

# Online Appendix to “Asset Pricing in Large Information Networks”

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The proofs of Propositions 1-12 have been omitted from the main text in the interest of brevity. This Online Appendix contains these proofs. Equation numbering continues in sequence with that established in the main text. See the main text for references to Theorem 1 and equations (1)-(46).

*Proof of Proposition 1:* We construct a growing sequence of “caveman” networks that converge to a given degree distribution. A caveman network is one which partitions the set of agents in the sense that if agent  $i$  is connected with  $j$  and  $j$  is connected with  $k$ , then  $i$  is connected with  $k$  (see Watts [2]).

We proceed as follows: First we observe that for  $d(1) = 1$ , the result is trivial, so we assume that  $d(1) \neq 1$ . For a given  $d \in S^\infty$ , define  $k = \min_i \{i \neq 1 : i \in \text{supp}[d]\}$ . For  $m > k$ , we define  $\hat{d}^m \in S^m$  by  $\hat{d}^m(i) = d(i) / \sum_{j=1}^m d(j)$ . Clearly,  $\lim_{m \rightarrow \infty} \sum_{i=1}^m |\hat{d}^m(i) - d(i)| = 0$ . For an arbitrary  $n \geq k^3$ , choose  $m = \lfloor n^{1/3} \rfloor$ . For  $1 < \ell \leq m$ ,  $\ell \neq k$ , choose  $z_\ell^n = \lfloor \hat{d}^m(\ell) \times n / \ell \rfloor$ , and  $z_k^n = \lfloor (n - \sum_{\ell \neq k} z_\ell^n) / k \rfloor$ .

Now, define  $\mathcal{G}^n$ , with degree distribution  $d^n$ , as a network in which there are  $z_\ell^n$  clusters of tightly connected sets of agents, with  $\ell$  members,  $1 < \ell \leq m$  and  $n - \sum_{\ell=2}^m \ell z_\ell^n$  singletons. With this construction,  $|z_\ell^n \ell / n - \hat{d}^m(i)| \leq \ell / n$  for  $\ell > 2$  and  $\ell \neq k$ . Moreover,  $|z_1^n / n - \hat{d}^m(1)| \leq (k + 1) / n$ , and  $|z_k^n k / n - \hat{d}^m(k)| \leq (k + 1) / n + m^2 / n$ , so  $\sum_{\ell=1}^m |z_\ell^n \ell - \hat{d}^m(\ell)| \leq 2(k + 1) / n + 2m^2 / n = O(n^{-1/3})$ .

Thus,  $\sum_{i=1}^{\lfloor n^{1/3} \rfloor} |d^n(i) - \hat{d}^{\lfloor n^{1/3} \rfloor}(i)| \rightarrow 0$ , when  $n \rightarrow \infty$  and since  $\sum_{i=1}^{\lfloor n^{1/3} \rfloor} |\hat{d}^{\lfloor n^{1/3} \rfloor}(i) - d(i)| \rightarrow 0$ , when  $n \rightarrow \infty$ , this sequence of caveman networks indeed provides a constructive example for which the degree distribution converges to  $d$ .

Moreover, it is straightforward to check that if  $d(i) = O(i^{-\alpha})$ ,  $\alpha > 1$ , then (10) is satisfied in the previously constructed sequence of caveman networks, and that if  $\alpha > 2$ , then (11) is satisfied.

If  $d(i) \sim i^{-\alpha}$ ,  $\alpha \leq 2$ , on the other hand, then clearly  $\sum_i d(i)i = \infty$ , so (11) will fail. ■

*Proof of Proposition 2:* We first show the form for  $\beta$ . We have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n (\mathbf{W}^n)_{ii}}{s^2 n} &= \lim_{n \rightarrow \infty} \sum_k k \times c_\alpha^n k^{-\alpha} \\ &= \zeta(\alpha)^{-1} \sum_{k=1}^{\infty} k^{-(\alpha-1)} = \zeta(\alpha)^{-1} \zeta(\alpha - 1). \end{aligned}$$

For (10), we notice that for a network with  $n = m^\alpha$  nodes, the maximum degree,  $(\mathbf{W}^n)_{ii}$  will not be larger than  $m$ . However, since each of the neighbors to that node has no more than  $m$  neighbors,  $\|\mathbf{W}^n\|_\infty = \sum_j (\mathbf{W}^n)_{ij} \leq m^2 = n^{2/\alpha} = o(n)$  when  $\alpha > 2$ .  $\blacksquare$

