\mathbf{s}_t = \sigma_s\,\mathrm{d}\mathbf{B}_t^s. \tag{34}$$

Here, all the Brownian motions are independent. Thus, the $\mathbf{s}$ process contains no information about $\mathbf{g}$ and elements of $\mathbf{s}$ are pairwise independent (generalizations to correlated signals are possible). The rational agent understands that the signal process is uninformative about $\mathbf{g}$. The irrational agent, on the other hand, is overconfident about the signal process and believes that it contains information about $\mathbf{g}$ through the following relation:

$$d\mathbf{s}_t = \phi\sigma_s \, d\mathbf{B}_t^g + \sqrt{1-\phi^2}\sigma_s \, d\mathbf{B}_t^s, \tag{35}$$

where $0 < \phi < 1$ is a constant. For simplicity, we assume that both agents know $\mathbf{g}_0$.

When extending the analysis to multiple assets, our main additional assumption compared with the previous literature is that the innovations in $\mathbf{s}$ across stocks are idiosyncratic (or at least not perfectly correlated). As discussed in Section 2.2, this idiosyncratic component could, for example, represent that the irrational agent wrongly believes that the age, accent, or fashion tastes of the CEO, the physical location of company headquarters, the sound of the company name, and so on are informative about the future prospects of the firm. As a specific example, the extensively documented home bias may make the irrational agent overestimate, or be overconfident about, the prospects of firms located in a geographical proximity. Our results may therefore apply to agents who live in regions where the density of firms is high.

We now have the following result, which is completely analogous to our previous results.

Proposition 11. *The expected log-consumption ratio when Agent 1 is rational and Agent 2 wrongly believes in the relation expressed in* Equation (35) *is bounded below by*

$$E[h_t - h_0] \geq \frac{CN}{2\gamma} t, \tag{36}$$

where $C > 0$ is a constant, that is, does not depend on N or t.

Thus, again, the result is a market selection process that scales linearly with N. The details of the analysis are given in the Appendix.

As an example, we use the same parameters as before, $g = 0.02$, $\sigma = 0.03218$, and $\gamma = 6$. We choose $\sigma_g = 0.005$, which leads to a roughly similar ratio between σ and σ_g as in Dumas, Krushev, and Uppal (2009), and vary ϕ between 0.25 (a low degree of overconfidence) and 0.95 (a very high degree of overconfidence). The results are shown in Table I. We see that the expected log-consumption ratio for Agent 1 increases linearly with N, in line with Proposition 11. Furthermore, the results for the intermediate case, when Agent 2 believes in "half" of the signal ($\phi = 0.5$) are quantitatively similar to our previous results. For example, the

Table I. Expected log-consumption share, $E[h_t]$ for Agent 1 after 25 years, with $N = 10, 50$, and 100 stocks, when the overconfidence parameter, ϕ, of Agent 2 varies between $\phi = 0.25$ and $\phi = 0.95$

	ϕ		
N	0.25	0.5	0.95
10	0.114	0.478	2.096
50	0.573	2.394	10.375
100	1.147	4.792	20.744

expected log-consumption ratio after 25 years with 100 stocks is 4.792, which is of similar size as in the base model, in which case it is 3.25.

Thus, our results continue to hold when the model is extended to explicitly account for overconfidence as a source of Agent 2's irrationality.

4.5 MISTAKES ABOUT THE COVARIANCE MATRIX

In line with the previous literature, we have focused on the irrationality of estimating drift terms. In practice, agents in the economy also need to estimate the variance–covariance matrix. Of course, in our continuous time setting, any disagreement about Σ immediately leads to an arbitrage opportunity; the agents would take different sides on arbitrarily large bets on the quadratic variation of the process under disagreement, and the irrational agent would immediately become infinitely indebted to the rational agent. The impossibility to mistake any variance or covariance term is an artifact of the continuous time model that is not reflected in practice.

If we go outside of the continuous time setting, it is straightforward to derive similar results when agents also mistake covariance terms. Specifically, in line with the drift term results, an agent who mistakes covariance terms and does not update his beliefs will severely underperform in a multi-asset market in this case too. In the Appendix, we introduce a discrete time version of the model, following the approach in Jouini and Napp (2006), and show that the same type of increasing underperformance with number of stocks as we have focused on so far also arise when mistakes are made about covariances.

We leave the details of the model to the Appendix, but in summary, the state of the world now evolves according to the discrete time random process

$$\omega_{t+1} = \omega_t + \mathbf{g} + \sigma_\omega \epsilon_{t+1},$$

where ϵ has a standard multivariate normal distribution, $\epsilon \sim N(0, I_N)$, and dividends [Equation (3)] and expected utility [Equation (5)] are now defined in discrete time. We focus on the case when the agents have the same personal discount rates.

A complete market is implemented by having many assets since dynamic replication with only $N+1$ assets is no longer possible. The two agents, $k \in \{1, 2\}$, mistake the drift term, $\mathbf{g}$, for $\mathbf{g}_k = \mathbf{g} + \delta_k$ and the variance–covariance matrix ($\Sigma = \sigma_\omega \sigma'_\omega$) for Σ_k.

The following result governs the consumption dynamics.

Proposition 12. *The expected log-consumption ratio, h_t, satisfies*

$$E[h_t - h_0] = \frac{1}{2\gamma}\left(\delta_2' \Sigma_2^{-1} \delta_2 - \delta_1' \Sigma_1^{-1} \delta_1 - \log\left(\frac{|\Sigma_1|}{|\Sigma_2|}\right) + \mathrm{tr}(\Sigma_2^{-1}\Sigma - \Sigma_1^{-1}\Sigma)\right) t. \quad (37)$$

Here, $|A|$ denotes the determinant and $\mathrm{tr}(A)$ *denotes the trace of a general square matrix, A, respectively.*

We see that Equation (37) contains two new terms compared with the original expression for the expected log-consumption ratio (Equation 31), representing the influence of irrationality about the variance–covariance matrix.

To see how irrationality about Σ affects the market selection process, we study three different scenarios. In the first scenario, Agent 1 knows both $\mathbf{g}$ and Σ, whereas Agent 2 makes mistakes about Σ. In the second scenario, Investor 1 still knows both the drift vector and the covariance matrix, whereas Agent 2 now makes mistakes about both $\mathbf{g}$ and Σ. Finally, in the third scenario, Agent 1 also makes mistakes about the drift vector and covariance matrix, although these mistakes are smaller than those made by Agent 2.

We choose the same parameters for the ω process (so that the process has the same distribution as the continuous time distribution, at the discrete points in time), as in Section 3.1, and (in scenarios 2 and 3) make the same assumptions about drift term mistakes for Agent 2. Thus, we assume a one-factor risk structure, with $g = 0.02$, $\sigma = 0.03218$, and irrationality parameter $q = 0.2$ for the drift term. We assume that the agents start with equal consumption shares.

To introduce mistakes about Σ, we go back to the one-factor structure, that is, we can write

$$(\omega_{t+1})_i = (\omega_t)_i + (\mathbf{g})_i + \beta^M \xi_t^M + \beta^i \xi_t^i, \quad (38)$$

where ξ_t^M is the market-wide shock and ξ_t^i the idiosyncratic shock to stock i, ξ_t^M and ξ_t^i are i.i.d. standard normal variables for all $1 \le i \le N$, $t = 1, 2, 3, \ldots$, the coefficients $\beta^M = \beta^i = \sigma\sqrt{\frac{k_N}{2}}$ for all i, and where $k_N = \frac{N}{N+1}$ is the scaling factor introduced to make the total risk independent of N. It immediately follows that this implies that the covariance matrix Σ has $(\Sigma)_{ii} = 2k_N\sigma^2$ and $(\Sigma)_{ij} = k_N\sigma^2$, in line with the one-factor model of Section 3.1. Now, along similar lines as when introducing drift term mistakes, we assume that Agent 2 mistakes β^M and β^i in

Equation (38) for $\widehat{\beta}^M = (1 + q\alpha^M/2)\beta^M$ and $\widehat{\beta}^i = (1 + q\alpha^i/2)\beta^i$, where α^M and α^i are i.i.d. random variables with a standard normal distribution. Thus, Agent 2 makes mistakes about the factor structure of the covariance matrix, of the same order of magnitude as the mistakes we introduced with respect to drift terms. In the third scenario, when Investor 1 also makes mistakes, we assume that his mistakes are similar to Investor 2's but that his error term is only half of Agent 2's, $q_1 = q_2/2 = 0.1$.

We calculate the expected log-consumption share of Agent 1 after 25 years in the three different scenarios, for $N = 10$, 50, and 100 stocks, using Monte-Carlo simulations for Equation (38). The results are shown in Table II. We see that the mistakes about Σ have very similar effects as mistakes about drift terms. In Scenario 1, the expected log-consumption ratio after 25 years with 100 stocks is 4.41, which is of similar size as when the agent makes mistakes about drift terms (in which case it is 3.25). In Scenario 2, when Agent 2 makes mistakes about both drifts and covariances, the mistakes basically add up, and $E[h_{25}] = 7.78$. Finally, in Scenario 3, when Agent 1 also makes mistakes about drifts and covariances, although only half as large as the mistakes made by Agent 2, $E[h_{25}]$ decreases by about 25% compared with Scenario 2, in line with the discussion in Section 4.3. Also along the lines of Section 4.3, the argument could be extended to include Bayesian learning about covariances, although the analysis would be much less tractable than the analysis of drift terms.

Thus, although we have focused on mistakes about drift terms in line with the previous literature, our argument that unsophisticated probabilistic behavior may be very costly for investors in markets with high-dimensional risk structures is not restricted to such mistakes.

4.6 GENERAL RISK STRUCTURES

So far we have focused on a textbook style economy with one systematic source of risk that affects all firms equally, symmetric idiosyncratic risk, and an irrational investor with symmetric sentiments. Our results are much more general, however. For example, the irrational investor does not need to be symmetric in his

Table II. Expected log-consumption share, $E[h_t]$ for Agent 1 after 25 years, with $N = 10, 50$, and 100 stocks, in three different scenarios. In Scenario 1, Agent 2 mistakes covariances; in Scenario 2, he mistakes both drifts and covariances; and in Scenario 3, Agent 1 also mistakes drifts and covariances

	Scenario		
N	1	2	3
10	0.403	0.764	0.577
50	2.18	3.87	2.94
100	4.41	7.78	5.91

sentiments. To see this, consider the economy with the same growth rate and risk as in Section 3.1 but with asymmetric sentiments. Specifically, the irrational investor believes that the drift is $g(1+q_1)$ for a fraction, a, of the stocks and $g(1+q_2)$ for the remaining stocks, where we assume that aN and $(1-a)N$ are both integers. Without loss of generality, we assume that $0<a<1$ (as the cases $a=0$ and $a=1$ are covered by taking $q_1=q_2$).

From the definition of the transfer index, K, in Proposition 2, it is straightforward to show that

$$K=\frac{g^2}{\sigma^2}(q_1^2a+q_2^2(1-a)+(q_1^2a+q_2^2(1-a)-(q_1a+q_2(1-a))^2)\times N).$$

Thus, by Proposition 3, the underperformance of the irrational investor will be severe for large N, unless the term $q_1^2a+q_2^2(1-a)-(q_1a+q_2(1-a))^2$ is equal to zero. It is easy to show that this term is equal to zero if and only if $q_1=q_2$. If $q_1=q_2$, then $K=\frac{g^2}{\sigma^2}q_1^2$ so the irrational investor's underperformance does not depend on N. Effectively, since his sentiment is uniform, he holds the same portfolio as the rational agent, and the model collapses to the representative firm model.