*Proof of Proposition 3:* It follows from Theorem 1 that

$$(\pi^*)^2 \sigma^2 = \frac{\beta^2 (\beta + \Delta^2)^2 \sigma^6}{(\Delta^2 + \beta (\beta + \Delta^2) \sigma^2)^2}, \quad (47)$$

$$(\gamma^*)^2 \Delta^2 = \frac{\Delta^2 (\beta + \Delta^2)^2 \sigma^4}{(\Delta^2 + \beta (\beta + \Delta^2) \sigma^2)^2}, \quad (48)$$

$$\text{var}(\tilde{p}) = \frac{(\beta + \Delta^2)^2 \sigma^4 (\Delta^2 + \beta^2 \sigma^2)}{(\Delta^2 + \beta (\beta + \Delta^2) \sigma^2)^2}. \quad (49)$$

(47) implies that

$$\frac{\partial (\pi^*)^2 \sigma^2}{\partial \beta} = \frac{2\beta \Delta^2 (\beta + \Delta^2) (2\beta + \Delta^2) \sigma^6}{(\Delta^2 + \beta (\beta + \Delta^2) \sigma^2)^3} > 0,$$

and this proves part (a).

(48) implies that

$$\frac{\partial (\gamma^*)^2 \Delta^2}{\partial \beta} = \frac{2\Delta^4 (\beta + \Delta^2) \sigma^4 - 2\Delta^2 (\beta + \Delta^2)^3 \sigma^6}{(\Delta^2 + \beta (\beta + \Delta^2) \sigma^2)^3}.$$

The expression above is strictly negative if and only if  $\beta > \frac{\Delta}{\sigma} - \Delta^2$ . This proves part (b).

Finally, (49) implies that

$$\frac{\partial \text{var}(\tilde{p})}{\partial \beta} = \frac{2\Delta^4 (\beta + \Delta^2) \sigma^4 - 2\Delta^2 (-\beta^3 + 2\beta \Delta^4 + \Delta^6) \sigma^6}{(\Delta^2 + \beta (\beta + \Delta^2) \sigma^2)^3}.$$

The expression above is strictly positive if and only if  $\Delta^2 < \frac{1-\beta\sigma^2}{2\sigma^2} + \frac{1}{2}\sqrt{\frac{1-2\beta\sigma^2+5\beta^2\sigma^4}{\sigma^4}}$ . This

proves part (c). ■

*Proof of Proposition 4:* It is straightforward from Theorem 1 and the projection theorem that

$$\begin{aligned} \text{var} \left( \tilde{X} | \tilde{p} \right) &= \sigma^2 - \frac{\left( \beta \frac{\sigma^2 \Delta^2 + \sigma^2 \beta}{\beta \sigma^2 \Delta^2 + \Delta^2 + \beta^2 \sigma^2} \sigma^2 \right)^2}{\left( \beta \frac{\sigma^2 \Delta^2 + \sigma^2 \beta}{\beta \sigma^2 \Delta^2 + \Delta^2 + \beta^2 \sigma^2} \right)^2 \sigma^2 + \left( \frac{\sigma^2 \Delta^2 + \sigma^2 \beta}{\beta \sigma^2 \Delta^2 + \Delta^2 + \beta^2 \sigma^2} \right)^2 \Delta^2} \\ &= \frac{\Delta^2 \sigma^2}{\Delta^2 + \beta^2 \sigma^2}. \end{aligned}$$

Hence the result follows. ■

*Proof of Proposition 5:* From (23), we know that agent  $i$ 's demand will take the form

$$\psi_i(\tilde{x}_i, \tilde{p}) = \frac{\alpha_{0i}}{\beta_i} + \frac{\alpha_{1i}}{\beta_i} \tilde{x}_i + \left( \frac{\alpha_{2i}}{\beta_i} - \frac{1}{\beta_i} \right) \tilde{p}.$$

Similar arguments as in the proof of Theorem 1 shows that

$$\frac{\alpha_{0i}}{\beta_i} = \frac{\bar{X}}{\sigma^2} - \left( \frac{\pi_0}{\gamma n} - \bar{Z} \right) A_i,$$

where  $A_i = \gamma n \frac{(\mathbf{1}^T \boldsymbol{\pi}) s_i^2 - (\boldsymbol{\pi}^T \mathbf{s}_i)}{s_i^2 (\boldsymbol{\pi}^T \mathbf{S} \boldsymbol{\pi} + \gamma^2 n^2 \Delta^2) - (\boldsymbol{\pi}^T \mathbf{s}_i)^2}$ , converges to  $\frac{\beta}{\Delta^2}$  for large  $n$ . Therefore