To study the underperformance in an even more general setting, we introduce a sequence of economies $\mathcal{M}=(\mathcal{E}_1,\ldots,\mathcal{E}_N,\ldots)$, where $\mathcal{E}_N=(\delta_N,\mathbf{g}_N,\Sigma_N,\mathbf{D}_N)$. The idea is now to see if the irrational investor's underperformance becomes severe as N tends to infinity, in the sense that the expected time to reach any prescribed consumption share approaches zero. In this case, we say that high-speed market selection occurs for large N. Formally, for a sequence of economies, $\mathcal{M}$, we have the following definitions.

Definition 2. *High-speed market selection occurs if, in market $\mathcal{E}_N$, the expected time to reach the consumption share f when the initial consumption share is f_0 satisfies $E(\tau_f)\le G(f_0,f,N)$ for some function $G:(0,1)\times(0,1)\times\{1,2,3,\ldots\}\to\mathbb{R}_+$, which for all f_0 and $f>f_0$ satisfies*

$$\lim_{N\to\infty}G(f_0,f,N)=0.$$

Definition 3. *High-speed market selection of order v (where $v>0$ is a constant) occurs if the function, G, in Definition 2 can be written in the form $G(f_0,f,N)=H(f_0,f)/N^v$.*

We let K_N denote the transfer index term in economy $\mathcal{E}_N$. Proposition 3 implies that high-speed market selection of order v occurs if and only if

$$k\stackrel{\text{def}}{=}\liminf_{N\to\infty}\frac{K_N}{N^v},\quad\text{where } K_N=\delta_N'\Sigma_N^{-1}\delta_N,$$

is greater than zero, that is, if and only if $0<k\le\infty$.

We define $\rho(\Sigma)$ to be the spectral radius of the covariance matrix, Σ.[20] We have a couple of immediate results, relating the spectral radius of the covariance matrix, $\rho(\Sigma)$, the sentiment, δ, and the transfer index, K. From Equations (18), (20), (21), it follows that high-speed market selection of order 1 occurs in the example in Section 3.1. The following two propositions can be used to show high-speed market selection in more general economies:

Proposition 13. *For a sequence of economies, $\mathcal{M}$, high-speed market selection of order v occurs if there are positive constants c and N_0 such that for all $N > N_0$: $\rho(\Sigma_N) \leq cN^{-v}\delta_N'\delta_N$.*

Proposition 14. *For an arbitrary vector, $\mathbf{q}$, define $Q_{\mathbf{q}}$ to be the Euclidean projection operator onto the orthogonal complement of $\mathbf{q}$ and the N-vector of ones, $\mathbf{1}_N = (1, 1, \ldots, 1)^T$.[21] For a sequence of economies, $\mathcal{M}$, high-speed market selection of order v occurs if there are positive constants c and N_0 such that the following two conditions are satisfied for all $N > N_0$:*

- *$\mathbf{1}_N$ is an eigenvector of Σ_N,*
- *$\rho(Q'_{1_N}\Sigma_N Q_{1_N}) \leq cN^{-v}(\delta_N'\delta_N - \frac{(1_N'\delta_N)^2}{N})$.*

It is straightforward to check that both the example in Section 3.1 and the example with asymmetric sentiments in this section satisfy the conditions of Proposition 14 with $v = 1$, with the exception of asymmetric sentiments with $q_1 = q_2$.[22] We can also use the proposition to study economies with general variance-covariance matrices of the form $\Sigma_N = N^{-1}(aI_N + b\mathbf{1}_N\mathbf{1}_N')/(a + bN)$, $a > 0$, $b \geq 0$, where I_N is the $N \times N$ identity matrix. For such covariance matrices, the first condition of Proposition 14 is always satisfied. Moreover, the second condition is satisfied with $v = 1$, as long as $b > 0$, that is, as long as the economy has a systematic risk component. If there is no systematic component, there are effectively N separate financial markets, and in each of these markets there is a representative firm. It is easy to show that the market selection process will be slow in this case.

4.7 SEQUENCES OF RANDOM ECONOMIES

Propositions 13 and 14 can be used to prove high-speed market selection for a specific sequence of markets but do not say how "often" high-speed market selection occurs. Is high-speed market selection the norm or are the previous examples just exceptional special cases? To answer this question, we study how often high-speed

[20] Since Σ is symmetric and positive definite, its spectral radius is simply its largest eigenvalue. We use the standard notation $\rho(\Sigma)$ since it should create no confusion with the personal discount rate, ρ.

[21] That is, in matrix notation, $Q_{\mathbf{q}} = I - \frac{\mathbf{q}\mathbf{q}'}{\mathbf{q}'\mathbf{q}}$, where I is the identity matrix.

[22] If $q_1 = q_2$, the second condition of Proposition 14 fails since the right-hand side is identically equal to zero.

market selection occurs in randomly generated economies. We look at economies with one systematic risk component affecting all firms and random parameters. In the Appendix, we discuss the generalization of the results to $Q > 1$ common risk components.

We make several assumptions about the randomness of the economies, $\mathcal{E}_N = (\delta_N, \mathbf{g}_N, \Sigma_N, \mathbf{D}_N)$. We assume that $(\mathbf{g}_N)_i = \tilde{p}_i^N$, where $\tilde{p}_i^N$ are i.i.d. random variables, $E(\tilde{p}_1^1) = \bar{p} > 0$ and $\mathrm{Var}(\tilde{p}_1^1) = \sigma_p^2 > 0$. Similarly, $(\delta_N)_i = \tilde{q}_i^N$, where $\tilde{q}_i^N$ are i.i.d. random variables, $E(\tilde{q}_1^1) = \bar{q}$ and $\mathrm{Var}(\tilde{q}_1^1) = \sigma_q^2 > 0$. Furthermore, we assume that the randomness of the ith asset, $\sigma_i \, dB_{it}$, is of the form $\sigma_i \, dB_{it} = (\beta_i^N \, d\xi_{0t} + \alpha_i^N \, d\xi_{it}^N)$, where ξ_{it} are i.i.d. standard Brownian motions and α_i^N and β_i^N are i.i.d. random variables: $E(\alpha_1^1) = \bar{\alpha} > 0$, $\mathrm{Var}(\alpha_1^1) = \sigma_\alpha^2 > 0$, $E(\beta_1^1) = \bar{\beta}$, and $\mathrm{Var}(\beta_1^1) = \sigma_\beta^2 > 0$. All random variables, $\tilde{p}_i^N$, $\tilde{q}_i^N$, β_i^N, α_i^N, ξ_{it}^N are jointly independent. For simplicity, we furthermore assume that all random variables are absolutely continuous (with respect to Lebesgue measure) and that the β's are (a.s.) bounded below by a strictly positive constant, $\epsilon > 0$.[23] We also require the $\tilde{p}$'s to be strictly positive (a.s.).

For a fixed N, the economy $\mathcal{E}_N$ will thus be characterized by $\mathbf{g}_N = (\tilde{p}_1^N, \ldots, \tilde{p}_N^N)'$, $\delta_N = (\tilde{q}_1^N, \ldots, \tilde{q}_N^N)'$, $\Sigma_N = (\mathrm{diag}(\alpha_1^N, \ldots, \alpha_N^N)^2 + \mathbf{b}_N \mathbf{b}_N')$, where $\mathbf{b}_N = (\beta_1^N, \ldots, \beta_N^N)'$ and the $\mathbf{D}_N$'s are arbitrary weakly positive vectors, with at least one nonzero element. Under these conditions, we have the following proposition.

Proposition 15. *In a sequence of economies, $\mathcal{M} = (\mathcal{E}_1, \mathcal{E}_2, \ldots)$, satisfying the previous assumptions, high-speed market selection of order 1 occurs almost surely.*

Thus, high-speed market selection is really the norm in such economies and the exception is when it breaks down.

From the discussions in this and the previous section, it is clear that as long as it is possible for the rational investor to "diversify" over the irrational investor's mistakes across assets, the market selection process will be fast. Specifically, as long as the risk processes are dependent, and the sentiment vector of the irrational investor has a dispersion across stocks, the market selection process will be fast, both in a deterministic economy, as in Section 4.6, and in a random economy, as in this section. When the risk processes are independent or the sentiment of the irrational investor is the same for all stocks, on the other hand, there is no opportunity for the rational agent to diversify across the irrational agent's mistakes, and the results collapse to those of a one-asset model (as follows from the discussion subsequent to Proposition 14).

[23] These assumptions can be relaxed in several directions but at the expense of increased complexity.

5. Concluding Remarks

As follows from our analysis, irrational agents who make probabilistic mistakes that *a priori* seem to be very small may be severely punished in stock markets with high-dimensional risk structures. In theory, stock markets may indeed be as dangerous as derivative markets, even though they offer no pure arbitrage opportunities. In our calibration, an irrational investor loses almost 95% of his consumption and wealth shares to a rational investor over a 25-year horizon. Moreover, the realized consumption paths of the two investors are severely suboptimal. The *ex post* welfare costs are about 40% of the total wealth in the economy in a 25-year horizon. Our results therefore highlight the value of financial education and also suggest that delegated investment management, as well as restrictions on the asset span in the market, under some circumstances may be welfare increasing when unsophisticated investors are present.

Although, for simplicity, strong assumptions were made in our base model, our results are robust to several extensions and generalizations. The results also hold when agents have different personal discount rates and risk aversion, when rational agents learn about the parameters of the economy, when agents also mistake covariance terms, and for general risk structures. This suggests that severe underperformance by unsophisticated investors may be the norm rather than the exception in markets with high-dimensional risk structures.

Appendix A

A.1 PROOFS

Proof of Proposition 1: To solve for equilibrium, we use the martingale approach (Karatzas, Lehoczky, and Shreve, 1987; Cox and Huang, 1989). Each agent solves the static optimization problem

$$\max_{c_k} E_k\left[\int_0^T e^{-\rho_k t}\frac{c_{k,t}^{1-\gamma}}{1-\gamma}\mathrm{d}t\right] \tag{A.1}$$

s.t.

$$E_k\left[\int_0^T \xi_{k,t}c_{k,t}\,\mathrm{d}t\right] \le f_{W_k,0}E_k\left[\int_0^T \xi_{k,t}C_t\,\mathrm{d}t\right], \tag{A.2}$$

where $f_{W_k,0}$ is the initial wealth fraction of agent k and $\xi_{k,t}$, $k \in \{1,2\}$, are the agent-specific Stochastic Discount Factors (SDFs), yet to be defined. Necessary and sufficient conditions for optimality of the consumption streams are

$$c_{k,t} = \left(y_k\, e^{\rho_k t}\xi_{k,t}\right)^{-\frac{1}{\gamma}}, \tag{A.3}$$

where $y_k > 0$ is such that the budget constraint holds with equality

$$E_k\left[\int_0^T \xi_{k,t}\left(y_k\,\mathrm{e}^{\rho_k t}\xi_{k,t}\right)^{-\frac{1}{\gamma}}\mathrm{d}t\right] = f_{W_k,0}E_k\left[\int_0^T \xi_{k,t}C_t\,\mathrm{d}t\right]. \tag{A.4}$$

To solve for the optimal consumption streams of the two agents, we introduce the central planner's problem

$$u(C_t,\lambda_t,t) = \max_{c_{1,t},c_{2,t}}\left\{\mathrm{e}^{-\rho_1 t}\frac{c_{1,t}^{1-\gamma}}{1-\gamma} + \lambda_t\,\mathrm{e}^{-\rho_2 t}\frac{c_{2,t}^{1-\gamma}}{1-\gamma}\right\} \tag{A.5}$$

s.t.

$$c_{1,t} + c_{2,t} = C_t, \tag{A.6}$$

where

$$\begin{aligned}\lambda_t &= \left(\frac{y_1\xi_{1,t}}{y_2\xi_{2,t}}\right)\\ &= \lambda_0\eta_t.\end{aligned} \tag{A.7}$$

Here, $\lambda_0 = \frac{y_1}{y_2}$ as $\xi_{1,0} = \xi_{2,0} = 1$ and $\eta_t = \exp(-\frac{1}{2}\int_0^t\Delta_s'\Delta_s\,\mathrm{d}s + \int_0^t\Delta_s'\,\mathrm{d}\mathbf{B}_s)$. From the first-order conditions of the central planner's problem in Equation (A.5), we have

$$c_{2,t} = \mathrm{e}^{(\rho_1-\rho_2)t/\gamma}\lambda_t^{\frac{1}{\gamma}}c_{1,t}. \tag{A.8}$$

Using Equations (A.6) and (A.8), we get

$$c_{1,t} + \mathrm{e}^{(\rho_1-\rho_2)t/\gamma}\lambda_t^{\frac{1}{\gamma}}c_{1,t} = C_t. \tag{A.9}$$

Rearranging, we get

$$\begin{aligned}c_{1,t} &= f_tC_t,\\ c_{2,t} &= (1-f_t)C_t,\end{aligned} \tag{A.10}$$

where $f_t = \frac{1}{1+\mathrm{e}^{(\rho_1-\rho_2)t/\gamma}\lambda_t^{\frac{1}{\gamma}}}$. Finally, solving Equation (A.3) for $\xi_{1,t}$, we get

$$\begin{aligned}\xi_{1,t} &= \mathrm{e}^{-\rho_1 t}\left(\frac{c_{1,t}}{c_{1,0}}\right)^{-\gamma}\\ &= \mathrm{e}^{-\rho_1 t}\left(\frac{f_t}{f_0}\right)^{-\gamma}\left(\frac{C_t}{C_0}\right)^{-\gamma}.\end{aligned} \tag{A.11}$$

The expressions for the wealth of the two agents follow from discounting future optimal consumption using the SDF.

Proof of Proposition 2: From the optimal consumption allocations in Proposition 1, we have

$$\begin{aligned} h_t &= \log\left(\frac{c_{1,t}}{c_{2,t}}\right) \\ &= \log\left(\mathrm{e}^{-(\rho_1-\rho_2)t/\gamma}\lambda_t^{-\frac{1}{\gamma}}\right) \\ &= -\frac{1}{\gamma}\log(e^{(\rho_1-\rho_2)t}\lambda_t), \quad \text{so} \\ \mathrm{d}h_t &= \left(\frac{1}{2\gamma}\Delta'\Delta + \frac{1}{\gamma}(\rho_2-\rho_1)\right)\mathrm{d}t + \frac{1}{\gamma}\Delta'\,\mathrm{d}\mathbf{B}_t \\ &= \left(\frac{1}{2\gamma}K + \frac{1}{\gamma}(\rho_2-\rho_1)\right)\mathrm{d}t + \frac{1}{\gamma}\sqrt{K}\,\mathrm{d}\tilde{B}_t, \end{aligned}$$

where $K = \delta'\Sigma^{-1}\delta$ and $\mathrm{d}\tilde{B}_t = -\frac{1}{\sqrt{K}}\Delta'\,\mathrm{d}\mathbf{B}_t$ is a standardized Brownian motion.

Proof of Proposition 3: The first passage probability density distribution for the time it takes for $\log(\frac{f_t}{1-f_t})$ to reach $\log(\frac{f}{1-f})$ with initial condition $\log(\frac{f_0}{1-f_0})$ is (Ingersoll, 1987)

$$\text{p.d.f.}(\tau_f) = \frac{\log\left(\frac{f}{1-f}\right) - \log\left(\frac{f_0}{1-f_0}\right)}{\left(2\pi\frac{K}{\gamma^2}t^3\right)^{1/2}}\exp\left[-\left(\log\left(\frac{f}{1-f}\right) - \log\left(\frac{f_0}{1-f_0}\right) - \left(\frac{1}{2\gamma}K + \frac{1}{\gamma}(\rho_2-\rho_1)\right)t\right)^2 \Big/ \left(2\frac{K}{\gamma^2}t\right)\right]. \tag{A.12}$$

The expected time is (see Ingersoll, 1987, p. 354)

$$E(\tau_f) = \frac{2\gamma\left(\log\left(\frac{f}{1-f}\right) - \log\left(\frac{f_0}{1-f_0}\right)\right)}{K + 2(\rho_2-\rho_1)}, \tag{A.13}$$

and the variance is

$$\mathrm{Var}(\tau_f) = \frac{8\gamma K\left(\log\left(\frac{f}{1-f}\right) - \log\left(\frac{f_0}{1-f_0}\right)\right)}{(K + 2(\rho_2-\rho_1))^3}. \tag{A.14}$$

Proof of Proposition 4: The result follows from the fact that the wealth–consumption ratio is constant for investors with logarithmic preferences.

Proof of Proposition 5: For generality, we prove the proposition for the case when the personal discount factors are agent specific. From Equation (14),

$$\begin{aligned}
W_{1,t} &= E_t\left[\int_t^T \frac{\xi_s}{\xi_t} f_s C_s \, \mathrm{d}s\right] \\
&= \frac{C_t}{\left(1+\mathrm{e}^{(\rho_1-\rho_2)t/\gamma}\lambda_t^{\frac{1}{\gamma}}\right)^{\gamma}} \int_t^T \mathrm{e}^{-\rho_1(s-t)} E_t\left[\left(1 + \mathrm{e}^{(\rho_1-\rho_2)s/\gamma}\lambda_s^{\frac{1}{\gamma}}\right)^{\gamma-1}\left(\frac{C_s}{C_t}\right)^{1-\gamma}\right]\mathrm{d}s \\
&= \frac{C_t}{\left(1+\mathrm{e}^{(\rho_1-\rho_2)t/\gamma}\lambda_t^{\frac{1}{\gamma}}\right)^{\gamma}} \int_t^T \mathrm{e}^{-\rho_1(s-t)} \sum_{k=0}^{\gamma-1}\binom{\gamma-1}{k}\mathrm{e}^{(\rho_1-\rho_2)sk/\gamma} E_t\left[\lambda_s^{\frac{k}{\gamma}}\left(\frac{C_s}{C_t}\right)^{1-\gamma}\right]\mathrm{d}s \\
&= \frac{C_t}{\left(1+\mathrm{e}^{(\rho_1-\rho_2)t/\gamma}\lambda_t^{\frac{1}{\gamma}}\right)^{\gamma}} \int_t^T \mathrm{e}^{-\rho_1(s-t)} \sum_{k=0}^{\gamma-1}\binom{\gamma-1}{k}\mathrm{e}^{(\rho_1-\rho_2)sk/\gamma}\lambda_t^{\frac{k}{\gamma}} E_t\left[\left(\frac{\lambda_s}{\lambda_t}\right)^{\frac{k}{\lambda}}\left(\frac{C_s}{C_t}\right)^{1-\gamma}\right]\mathrm{d}s.
\end{aligned} \tag{A.15}$$

Note that since $(\frac{\lambda_s}{\lambda_t})^{\frac{k}{\gamma}}$ and $(\frac{C_s}{C_t})^{1-\gamma}$ are jointly lognormal, we get $E_t\big[(\frac{\lambda_s}{\lambda_t})^{\frac{k}{\gamma}}(\frac{C_s}{C_t})^{1-\gamma}\big] = \mathrm{e}^{A_k(s-t)}$, where $A_k = \frac{k}{2\gamma}(\frac{k}{\gamma}-1)\Delta'\Delta + (1-\gamma)(g_1 + \frac{1-\gamma}{2}\sigma'_{\omega_1}\sigma_{\omega_1} + \frac{k}{\gamma}\sigma'_{\omega_1}\Delta)$. Inserting into Equation (A.15) and solving the integral, we get

$$\begin{aligned}
W_{1,t} &= \frac{C_t}{\left(1 + \mathrm{e}^{(\rho_1-\rho_2)t/\gamma}\lambda_t^{\frac{1}{\gamma}}\right)} \sum_{k=0}^{\gamma-1}\binom{\gamma-1}{k}\mathrm{e}^{(\rho_1-\rho_2)t/\gamma}\lambda_t^{\frac{k}{\gamma}} \frac{1}{\alpha_k - (\rho_1-\rho_2)k/\gamma} \\
&\quad \times\left(1 - \mathrm{e}^{-(\alpha_k-(\rho_1-\rho_2)k/\gamma)(T-t)}\right),
\end{aligned}$$

where $\alpha_k = \rho_1 - A_k$. Following a similar calculation, we get

$$\begin{aligned}
W_{2,t} &= \frac{C_t}{\left(1 + \mathrm{e}^{(\rho_1-\rho_2)t/\gamma}\lambda_t^{\frac{1}{\gamma}}\right)^{\gamma}} \sum_{k=0}^{\gamma-1}\binom{\gamma-1}{k}\mathrm{e}^{(\rho_1-\rho_2)t(k+1)/\gamma}\lambda_t^{\frac{k+1}{\gamma}} \\
&\quad \times \frac{1}{\alpha_{k+1} - (\rho_1-\rho_2)(k+1)/\gamma}\left(1 - \mathrm{e}^{-(\alpha_{k+1}-(\rho_1-\rho_2)(k+1)/\gamma)(T-t)}\right).
\end{aligned}$$

Proof of Proposition 7: Using a similar approach as in the proof of Proposition 5, the objective expected utility of Agents 1 and 2 can be written as

$$\begin{aligned}
U_1^{\mathrm{OBJ}} &= E\left[\int_0^T \mathrm{e}^{-\rho_1 t}\frac{c_{1,t}^{1-\gamma}}{1-\gamma}\mathrm{d}t\right] \\
&= \frac{1}{1-\gamma}\int_0^T \mathrm{e}^{-\rho_1 t} E\left[C_t^{1-\gamma}\left(1 + \lambda_t^{\frac{1}{\gamma}}\right)^{\gamma-1}\right]\mathrm{d}t \\
&= \frac{C_0^{1-\gamma}}{1-\gamma}\sum_{k=0}^{\gamma-1}\binom{\gamma-1}{k}\lambda_0^{\frac{k}{\gamma}}\frac{1}{\alpha_k}\left(1 - \mathrm{e}^{-\alpha_k T}\right)
\end{aligned}$$

and

$$U_2^{\mathrm{OBJ}} = \frac{C_0^{1-\gamma}}{1-\gamma}\sum_{k=0}^{\gamma-1}\binom{\gamma-1}{k}\lambda_0^{\frac{k+1-\gamma}{\gamma}}\frac{1}{\alpha_{k+1-\gamma}}(1-e^{-\alpha_{k+1-\gamma}T}).$$

Using the above expressions together with Equation (26), we get

$$\beta = \frac{\left(\frac{1}{\alpha_0}(1-\mathrm{e}^{-\alpha_0 T})\right)^{\frac{1}{1-\gamma}}}{\left(\sum_{k=0}^{\gamma-1}\binom{\gamma-1}{k}\lambda_0^{\frac{k}{\gamma}}\frac{1}{\alpha_k}(1-\mathrm{e}^{-\alpha_k T})\right)^{\frac{1}{1-\gamma}} + \left(\sum_{k=0}^{\gamma-1}\binom{\gamma-1}{k}\lambda_0^{\frac{k+1-\gamma}{\gamma}}\frac{1}{\alpha_{k+1-\gamma}}(1-\mathrm{e}^{-\alpha_{k+1-\gamma}T})\right)^{\frac{1}{1-\gamma}}},$$

and since $\theta = 1 - \frac{1}{\beta}$, Equation (27) follows.