$$\begin{aligned} \frac{\alpha_{0i}}{\beta_i} &\xrightarrow{n \rightarrow \infty} \frac{\bar{X} \Delta^2 + \bar{Z} \beta \sigma^2}{\sigma^2 \Delta^2 + \sigma^2 \beta}, \\ \frac{\alpha_{1i}}{\beta_i} &= \frac{\boldsymbol{\pi}^T \mathbf{S} \boldsymbol{\pi} + \gamma^2 n^2 \Delta^2 - (\mathbf{1}^T \boldsymbol{\pi})(\boldsymbol{\pi}^T \mathbf{s}_i)}{s_i^2 (\boldsymbol{\pi}^T \mathbf{S} \boldsymbol{\pi} + \gamma^2 n^2 \Delta^2) - (\boldsymbol{\pi}^T \mathbf{s}_i)^2} \\ &\xrightarrow{n \rightarrow \infty} \frac{1}{s_i^2} = \frac{W_i}{s^2}, \\ \frac{\alpha_{2i}}{\beta_i} &= \frac{(\mathbf{1}^T \boldsymbol{\pi}) s_i^2 - (\boldsymbol{\pi}^T \mathbf{s}_i)}{s_i^2 (\boldsymbol{\pi}^T \mathbf{S} \boldsymbol{\pi} + \gamma^2 n^2 \Delta^2) - (\boldsymbol{\pi}^T \mathbf{s}_i)^2} \\ &\xrightarrow{n \rightarrow \infty} \frac{\beta}{\Delta^2 \gamma^*}. \end{aligned}$$

Similarly, we have

$$\begin{aligned}
\frac{1}{\beta_i} &= \frac{(\sigma^2 + s_i^2) (\boldsymbol{\pi}^T \mathbf{S} \boldsymbol{\pi} + n^2 \Delta^2 \gamma^2 + (\mathbf{1}^T \boldsymbol{\pi})^2 \sigma^2) - ((\mathbf{1}^T \boldsymbol{\pi}) \sigma^2 + (\boldsymbol{\pi}^T \mathbf{s}_i))^2}{\sigma^2 (s_i^2 (\boldsymbol{\pi}^T \mathbf{S} \boldsymbol{\pi} + \gamma^2 n^2 \Delta^2) - (\boldsymbol{\pi}^T \mathbf{s}_i)^2)} \\
&= \frac{(\sigma^2 + s_i^2) (\mathbf{q}^T \mathbf{S} \mathbf{q} / n^2 + \Delta^2 + (\mathbf{1}^T \mathbf{q})^2 \sigma^2 / n^2) - ((\mathbf{1}^T \mathbf{q}) \sigma^2 + (\mathbf{q}^T \mathbf{s}_i))^2 / n^2}{\sigma^2 (s_i^2 (\mathbf{q}^T \mathbf{S} \mathbf{q} / n^2 + \Delta^2) - (\mathbf{q}^T \mathbf{s}_i)^2 / n^2)} \\
&\xrightarrow{n \rightarrow \infty} \frac{(\sigma^2 + s_i^2) (\Delta^2 + \beta^2 \sigma^2) - (\beta \sigma^2)^2}{\sigma^2 s_i^2 \Delta^2} = \frac{1}{s_i^2} + \frac{1}{\sigma^2} + \frac{\beta^2}{\Delta^2}. \tag{50}
\end{aligned}$$

Thus,

$$\psi_i(\tilde{x}_i, \tilde{p}) = \frac{\bar{X} \Delta^2 + \bar{Z} \beta \sigma^2}{\sigma^2 \Delta^2 + \sigma^2 \beta} + \frac{W_i}{s^2} (\tilde{x}_i - \tilde{p}) + \left( \frac{\beta}{\Delta^2 \gamma^*} - \frac{1}{\sigma^2} - \frac{\beta^2}{\Delta^2} \right) \tilde{p}.$$

Since

$$\begin{aligned}
\frac{\beta}{\Delta^2 \gamma^*} - \frac{1}{\sigma^2} - \frac{\beta^2}{\Delta^2} &= \frac{\beta(\beta \sigma^2 \Delta^2 + \Delta^2 + \beta^2 \sigma^2)}{\Delta^2 (\sigma^2 \Delta^2 + \sigma^2 \beta)} - \frac{\Delta^4 + \beta \Delta^2}{\Delta^2 (\sigma^2 \Delta^2 + \sigma^2 \beta)} - \frac{\beta^2 \sigma^2 (\Delta^2 + \beta)}{\Delta^2 (\sigma^2 \Delta^2 + \sigma^2 \beta)} \\
&= -\frac{\Delta^2}{\sigma^2 (\Delta^2 + \beta)},
\end{aligned}$$

the expression for the demand function reduces to

$$\psi_i(\tilde{x}_i, \tilde{p}) = \frac{\bar{X} \Delta^2 + \bar{Z} \beta \sigma^2}{\sigma^2 \Delta^2 + \sigma^2 \beta} - \frac{\Delta^2}{\sigma^2 (\Delta^2 + \beta)} \tilde{p} + \frac{W_i}{s^2} (\tilde{x}_i - \tilde{p}). \tag{51}$$