Proof of Proposition 8: Assume that $\gamma_1 > \gamma_2$. From the agents' first-order conditions, it follows that

$$\frac{(c_{1t})^{-\gamma_1}}{(c_{2t})^{-\gamma_2}} = \mathrm{e}^{-(\rho_2-\rho_1)t}\lambda_t,$$

and therefore,

$$\frac{c_{1t}^{\gamma_1/\gamma_2}}{c_{2t}} = \mathrm{e}^{(\rho_2-\rho_1)t/\gamma_2}\lambda_t^{-1/\gamma_2} \stackrel{\mathrm{def}}{=} q_t.$$

Now, $c_{1t}^{\gamma_1/\gamma_2-1} = C_t^{\gamma_1/\gamma_2-1}(1+c_{2t}/c_{1t})^{1-\gamma_1/\gamma_2}$, so we have

$$q_t = \frac{c_{1t}^{\gamma_1/\gamma_2}}{c_{2t}} = \frac{c_{1t}}{c_{2t}}C_t^{\gamma_1/\gamma_2-1}(1 + c_{2t}/c_{1t})^{1-\gamma_1/\gamma_2},$$

so

$$\frac{c_{1t}}{c_{2t}}(1 + c_{2t}/c_{1t})^{1-\gamma_1/\gamma_2} = q_t C_t^{1-\gamma_1/\gamma_2}, \tag{A.16}$$

and therefore,

$$\left(\frac{c_{1t}/c_{2t}}{c_{10}/c_{20}}\right)\frac{(1 + c_{2t}/c_{1t})^{1-\gamma_1/\gamma_2}}{(1 + c_{20}/c_{10})^{1-\gamma_1/\gamma_2}} = \left(\frac{q_t}{q_0}\right)\left(\frac{C_t}{C_0}\right)^{1-\gamma_1/\gamma_2}.$$

Since $\gamma_1 > \gamma_2$, it follows that $(1+c_{2t}/c_{1t})^{1-\gamma_1/\gamma_2} \leq 1$, and therefore,

$$\left(\frac{c_{1t}/c_{2t}}{c_{10}/c_{20}}\right)\frac{1}{(1 + c_{20}/c_{10})^{1-\gamma_1/\gamma_2}} \geq \left(\frac{q_t}{q_0}\right)\left(\frac{C_t}{C_0}\right)^{1-\gamma_1/\gamma_2}.$$

Taking logarithms on both sides leads to

$$h_t - h_0 - \frac{\gamma_2 - \gamma_1}{\gamma_2}\log(1 + e^{-h_0}) \geq \log\left(\frac{q_t}{q_0}\right) + \frac{\gamma_2 - \gamma_1}{\gamma_2}\log\left(\frac{C_t}{C_0}\right).$$

Taking expectations on both sides and rearranging, using the method in the proof of Proposition 2 for q_t and that $E[\log(\frac{C_t}{C_0})] \leq \bar{g}t$, the result follows.

Now, assume that $\gamma_2 > \gamma_1$. Define $z_t = \mathrm{e}^{h_t}$, $y_t = \log(q_t) + (1 - \gamma_1/\gamma_2)\log(C_t)$, and $\alpha = \gamma_1/\gamma_2 < 1$. From Equation (A.16), it follows that $z_t^{\alpha}(1 + z_t)^{1-\alpha} = \mathrm{e}^{y_t}$, so, taking logarithms on both sides,

$$y_t = \alpha h_t + (1 - \alpha)\log(1 + e^{h_t}).$$

It is easily seen that y is a strictly convex function of h such that $y'(-\infty) = \alpha$, $y'(\infty) = 1$, and $y(0) = (1 - \alpha)\log(2)$. Therefore, $h = f(y)$ where f is a strictly concave function such that $f'(-\infty) = \frac{1}{\alpha}$, $f'(\infty) = 1$, and $f((1 - \alpha)\log(2)) = 0$.

It therefore follows that $h = f(y) \geq y - (1 - \alpha)\log(2) + (\frac{1}{\alpha} - 1)(y - (1 - \alpha)\log(2))_-$, where $(x)_-$ defines a function of x that is equal to x when $x < 0$ and to 0 when $x \geq 0$.

Now, since $y_t = \log(q_t) + (1 - \gamma_1/\gamma_2)\log(C_t)$, using the same approach as in Proposition 3, it follows that $(y_t - (1 - \alpha)\log(2))_-$ first order stochastically dominates $(v_t - (1 - \alpha)\log(2))_-$, where

$$v_t \sim N\left(\log(q_0) + (1 - \gamma_1/\gamma_2)\log(C_0) + \left(\frac{K}{2\gamma_2} + \frac{\rho_2 - \rho_1}{\gamma_2}\right)t, 4\frac{K}{\gamma^2}t\right),$$

so $E[(y_t - (1 - \alpha)\log(2))_-] \geq E[(v_t - (1 - \alpha)\log(2))_-]$.

It is easy to show that for a random variable, $x \sim N(\mu, \sigma^2)$, and constant $\beta < \mu$, it is the case that $E[(x - \beta)_-] = O(\frac{\sigma^3}{(\mu-\beta)^2}\mathrm{e}^{-\frac{(\mu-\beta)^2}{2\sigma^2}})$, from which it follows that $E[(v_t - (1 - \alpha)\log(2))_-] = -O(K^{-1/2}\mathrm{e}^{-qKt}) = -O(\mathrm{e}^{-qKt})$. Therefore, $E[h_t] \geq E[y_t] - (1 - \alpha)\log(2) - O(\mathrm{e}^{-qKt})$.

Moreover, from Equation (A.16), it is clear that $h_0 + (1 - \gamma_1/\gamma_2)\log(1 + \mathrm{e}^{-h_0}) = y_0$, so $h_0 \leq y_0$. Together these two inequalities imply that

$$E[h_t - h_0] \geq E[y_t - y_0] - (1 - \alpha)\log(2) - O(\mathrm{e}^{-cKt}),$$

and since

$$\begin{aligned} E[y_t - y_0] &= \frac{-1}{\gamma_2}E\left[\log\left(\frac{q_t}{q_0}\right)\right] + \frac{\gamma_2 - \gamma_1}{\gamma_2}E\left[\log\left(\frac{C_t}{C_0}\right)\right] \\ &\geq \frac{1}{\gamma_2}\left(\frac{K}{2} + \rho_2 - \rho_1\right)t + \frac{\gamma_2 - \gamma_1}{\gamma_2}\underline{g}t, \end{aligned}$$

the result follows.

Proof of Proposition 9: From the agents' first-order conditions, it follows that

$$\frac{c_{i,t}}{c_{j,t}} = \left(\frac{\lambda_{i,t}}{\lambda_{j,t}}\right)^{\frac{1}{\gamma}} \mathrm{e}^{(\rho_j-\rho_i)t/\gamma}.$$

Consequently, we have that

$$\begin{aligned}\log\left(\tfrac{c_{i,t}}{c_{j,t}}\right) &= \left(\frac{1}{2\gamma}\left(K_j - K_i\right) + \frac{1}{\gamma}\left(\rho_j - \rho_i\right)\right)t + \frac{1}{\gamma}(\Delta_i - \Delta_j)'\mathbf{B}_t \\ &= \left(\frac{1}{2\gamma}\left(K_j - K_i\right) + \frac{1}{\gamma}\left(\rho_j - \rho_i\right)\right)t + \frac{1}{\gamma}\sqrt{(K_i + K_j - 2K_{ij})}\tilde{B}_t,\end{aligned}$$

where $\tilde{B}_t = \frac{1}{\sqrt{(K_i+K_j-2K_{ij})}}(\Delta_i - \Delta_j)'\mathbf{B}_t$ is a standard Brownian motion. Using the same approach as in the proof of Proposition 3, we get

$$E(\tau_f) = \frac{2\gamma v}{K_j - K_i + 2\left(\rho_j - \rho_i\right)}$$

and

$$\mathrm{Var}(\tau_f) = \frac{8\gamma v(K_i + K_j - 2K_{ij})}{\left(K_j - K_i + 2\left(\rho_j - \rho_i\right)\right)^3},$$

where $v = \log(\frac{f}{1-f}) - \log(\frac{f_0}{1-f_0})$.

Proof of Proposition 10: From Proposition 16, we have that

$$h_t - h_0 = \frac{1}{2\gamma}\int_0^t \left(K_{2,s} - K_{1,s}\right) \mathrm{d}s + \frac{1}{\gamma}\int_0^t \left(\Delta_{2,s} - \Delta_{1,s}\right)' \mathrm{d}\mathbf{B}_s. \tag{A.17}$$

Taking expectation on both sides of Equation (A.17),

$$\begin{aligned}E(h_t - h_0) &= \tfrac{1}{2\gamma}\textstyle\int_0^t E\left(K_{2,s} - K_{1,s}\right)\mathrm{d}s \\ &= \tfrac{1}{2\gamma}\textstyle\int_0^t E\Big(\left(\tfrac{1}{T}\sigma_\omega(\mathbf{B}_0 - \mathbf{B}_{-T})\right)'\Sigma^{-1}\left(\tfrac{1}{T}\sigma_\omega\left(\mathbf{B}_0 - \mathbf{B}_{-T}\right)\right)\Big)\mathrm{d}s \\ &\quad - \tfrac{1}{2\gamma}\textstyle\int_0^t E\Big(\left(\tfrac{1}{T+s}\sigma_\omega(\mathbf{B}_s - \mathbf{B}_{-T})\right)'\Sigma^{-1}\left(\tfrac{1}{T+s}\sigma_\omega\left(\mathbf{B}_s - \mathbf{B}_{-T}\right)\right)\Big)\mathrm{d}s \\ &= \tfrac{N}{2\gamma}\left(\tfrac{t}{T} - \log\left(1 + \tfrac{t}{T}\right)\right).\end{aligned} \tag{A.18}$$

Proof of Proposition 11: We build on the model specification in the Appendix that led to Proposition 16. To calculate the expected log-consumption ratio, $E[h_t - h_0]$, for our specific example with $\Sigma = \sigma k_N (I + \mathbf{11}')$, we calculate the distributions of δ_t^1 and δ_t^2 (and thus the variance–covariance matrices) for this case. To this end, define $\Sigma_{\delta^1} = V^{1'}\Sigma^{-1}V^1 + \sigma_G^2 I_{N\times N}$ and

$\Sigma_{\delta^2} = V^{2'}\Sigma^{-1}V^2 + \sigma_G^2 I_{N\times N} + \phi^2\sigma_G^2 I_{N\times N}$. The variance–covariance matrix of δ_t^1 can then be calculated as follows:

$$\begin{aligned}\mathrm{VAR}_0(\delta_t^1) &= E[(\delta_t^1 - E[\delta_t^1])(\delta_t^1 - E[\delta_t^1])']\\ &= \int_0^t e^{\Psi^1(t-u)}\Sigma_{\delta^1}\left(e^{\Psi^1(t-u)}\right)' du\\ &= \int_0^t T\, e^{-\Lambda^1(t-u)}T^{-1}T\Lambda_{\delta^1}T^{-1}\left(T\, e^{-\Lambda^1(t-u)}T^{-1}\right)' du\\ &= T\int_0^t e^{-\Lambda^1(t-u)}\Lambda_{\delta^1}\left(e^{-\Lambda^1(t-u)}\right)' du\, T^{-1}\\ &= TG^1(t)T^{-1},\end{aligned}$$

where $G^1(t)$ is a diagonal matrix with element $[G^1(t)]_{ii} = \frac{\lambda_i^{\delta^1}}{2\lambda_i^1}(1 - e^{-2\lambda_i^1 t})$ with λ^1 and λ^{δ^1} being the vector of eigenvalues of Ψ^1 and Σ_{δ^1}, respectively. In the above, we have used the decompositions $\Psi^1 = T\Lambda^1 T^{-1}$ and $\Sigma_{\delta^1} = T\Lambda^{\delta^1}T^{-1}$. Note that Ψ^1 and Σ_{δ^1} can both be diagonalized by the same transformation matrix, T. This follows from the fact that the same transformation matrix T diagonalizes any matrix of the form $aI + b\mathbf{1}\mathbf{1}'$ (where a and b are arbitrary constants). We can similarly calculate the variance–covariance matrix of δ_t^2:

$$\mathrm{VAR}_0(\delta_t^2) = TG^2(t)T^{-1},$$

where $G^2(t)$ is a diagonal matrix with elements $[G^2(t)]_{ii} = \frac{\lambda_i^{\delta^2}}{2\lambda_i^2}(1 - e^{-2\lambda_i^2 t})$, where λ^2 and λ^{δ^2} are the vector of eigenvalues of Ψ^2 and Σ_{δ^2}, respectively. Next, it is straightforward to show that the eigenvalues λ^1, λ^2, λ^{δ^1}, and λ^{δ^2} take a particularly simple form:

$$\lambda_i^1 = \alpha + \left(\frac{\Delta v^1}{\sigma^2}\right)\left(1 - \frac{1}{N}\right), \quad i = 1, \ldots, N-1,$$

$$\lambda_N^2 = \alpha + \left(\upsilon_2^1 + \frac{\Delta\upsilon^1}{N}\right)\left(\frac{1}{\sigma^2}\right),$$

$$\lambda_i^2 = \alpha + \left(\frac{\Delta\upsilon^2}{\sigma^2}\right)\left(1 - \frac{1}{N}\right), \quad i = 1, \ldots, N-1,$$

$$\lambda_N^2 = \alpha + \left(\upsilon_2^2 + \frac{\Delta\upsilon^2}{N}\right)\left(\frac{1}{\sigma^2}\right),$$

and

$$\lambda_i^{\delta^1} = 2(\sigma_G^2 - \alpha\Delta\upsilon^1), \quad i = 1, \ldots, N-1,$$

$$\lambda_N^{\delta^1} = 2(\sigma_G^2 - \alpha\Delta\upsilon^1 - N\alpha\upsilon_2^1),$$

$$\lambda_i^{\delta^2} = 2(\sigma_G^2 - \alpha\Delta v^2) + \sigma_G \phi^2, \quad i = 1, \ldots, N-1,$$

$$\lambda_N^{\delta^2} = 2(\sigma_G^2 - \alpha\Delta v^2 - N\alpha v_2^2) + \sigma_G^2\phi^2.$$

It then follows that

$$\lim_{N\to\infty} \frac{\lambda_1^{\delta^2}}{\lambda_1^2} - \frac{\lambda_1^{\delta^1}}{\lambda_1^1} = \frac{2(\sigma_G^2 - \alpha\Delta v_\infty^2)}{\alpha + \frac{\Delta v_\infty^2}{\sigma^2}} - \frac{2(\sigma_G^2 - \alpha\Delta v_\infty^1)}{\alpha + \frac{\Delta v_\infty^1}{\sigma^2}} + \frac{\sigma_G^2\phi^2}{\alpha + \frac{\Delta v_\infty^2}{\sigma^2}} \stackrel{\text{def}}{=} c > 0,$$

where it follows that $c > 0$ from the fact that $\Delta v_\infty^1 \geq \Delta v_\infty^2$.

Since the covariance matrix $\mathrm{VAR}_0(\delta_t^1) = TG(t)T^{-1}$ has the same off-diagonal elements and structure as Ψ^1, we can calculate the covariance of $\delta_{i,t}^1$ and $\delta_{j,t}^1$ as

$$\mathrm{cov}_0(\delta_{i,t}^1, \delta_{j,t}^1) = \left(\frac{\lambda_N^{\delta^1}}{2\lambda_N^1}\left(1 - \mathrm{e}^{-2\lambda_N^1 t}\right) - \frac{\lambda_1^{\delta^1}}{2\lambda_1^1}\left(1 - \mathrm{e}^{-2\lambda_1^1 t}\right)\right)\Big/ N.$$

The variance of $\delta_{i,t}^1$ is

$$\mathrm{var}_0(\delta_{i,t}^1) = \frac{\lambda_1^{\delta^1}}{2\lambda_1^1}\left(1 - \mathrm{e}^{-2\lambda_1^1 t}\right)\left(1 - \frac{1}{N}\right) + \frac{\lambda_N^{\delta^1}}{2\lambda_N^1}\left(1 - \mathrm{e}^{-2\lambda_N^1 t}\right)\Big/ N.$$

Similarly, we calculate the variances and covariances of δ_t^2:

$$\mathrm{cov}_0(\delta_{i,t}^2, \delta_{j,t}^2) = \left(\frac{\lambda_N^{\delta^2}}{2\lambda_N^2}\left(1 - \mathrm{e}^{-2\lambda_N^2 t}\right) - \frac{\lambda_1^{\delta^2}}{2\lambda_1^2}\left(1 - \mathrm{e}^{-2\lambda_1^2 t}\right)\right)\Big/ N$$

and

$$\mathrm{var}_0(\delta_{i,t}^1) = \frac{\lambda_1^{\delta^2}}{2\lambda_1^2}\left(1 - \mathrm{e}^{-2\lambda_1^2 t}\right)\left(1 - \frac{1}{N}\right) + \frac{\lambda_N^{\delta^2}}{2\lambda_N^2}\left(1 - \mathrm{e}^{-2\lambda_N^2 t}\right)\Big/ N.$$

Note that the variance and covariances are all the same, for $i = 1, \ldots, N$.

Since both agents start with the correct estimate of $\mathbf{g}$, and thus $\delta_0^i = 0$, we have that

$$E[K_t^i] = \mathrm{trace}(\Sigma^{-1}\mathrm{VAR}_0(\delta_t^i)).$$

Using the above, we can calculate this as

$$\mathrm{trace}(\Sigma^{-1}\mathrm{VAR}_0(\delta_t^i)) = \left(\frac{1}{\sigma}\right)^2\left(N\left(\mathrm{var}_0^{\delta^i}(t) - \mathrm{cov}_0^{\delta^i}(t)\right) + \mathrm{cov}_0^{\delta^i}(t)\right).$$

Therefore, the expected log-consumption ratio is

$$\begin{aligned}
E[h_t - h_0] &= \int_0^t \tfrac{1}{2\gamma\sigma^2}\left(N\left(\mathrm{var}_0^{\delta^2}(u) - \mathrm{cov}_0^{\delta^2}(u)\right) + \mathrm{cov}_0^{\delta^2}(u)\right)\mathrm{d}u \\
&\quad - \int_0^t \tfrac{1}{2\gamma\sigma^2}\left(N\left(\mathrm{var}_0^{\delta^1}(u) - \mathrm{cov}_0^{\delta^1}(u)\right) + \mathrm{cov}_0^{\delta^1}(u)\right)\mathrm{d}u \\
&= \tfrac{N}{2\gamma\sigma^2}\left(\frac{\lambda_1^{\delta^2}}{2\lambda_1^2}\left(t - \frac{1}{2\lambda_1^2}\left(1 - \mathrm{e}^{-2\lambda_1^2 t}\right)\right) - \frac{\lambda_1^{\delta^1}}{2\lambda_1^1}\left(t - \frac{1}{2\lambda_1^1}\left(1 - \mathrm{e}^{-2\lambda_1^1 t}\right)\right)\right) \\
&\quad + \tfrac{1}{2N\gamma\sigma^2}\left(\frac{\lambda_N^{\delta^2}}{2\lambda_N^2}\left(t - \frac{1}{2\lambda_N^2}\left(1 - \mathrm{e}^{-2\lambda_N^2 t}\right)\right) - \frac{\lambda_1^{\delta^1}}{2\lambda_N^1}\left(t - \frac{1}{2\lambda_N^1}\left(1 - \mathrm{e}^{-2\lambda_N^1 t}\right)\right)\right).
\end{aligned}$$

Next note that, as $\Delta v^1 > \Delta v^2$ and $v_2^1 > v_2^2$, it follows that $\lambda_i^1 > \lambda_i^2$ and $\lambda_i^{\delta^1} < \lambda_i^{\delta^2}$. We then have that

$$\begin{aligned}
E[h_t - h_0] &\geq \tfrac{N}{2\gamma\sigma^2}\left(\frac{\lambda_1^{\delta^2}}{2\lambda_1^2} - \frac{\lambda_1^{\delta^1}}{2\lambda_1^1}\right)\left(t - \frac{1}{2\lambda_1^2}\left(1 - \mathrm{e}^{-2\lambda_1^2 t}\right)\right) \\
&\quad + \tfrac{1}{2N\gamma\sigma^2}\left(\frac{\lambda_N^{\delta^2}}{2\lambda_N^2} - \frac{\lambda_N^{\delta^1}}{2\lambda_N^1}\right)\left(t - \frac{1}{2\lambda_N^2}\left(1 - \mathrm{e}^{-2\lambda_N^2 t}\right)\right) \\
&\geq \tfrac{1}{2\gamma\sigma^2}\left(\frac{\lambda_1^{\delta^2}}{2\lambda_1^2} - \frac{\lambda_1^{\delta^1}}{2\lambda_1^1}\right)\left(t - \frac{1}{2\lambda_1^2}\left(1 - \mathrm{e}^{-2\lambda_1^2 t}\right)\right) \\
&= \tfrac{Nt}{2\gamma\sigma^2}\left(\frac{\lambda_1^{\delta^2}}{2\lambda_1^2} - \frac{\lambda_1^{\delta^1}}{2\lambda_1^1}\right)\left(1 - \frac{1}{2\lambda_1^2}\left(1 - \mathrm{e}^{-2\lambda_1^2 t}\right)\right) \\
&\geq \tfrac{Nt}{2\gamma\sigma^2}\left(\frac{\lambda_1^{\delta^2}}{2\lambda_1^2} - \frac{\lambda_1^{\delta^2}}{2\lambda_1^1}\right)\left(1 - \frac{1}{2t\lambda_1^2}\right) \\
&= \tfrac{Nt}{2\gamma\sigma^2}\left(\frac{\lambda_1^{\delta^2}}{2\lambda_1^2} - \frac{\lambda_1^{\delta^1}}{2\lambda_1^2}\right)\left(1 - \frac{1}{2t(\alpha + o(1))}\right) \\
&\geq \tfrac{Nt}{4\gamma\sigma^2}\left(\frac{\lambda_1^{\delta^2}}{2\lambda_1^2} - \frac{\lambda_1^{\delta^1}}{2\lambda_1^1}\right) \\
&\geq \frac{c - o(1)}{4\sigma^2} \times \frac{Nt}{2\gamma} \\
&\geq C\tfrac{Nt}{2\gamma}
\end{aligned}$$

for large N and t. The second to last inequality follows since $\left(\frac{\lambda_1^{\delta^2}}{\lambda_1^2} - \frac{\lambda_1^{\delta^1}}{\lambda_1^1}\right) \to c > 0$ for large N (where c is independent of N and t). We are done.

Proof of Proposition 12: Taking the expectation of the expression for h_t with $\rho_1 = \rho_2$ in Proposition 20 yields the result.

Proof of Proposition 13: The proof is a straightforward application of spectral decomposition. The spectral theorem ensures that for each *N*, there is a real orthogonal transformation of Σ_N into a diagonal matrix with strictly positive elements, $\Sigma_N = R_N' \Lambda_N R_N$, $\Lambda_N = \text{diag}(\rho_1, \ldots, \rho_N)$, and $R_N^{-1} = R_N'$. Without loss of generality, we can assume that the ρ's are ordered increasingly, so the spectral radius of Σ_N is ρ_N. Standard matrix-norm theory implies that

$$\min_{\delta_N' \neq 0_N} \frac{\delta_N' \Sigma_N^{-1} \delta_N}{\delta_N' \delta_N} = \frac{1}{\rho_N},$$

so $\delta_N' \Sigma_N^{-1} \delta_N \geq \frac{\delta_N' \delta_N}{\rho_N}$, and by our assumptions $\frac{\delta_N' \delta_N}{\rho_N} \geq c^{-1} N^{\nu}$, so $K_N = \delta_N' \Sigma_N^{-1} \delta_N \geq c^{-1} N^{\nu}$.

Proof of Proposition 14: As in the proof of the previous proposition, the spectral theorem ensures that for each *N*, there is a real orthogonal transformation of Σ_N into a diagonal matrix with strictly positive elements, $\Sigma_N = R_N' \Lambda_N R_N$, $\Lambda = \text{diag}(\rho_1, \ldots, \rho_N)$, and $R_N^{-1} = R_N'$. Moreover, the first assumption ensures that there is an eigenvalue, ρ_i, with corresponding eigenvector $\mathbf{1}_N$. We define $\rho^* = \rho(Q_{1_N}' \Sigma_N^{-1} Q_{1_N})$. Also, let us denote by P_N, the projection operator onto the one-dimensional subspace spanned by $\mathbf{1}_N$, so $Q_{\mathbf{1}_N} \perp P_N$. Clearly, $P_N \delta_N = \frac{1_N' \delta_N}{N} 1_N$. We can decompose

$$\begin{aligned}
\delta_N' \Sigma_N^{-1} \delta_N &= (\delta_N - P_N \delta_N + P_N \delta_N)' \Sigma^{-1} (\delta_N - P_N \delta_N + P_N \delta_N) \\
&= (\delta_N - P_N \delta_N)' Q_{1_N}' \Sigma_N^{-1} Q_{1_N} (\delta_N - P_N \delta_N) + \frac{(P_N \delta_N)'(P_N \delta_N)}{\rho_i} \\
&\geq (\delta_N - P_N \delta_N)' Q_{1_N}' \Sigma_N^{-1} Q_{1_N} (\delta_N - P_N \delta_N) \\
&\geq \frac{(\delta_N - P_N \delta_N)'(\delta_N - P_N \delta_N)}{\rho^*} \\
&= \frac{\delta_N' \delta_N - \frac{(\mathbf{1}_N' \delta_N)^2}{N}}{\rho^*}.
\end{aligned}$$

By the assumptions of the proposition, we therefore have (for large enough *N*)

$$K_N = \delta_N' \Sigma_N^{-1} \delta_N \geq c^{-1} N^{\nu}.$$

Proof of Proposition 15: Define $\Lambda_{\alpha,N} = \text{diag}(\alpha_1^N, \ldots, \alpha_N^N)$ and $\lambda_N = \delta_N$. We use the inversion formula $(I + \mathbf{x}\mathbf{x}')^{-1} = I - \frac{1}{1+\mathbf{x}'\mathbf{x}} \mathbf{x}\mathbf{x}'$ for an arbitrary vector $\mathbf{x}$ to get

$$
\begin{aligned}
\frac{K_N}{N} &= \frac{1}{N}\delta_N' \Sigma^{-1} \delta_N = \frac{1}{N}\lambda_N'(\Lambda_{\alpha,N}^2 + \mathbf{b}_N \mathbf{b}_N')^{-1}\lambda_N \\
&= \tfrac{1}{N}(\Lambda_{\alpha,N}^{-1}\lambda_N)'(I_N + \Lambda_{\alpha,N}^{-1}(\mathbf{b}_N\mathbf{b}_N')\Lambda_{\alpha,N}^{-1})^{-1}(\Lambda_{\alpha,N}^{-1}\lambda_N) \\
&= \tfrac{1}{N}(\Lambda_{\alpha,N}^{-1}\lambda_N)'\left(I_N - \tfrac{1}{1+\mathbf{b}_N'\Lambda_{\alpha,N}^{-2}\mathbf{b}_N}\Lambda_{\alpha,N}^{-1}\left(\mathbf{b}_N\mathbf{b}_N'\right)\Lambda_{\alpha,N}^{-1}\right)\left(\Lambda_{\alpha,N}^{-1}\lambda_N\right) \\
&= \frac{\lambda_N'\Lambda_{\alpha,N}^{-2}\lambda_N}{N} - \frac{1}{\frac{1}{N} + \frac{\mathbf{b}_N'\Lambda_{\alpha,N}^{-2}\mathbf{b}_N}{N}}\left(\frac{\lambda_N'\Lambda_{\alpha,N}^{-2}\mathbf{b}_N}{N}\right)^2.
\end{aligned}
$$

The independence of these variables, together with the strong law of large numbers, implies that

$$
\begin{aligned}
\frac{K_N}{N} &\to \text{a.s. } E[(\tilde{q}_1^1)^2]E[(\alpha_1^1)^{-2}] - \frac{1}{E\left[(\beta_1^1)^2\right]E\left[(\alpha_1^1)^{-2}\right]}\left(E\left[\tilde{q}_1^1\right]E\left[\beta_1^1\right]E\left[(\alpha_1^1)^{-2}\right]\right)^2 \\
&= E[(\alpha_1^1)^{-2}]\left(E[(\tilde{q}_1^1)^2] - \frac{E[\tilde{q}_1^1]^2 E[\beta_1^1]^2}{E\left[(\beta_1^1)^2\right]}\right) \\
&= \tfrac{E[(\alpha_1^1)^{-2}]}{E\left[(\beta_1^1)^2\right]}(E[(\tilde{q}_1^1)^2]E[(\beta_1^1)^2] - E[\tilde{q}_1^1]^2 E[\beta_1^1]^2) \\
&= \tfrac{E[(\alpha_1^1)^{-2}]}{E\left[(\beta_1^1)^2\right]}((\sigma_q^2 + \bar{q}^2)(\sigma_\beta^2 + \bar{\beta}^2) - \bar{q}^2\bar{\beta}^2) \\
&= \tfrac{E[(\alpha_1^1)^{-2}]}{\sigma_\beta^2 + \bar{\beta}^2}(\sigma_q^2\sigma_\beta^2 + \bar{q}^2\sigma_\beta^2 + \sigma_q^2\bar{\beta}^2) = k \in (0, \infty].
\end{aligned}
$$

The strict positivity of k is ensured, as $\sigma_\beta > 0$, $\sigma_q > 0$, and Jensen's inequality ensures that $E[(\alpha_1^1)^{-2}] \geq \frac{1}{\sigma_\alpha^2 + \bar{\alpha}^2}$. Thus, K_N grows like kN a.s. as N becomes large. This completes the proof. If $E[(\alpha_1^1)^{-2}] < \infty$ (which is not guaranteed by our assumptions) then $k < \infty$, so in this case the order of the natural selection process is exactly one. Otherwise it can be faster.

We note that the argument is easy to generalize to more general random structures. For example, a similar result can be derived for Q-factor models, $Q > 1$, using the same argument as above but with the inversion rule $(I_N + \mathbf{X}\mathbf{X}')^{-1} = I_N - \mathbf{X}(I_Q + \mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$. Here, $\mathbf{X}$ is an $N \times Q$ random matrix, representing the factor loadings of the N stocks on Q factors, I_N is the $N \times N$ identity matrix, and I_Q is the $Q \times Q$ identity matrix.

A.2 LEARNING (SECTION 4.3)

We show that a Bayesian investor, who originally is uncertain about the growth term, quickly will learn enough to avoid the severe underperformance experienced by the irrational investor.

The general linear filtering in continuous time is given by the system equations and the observation equations (see Øksendal, chapter 6). The system equations are

$$\mathrm{d}X_t = F_t X_t \,\mathrm{d}t + C_t \,\mathrm{d}U_t.$$

The observations equations are

$$\mathrm{d}Z_t = G_t X_t \,\mathrm{d}t + D_t \,\mathrm{d}V_t,$$

where $F \in R^{n\times n}$, $C \in R^{n\times p}$, $G \in R^{m\times n}$, and $D \in R^{m\times r}$. Define

$$\widehat{X}_t = E_t(X_t)$$

and

$$S_t = E_t[(X_t - \widehat{X}_t)(X_t - \widehat{X}_t)'].$$

Then, we have that

$$\mathrm{d}\widehat{X}_t = (F - SG'(DD')^{-1}G)\widehat{X}_t \,\mathrm{d}t + SG'(DD')^{-1}\,\mathrm{d}Z_t$$

and

$$\frac{\mathrm{d}S}{\mathrm{d}t} = FS + SF' - SG'(DD')^{-1}GS + CC',$$

where $\widehat{X}_0$ and S_0 are given.

Let us now apply this to the model in the paper in which there is an N-dimensional state vector, $\omega_t \in \mathbb{R}^N$, which evolves according to

$$\mathrm{d}\omega_t = \mathbf{g}\,\mathrm{d}t + \sigma_\omega \,\mathrm{d}\mathbf{B}_t.$$

A Bayesian agent does not observe $\mathbf{g}$ but must estimate it from ω, which he observes from time $-T$ and forward, where $T \geq 0$. Let $\widehat{\mathbf{g}}_{-T} = \mathbf{g} + \delta_{-T} \in \mathbb{R}^N$ and $V_{-T} \in \mathbb{R}^{N\times N}$ be the agent's prior mean and variance matrix (i.e., the agent's prior at $t = -T$ is that $\mathbf{g} \sim N(\widehat{\mathbf{g}}_{-T}, V_{-T})$). Then the filtering equations look like

$$\mathrm{d}\widehat{\mathbf{g}}_t = V_t(\sigma'_\omega)^{-1}\,\mathrm{d}\tilde{B}_t, \tag{A.19}$$

and

$$\frac{\mathrm{d}V_t}{\mathrm{d}t} = -V_t\Sigma^{-1}V_t, \tag{A.20}$$

where $\tilde{\mathbf{B}}_t$ is the "observed" Brownian motion. Solving the ordinary differential equation in Equation (A.20), we get

$$V_t = \left(V_{-T}^{-1} + \Sigma^{-1}(t - T)\right)^{-1}, \quad t > -T. \tag{A.21}$$

Note that Equation (A.19) can be written in terms of ω

$$d\widehat{\mathbf{g}}_t = -V_t\Sigma^{-1}\widehat{\mathbf{g}}_t\, dt + V_t\Sigma^{-1}\, d\omega_t. \tag{A.22}$$

Solving the Stochastic Differential Equation (SDE) in Equation (A.22), we get

$$\widehat{\mathbf{g}}_t = V_t V_{-T}^{-1}\widehat{\mathbf{g}}_0 + V_t\Sigma^{-1}\omega_t.$$

Therefore, $\widehat{\mathbf{g}}_t = \mathbf{g} + \delta_t$, where

$$\delta_t = V_t(V_{-T}^{-1}\delta_0 + \Sigma^{-1}\sigma_\omega(\mathbf{B}_t - \mathbf{B}_{-T})). \tag{A.23}$$

We now use the analysis to compare the two-agent economy in which Agent 1 is Bayesian and does not know the drift term (as opposed to in the main paper, where he does) and Agent 2 is irrational. Specifically, we assume that the Bayesian agent's beliefs at $t = 0$ are formed by Bayesian updating for T previous years, starting with a diffuse prior at $t = -T$. The diffuse prior is modeled by assuming that $V_{-T} = \tau^2 I$, where we formally let $\tau \to \infty$, which via Equations (A.21) and (A.23) leads to $\delta_0^1 = \frac{1}{T}\sigma_\omega(\mathbf{B}_0 - \mathbf{B}_{-T})$. Agent 1 then keeps updating his beliefs, so that at $t > 0$, $\delta_t^1 = \frac{1}{T+t}\sigma_\omega(\mathbf{B}_t - \mathbf{B}_{-T})$.

Agent 2, on the other hand, does not update but stubbornly sticks to his initial estimate of the drift term. To focus on the effect of learning, we assume that Agent 2's initial estimate is the same as Agent 1's, $\delta_0^2 = \delta_0^1$. In practice, we would expect Agent 2 to have a larger error term since he would not be rational in forming his initial beliefs either, which would imply faster market selection.

We define $\Delta_{1,t}$ and $\Delta_{2,t}$ as

$$\Delta_{1,t} = \sigma_\omega^{-1}\delta_t^1, \quad \Delta_{2,t} = \sigma_\omega^{-1}\delta_t^2$$

and the two transfer indexes $K_{1,t} = \Delta'_{1,t}\Delta_{1,t}$, $K_{2,t} = \Delta'_{2,t}\Delta_{2,t}$.

The central planner's problem in this setting is

$$\begin{aligned} u(C_t, \lambda_t) &= \max_{c_1(t), c_2(t)} \lambda_{1,t}\frac{c_{1,t}^{1-\gamma}}{1-\gamma} + \lambda_2(t)\frac{c_{2,t}^{1-\gamma}}{1-\gamma} \\ \text{s.t.}& \\ c_{1,t} + c_{2,t} &= C_t. \end{aligned}$$

In the above, we have that

$$\lambda_{k,t} = \exp\left(-\frac{1}{2}\int_0^t \Delta'_{k,s}\Delta_{k,s}\, ds + \int_0^t \Delta'_{k,s}\, d\mathbf{B}_s\right),$$

and solving the central planner problem, we get

$$\begin{aligned} c_{2,t} &= f_t C_t, \\ c_{2,t} &= (1 - f_t)C_t, \end{aligned}$$

where

$$f(t) = \frac{1}{1 + \lambda_t^{\frac{1}{\gamma}}},$$

where

$$\lambda_t = \frac{\lambda_{2,t}}{\lambda_{1,t}}.$$

We then have the following proposition.

Proposition 16. *The dynamics of the log consumption ratio h is*

$$\mathrm{d}h_t = \frac{1}{2\gamma}(K_{2,t} - K_{1,t})\mathrm{d}t + \frac{1}{\gamma}(\Delta_{1,t} - \Delta_{2,t})' \, \mathrm{d}\mathbf{B}_t.$$

Proof of Proposition 16: The proof is similar to the proof of Proposition 2.

A.3 EXPLICITLY MODELING OVERCONFIDENCE (SECTION 4.4)

There is an N-dimensional state vector, $\omega_t \in \mathbb{R}^N$, which evolves according to

$$\mathrm{d}\omega_t = \mathbf{g}_t \, \mathrm{d}t + \sigma_\omega \, \mathrm{d}\mathbf{B}_t.$$

Here, $\mathbf{g} \in \mathbb{R}^N$ and $\sigma_\omega \in \mathbb{R}^{N \times N}$. The variance–covariance matrix of ω is $\Sigma = \sigma_\omega \sigma'_\omega$. We assume that Σ is invertible at all points in time. The expected growth follows

$$\mathrm{d}\mathbf{g}_t = \alpha(\bar{\mathbf{g}} - \mathbf{g}_t)\mathrm{d}t + \sigma_g \, \mathrm{d}\mathbf{B}_t^g. \tag{A.24}$$

Finally, there is an N-dimensional signal process that follows

$$\mathrm{d}\mathbf{s}_t = \sigma_s \, \mathrm{d}\mathbf{B}_t^s.$$

All the Brownian motions are uncorrelated. There are two agents in the economy. We assume that both investors use ω and the signal processes, $\mathbf{s}$, to filter out the current value of $\mathbf{g}$. Agent 1 correctly believes that the signal process is uninformative about the value of $\mathbf{g}$. Agent 2 is overconfident about the signal and believes that it follows the process

$$\mathrm{d}\mathbf{s}_t = \phi\sigma_s \, \mathrm{d}\mathbf{B}_t^g + \sqrt{1 - \phi^2}\sigma_s \, \mathrm{d}\mathbf{B}_t^s,$$

where $0 < \phi < 1$.

Using standard filtering, one can show that the conditional expected values of $\mathbf{g}$ as perceived by Agents 1 and 2 are given by

$$\mathrm{d}\mathbf{g}_t^1 = \alpha(\bar{\mathbf{g}} - \mathbf{g}_t^1)\mathrm{d}t + V^1\Sigma^{-1}(\mathrm{d}\omega_t - \mathbf{g}_t^1\,\mathrm{d}t) \tag{A.25}$$

and

$$\mathrm{d}\mathbf{g}_t^2 = \alpha(\bar{\mathbf{g}} - \mathbf{g}_t^2)\mathrm{d}t + V^2\Sigma^{-1}(\mathrm{d}\omega_t - \mathbf{g}_t^2\mathrm{d}t) + \left(\frac{\sigma_g}{\sigma_s}\right)\phi\,\mathrm{d}\mathbf{s}_t, \tag{A.26}$$

where V^1 and V^2 are the steady-state variance–covariance matrix of $\mathbf{g}$ as estimated by Agents 1 and 2, respectively. The steady-state variance is the solution to the following Matrix Riccati equations:

$$V^{1'}\Sigma^{-1}V^1 + 2\alpha V^1 - \sigma_G I_{N\times N} = 0$$

for the rational agent and

$$V^{2'}\Sigma^{-1}V^2 + 2\alpha V^2 - (1-\phi^2)\sigma_G I_{N\times N} = 0$$

for the irrational agent.

We focus on the solution for the rational agent; the solution for the irrational agent follows easily from identical arguments. Note that due to the symmetry of problem, we can reduce the Riccati equation above to a system of two equations and two unknowns. Define $v_1^1 = (V^1)_{ii}$, $v_1^2 = (V^2)_{ii}$, and $v_2^1 = (V^1)_{ij}$, $v_2^2 = (V^2)_{ij}$ for $i \neq j$. The equations then become

$$\left(v_1^1\right)^2 + 2\left(1-\frac{1}{N}\right)\left(v_2^1\right)^2 - 2\left(1-\frac{1}{N}\right)v_1^1v_2^1 + 2\alpha\sigma^2v_1^1 - \sigma_G^2\sigma^2 = 0, \tag{A.27}$$

$$-\frac{1}{N}\left(v_1^1\right)^2 + \left(1-\frac{3}{N}\right)\left(v_2^1\right)^2 + \frac{4}{N}v_1^1v_2^1 + 2\alpha\sigma^2v_2^1 = 0. \tag{A.28}$$

Subtracting the second equation from the first yields

$$\frac{1}{k_n}\left(\Delta v^1\right)^2 + 2\alpha\sigma^2\Delta v^1 - \sigma_G^2\sigma^2 = 0,$$

where $k_N = \frac{N}{N+1}$ (as before) and $\Delta v^1 = v_1^1 - v_2^1$. Solving for Δv^1 leads to

$$\Delta v^1 = \alpha\sigma^2 k_N\left(\sqrt{1 + \left(\frac{\sigma_G}{\alpha\sigma}\right)^2\frac{1}{k_N}} - 1\right).$$

Substituting $v_1^1 = v_2^1 + \Delta v^1$ into Equation (A.28) and solving for v_2^1 yield

$$v_2^1 = \sqrt{\frac{\Delta v^1}{N} + \left(\frac{\Delta v^1}{N} + \alpha\sigma^2\right)^2} - \left(\frac{\Delta v^1}{N} + \alpha\sigma^2\right).$$

Following the same procedure for the irrational agent, it follows that

$$\Delta v^2 = \alpha\sigma^2 k_N \left(\sqrt{1 + \left(\frac{\sigma_G}{\alpha\sigma}\right)^2 \frac{(1-\phi^2)}{k_N}} - 1 \right)$$

and

$$v_2^2 = \sqrt{\frac{\Delta v^2}{N} + \left(\frac{\Delta v^2}{N} + \alpha\sigma^2\right)^2} - \left(\frac{\Delta v^2}{N} + \alpha\sigma^2\right).$$

It is straightforward to show that $\Delta v^1 > \Delta v^2$, $v_2^1 > v_2^2$, and $v_1^1 > v_1^2$. This follows intuitively from the fact that the irrational agent is overconfident and believes he is learning from the signal and thus his posterior variance is lower.

It now follows that as N approaches infinity,

$$\Delta v^1 \to \alpha\sigma^2 \left(\sqrt{1 + \left(\frac{\sigma_G}{\alpha\sigma}\right)^2} - 1 \right) \stackrel{\text{def}}{=} \Delta v_\infty^1,$$

$$\Delta v^2 \to \alpha\sigma^2 \left(\sqrt{1 + \left(\frac{\sigma_G}{\alpha\sigma}\right)^2 (1-\phi^2)} - 1 \right) \stackrel{\text{def}}{=} \Delta v_\infty^2,$$

and both v_2^1 and v_2^2 approach zero. Thus, in the steady-state limit, V^1 and V^2 converge to diagonal matrices.

Neither of the two agents knows the true $\mathbf{g}$, and consequently both agents will be wrong on average. However, Agent 2 is overconfident and will on average be "more wrong" than Agent 1. Let $\delta_t^i = \mathbf{g}_t^i - \mathbf{g}_t$ be the error that agent i makes in his estimation of the expected growth rate, $\mathbf{g}$. Then, applying Itô's lemma to Equations (A.24)–(A.26), it follows that

$$\mathrm{d}\delta_t^1 = -\Psi^1 \delta_t^1 \,\mathrm{d}t + V^1 \Sigma^{-1} \sigma_\omega \,\mathrm{d}\mathbf{B}_t - \sigma_g \,\mathrm{d}\mathbf{B}_t^g, \tag{A.29}$$

and

$$\mathrm{d}\delta_t^2 = -\Psi^2 \delta_t^2 \,\mathrm{d}t + V^2 \Sigma^{-1} \sigma_\omega \,\mathrm{d}\mathbf{B}_t - \sigma_g \,\mathrm{d}\mathbf{B}_t^g + \sigma_g \phi \,\mathrm{d}\mathbf{B}_t^s, \tag{A.30}$$

where $\Psi^1 = \alpha I_{N\times N} + V^1\Sigma^{-1}$ and $\Psi^2 = \alpha I_{N\times N} + V^2\Sigma^{-1}$. Both δ^1 and δ^2 are N-dimensional Ornstein–Uhlenbeck processes that revert to zero. The solution of the SDEs [Equations (A.29) and (A.30)] are given by

$$\delta_t^1 = \mathrm{e}^{-\Psi^1 t}\delta_0^1 + \int_0^t \mathrm{e}^{-\Psi^1(t-u)} V^1 \Sigma^{-1} \sigma_\omega \,\mathrm{d}\mathbf{B}_u - \int_0^t \mathrm{e}^{-\Psi^1(t-u)} \sigma_g \,\mathrm{d}\mathbf{B}_u^g$$

and

$$\delta_t^2 = \mathrm{e}^{-\Psi^2 t}\delta_0^2 + \int_0^t \mathrm{e}^{-\Psi^2(t-u)} V^2 \Sigma^{-1}\sigma_\omega \, \mathrm{d}\mathbf{B}_u - \int_0^t \mathrm{e}^{-\Psi^2(t-u)}\sigma_g \, \mathrm{d}\mathbf{B}_u^g + \int_0^t \mathrm{e}^{-\Psi^2(t-u)}\sigma_g \phi \, \mathrm{d}\mathbf{B}_u^s.$$

A similar argument as in the proofs of Propositions 2 and 10 then leads to the following.

Proposition 17. *Define the instantaneous transfer index of Agent i as*

$$K_t^i = \delta_t^{i\prime}\Sigma^{-1}\delta_t^i$$

and the cross transfer index between Agents 1 and 2 as

$$K_t^{1,2} = \delta_t^{1\prime}\Sigma^{-1}\delta_t^2.$$

Then, the instantaneous dynamics of the log-consumption ratio, h_t, is

$$\mathrm{d}h_t = \frac{1}{2\gamma}(K_t^2 - K_t^1)\mathrm{d}t + \frac{1}{\gamma}\sqrt{K_t^2 + K_t^1 - 2K_t^{1,2}}\,\mathrm{d}\tilde{B}_t,$$

where $\tilde{B}$ is a standardized Brownian motion.

A.4. MISTAKES ABOUT THE COVARIANCE MATRIX (SECTION 4.5)

We consider a discrete time version of the model, allowing agents to disagree on the variance–covariance matrix of the state variables, following Jouini and Napp (2006). We consider a model with normally distributed shocks and can therefore no longer implement equilibrium with $N+1$ long-lived assets, as we could in the continuous time case. In line with the previous literature (see Jouini and Napp, 2006) we assume that there are enough assets such that the agents can implement the optimal consumption allocations, that is, that Arrow–Debreu securities exist for each state of the world.

We assume a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\{0,\ldots,T\}}, \mathbb{P})$ satisfying the usual assumptions. As in the continuous time case, T could be finite or infinite. There is an N-dimensional state vector, $\omega_t \in \mathbb{R}^N$, which evolves according to

$$\omega_{t+1} = \omega_t + \mathbf{g} + \sigma_\omega \boldsymbol{\epsilon}_{t+1}.$$

Here, $\mathbf{g} \in \mathbb{R}^N$ and $\sigma_\omega \in \mathbb{R}^{N\times N}$ and $\boldsymbol{\epsilon}_t \sim N(0, I_N)$. The variance–covariance matrix of $\Delta\omega_{t+1} = \omega_{t+1} - \omega_t$ is $\Sigma = \sigma_\omega \sigma'_\omega$, and consequently $\Delta\omega_{t+1} \sim N(\mathbf{g}, \Sigma)$. We assume that for each ω_i, there is a corresponding firm i that produces $D_{i,t}$ of a perishable consumption good, where

$$D_{i,t} = D_{i,0}\,\mathrm{e}^{\omega_{i,t}}, \quad D_{i,0} > 0,$$

and where $\omega_{i,t} = (\omega_t)_i$ is the ith element of the vector ω at time t. We define $\mathbf{D} = (D_{1,0}, D_{2,0}, \ldots, D_{N,0})'$. The aggregate consumption is given by

$$C_t = \sum_{i=1}^{N} D_{i,t}.$$

As in the continuous time case, there are two price-taking investors, $k \in \{1, 2\}$. We assume that both agents make mistakes in the expected growth and variance–covariance matrix of ω. That is, we assume that agent k believes that $\Delta\omega_{t+1} \sim N(\mathbf{g} + \delta_k, \Sigma_k)$

We further assume that investors $k \in \{1, 2\}$ have initial wealth W_k and CRRA preferences with time discount factors ρ_k and common relative risk aversion parameter, γ. For expositional reasons, we mainly focus on the case when $\gamma \neq 1$, although our results also hold under logarithmic utility. Thus, investor k optimizes

$$U_k = E_k\left[\sum_{t=0}^{T} \mathrm{e}^{-\rho_k t}\frac{c_{k,t}^{1-\gamma}}{1-\gamma}\right],$$

subject to his budget constraint, where $c_{k,t}$ is the consumption at t of investor k. Here, since the two investors have different expectations, the k subscript of the expectation operator is motivated. The total initial wealth is $W = W_1 + W_2$. The economic environment can be summarized by the tuplet $\mathcal{E} = (\delta_1, \delta_2, \Sigma_1, \Sigma_2, \mathbf{g}, \Sigma, \mathbf{D})$, whereas the agents' preferences are summarized by the triplet (γ, ρ_1, ρ_2).

We construct the social planner's problem with a representative agent state by state and time by time from

$$\begin{aligned} u(C_t, \lambda_t, t) &= \max_{c_{1,t}, c_{2,t}} \left\{ \mathrm{e}^{-\rho_1 t}\lambda_{2,t}\frac{c_{1,t}^{1-\gamma}}{1-\gamma} + \lambda_{2,t}\,\mathrm{e}^{-\rho_2 t}\frac{c_{2,t}^{1-\gamma}}{1-\gamma}\right\} \\ &\text{s.t.} \\ c_{1,t} + c_{2,t} &= C_t. \end{aligned}$$

Here, $\lambda_{k,t} = \lambda_{k,0}\eta_{k,t}$, $\eta_{k,t} = \frac{d\mathbb{P}_k}{d\mathbb{P}}$, and $\lambda_{k,0} \in \mathbb{R}_+$ determine agent k's weight in the social planner's problem. The solution is given by the following.

Proposition 18. *Agent 1's consumption share at time t is*

$$f_t = \frac{1}{1 + e^{(\rho_1-\rho_2)t/\gamma}\lambda_t^{\frac{1}{\gamma}}},$$

which determines the agents' consumption:

$$\begin{aligned} c_{1,t} &= f_t C_t, \\ c_{2,t} &= (1-f_t)C_t, \end{aligned}$$

In the above, $\lambda_t = \frac{\lambda_{2,t}}{\lambda_{1,t}}$.

Proof of Proposition 18: The proof is similar to the proof of Proposition 1.

Proposition 19. *The Radon–Nikodym derivative,* $\eta_{k,t}$, *is*

$$\eta_{k,t} = \exp\left(-\frac{1}{2}\left(\delta_k'\Sigma_k^{-1}\delta_k - \log\left(\frac{|\Sigma|}{|\Sigma_k|}\right)\right)t\right) \times \exp\left(\sum_{s=1}^{t}\left[u_s'\Sigma_k^{-1}\delta + \frac{1}{2}u_s'\left(\Sigma^{-1} - \Sigma_k^{-1}\right)u_s\right]\right), \tag{A.31}$$

where $u_s = \sigma_\omega \epsilon_s$.

Proof of Proposition 19: First note that $\Delta\omega_t$ and $\Delta\omega_s$ are independent for $s \neq t$. The Radon–Nikodym derivative of agent k is

$$\eta_{k,t} = \frac{d\mathbb{P}_k}{d\mathbb{P}} = \prod_{s=1}^{t}\frac{\phi_k(\Delta\omega_s)}{\phi(\Delta\omega_s)}, \tag{A.32}$$

where

$$\phi(\mathbf{x}) = \frac{1}{(2\pi)^{N/2}|\Sigma|^{1/2}}\exp\left(-\frac{1}{2}(\mathbf{x}-\mathbf{g})'\Sigma^{-1}(\mathbf{x}-\mathbf{g})\right) \tag{A.33}$$

and

$$\phi_k(\mathbf{x}) = \frac{1}{(2\pi)^{N/2}|\Sigma_k|^{1/2}} \exp\left(-\frac{1}{2}\left(\mathbf{x} - \left(\mathbf{g} + \delta_k\right)\right)' \Sigma_k^{-1} \left(\mathbf{x} - \left(\mathbf{g} + \delta_k\right)\right)\right) \tag{A.34}$$

Inserting Equations (A.33) and (A.34) into Equation (A.32) yields the result.

Proposition 20. *The log-consumption ratio,* $h_t = \log(\frac{c_{1,t}}{c_{2,t}})$, *is*

$$\begin{aligned} h_t &= h_0 + \left(\frac{1}{2\gamma}\left(\delta_2'\Sigma_2^{-1}\delta_2 - \delta_1'\Sigma_1^{-1}\delta_1 - \log\left(\frac{|\Sigma_1|}{|\Sigma_2|}\right)\right) + \frac{1}{\gamma}\left(\rho_2 - \rho_1\right)\right)t \\ &\quad + \frac{1}{\gamma}\sum_{s=1}^{t}\left[u_s'\Sigma_1^{-1}\delta_1 + \frac{1}{2}u_s'\left(\Sigma^{-1} - \Sigma_1^{-1}\right)u_s\right] - \frac{1}{\gamma}\sum_{s=1}^{t}\left[u_s'\Sigma_2^{-1}\delta_2 \right. \\ &\quad \left. + \frac{1}{2}u_s'\left(\Sigma^{-1} - \Sigma_2^{-1}\right)u_s\right]. \end{aligned} \tag{A.35}$$

Proof of Proposition 20: From the optimal consumption allocations in Proposition 18, we have

$$\begin{aligned} h_t &= \log\left(\frac{c_{1,t}}{c_{2,t}}\right) = \log\left(e^{-(\rho_1-\rho_2)t/\gamma}\left(\frac{\lambda_{2,t}}{\lambda_{1,t}}\right)^{-\frac{1}{\gamma}}\right) = -\frac{1}{\gamma}\log\left(e^{(\rho_1-\rho_2)t}\left(\frac{\lambda_{2,t}}{\lambda_{1,t}}\right)\right) \\ &= -\frac{1}{\gamma}\log\left(e^{(\rho_1-\rho_2)t}\left(\frac{\lambda_{2,0}\eta_{2,t}}{\lambda_{1,0}\eta_{1,t}}\right)\right) \\ &= h_0 + \left(\frac{1}{2\gamma}\left(\delta_2'\Sigma_2^{-1}\delta_2 - \delta_1'\Sigma_1^{-1}\delta_1 - \log\left(\left|\frac{\Sigma_1}{\Sigma_2}\right|\right)\right) + \frac{1}{\gamma}\left(\rho_2 - \rho_1\right)\right)t \\ &\quad + \frac{1}{\gamma}\sum_{s=1}^{t}\left[u_s'\Sigma_1^{-1}\delta_1 + \frac{1}{2}u_s'\left(\Sigma^{-1} - \Sigma_1^{-1}\right)u_s\right] - \frac{1}{\gamma}\sum_{s=1}^{t}\left[u_s'\Sigma_2^{-1}\delta_2 \right. \\ &\quad \left. + \frac{1}{2}u_s'\left(\Sigma^{-1} - \Sigma_2^{-1}\right)u_s\right]. \end{aligned} \tag{A.36}$$

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