Expected profits are of the form  $E[\psi_i(\tilde{x}_i, \tilde{p})(\tilde{X} - \tilde{p})]$ , and therefore (17) immediately follows.

■

*Proof of Proposition 6:* We define the average expected profit in economy  $n$ ,

$$\Pi^n = \frac{\sum_{i=1}^n E \left[ (\tilde{X} - \tilde{p}^n) \psi_i^n(\tilde{x}_i^n, \tilde{p}^n) \right]}{n}.$$

From Theorem 1, we know that the market clearing condition  $\sum_{i=1}^n \psi_i(\tilde{x}_i, \tilde{p}) / n \equiv \tilde{Z}_n$ . We

therefore have

$$\begin{aligned}
\Pi^n &= E \left[ \left( \tilde{X} - \tilde{p}^n \right) \tilde{Z}_n \right] \\
&= E \left[ \left( \tilde{X} - \pi_0^n - \sum_{i=1}^n \pi_i^n (\tilde{X} + \tilde{\eta}_i^n) + \gamma^n \tilde{Z}_n \right) \tilde{Z}_n \right] \\
&= \left( 1 - \sum_{i=1}^n \pi_i^n \right) E \left[ \tilde{X} \tilde{Z}_n \right] - \pi_0^n E \left[ \tilde{Z}_n \right] + \gamma^n E \left[ \tilde{Z}_n \tilde{Z}_n \right] \\
&= \left( 1 - \sum_{i=1}^n \pi_i^n \right) \bar{X} \bar{Z} - \pi_0^n \bar{Z} + \gamma^n (\Delta^2 + \bar{Z}^2) \\
&\xrightarrow{n \rightarrow \infty} (1 - \pi^*) \bar{X} \bar{Z} - \pi_0^* \bar{Z} + \gamma^* (\Delta^2 + \bar{Z}^2).
\end{aligned}$$

Now, since  $\bar{X} = \bar{Z} = 0$  it follows that

$$\Pi = \gamma^* \Delta^2 = \frac{\Delta^2 (\beta + \Delta^2) \sigma^2}{\Delta^2 + \beta (\beta + \Delta^2) \sigma^2}. \quad (52)$$

We also have

$$\begin{aligned}
\Pi_i &= \frac{\Delta^2}{\sigma^2 (\Delta^2 + \beta)} \left( (\gamma^*)^2 \Delta^2 - \pi^* (1 - \pi^*) \sigma^2 \right) + \frac{W_i}{s^2} \left( (1 - \pi^*)^2 \sigma^2 + (\gamma^*)^2 \Delta^2 \right) \\
&= \frac{\Delta^4 (W_i + s^2 \Delta^2) \sigma^2 + W_i \Delta^2 (\beta + \Delta^2)^2 \sigma^4}{s^2 (\Delta^2 + \beta (\beta + \Delta^2) \sigma^2)^2}.
\end{aligned} \quad (53)$$

It then follows from (53) that

$$\begin{aligned}
\frac{\partial \Pi_i}{\partial W_i} &= \frac{\Delta^4 \sigma^2 + \Delta^2 (\beta + \Delta^2)^2 \sigma^4}{s^2 (\Delta^2 + \beta (\beta + \Delta^2) \sigma^2)^2} > 0, \\
\frac{\partial \Pi_i}{\partial \beta} &= - \frac{2\Delta^4 (s^2 \Delta^4 + \beta (W + 2s^2 \Delta^2)) \sigma^4 + 2W_i \Delta^2 (\beta + \Delta^2)^3 \sigma^6}{s^2 (\Delta^2 + \beta (\beta + \Delta^2) \sigma^2)^3} < 0.
\end{aligned}$$

Hence the proposition follows. ■

*Proof of Proposition 7:* It follows from (52) that

$$\frac{\partial \Pi}{\partial \beta} = \frac{\Delta^4 \sigma^2 - \Delta^2 (\beta + \Delta^2)^2 \sigma^4}{(\Delta^2 + \beta (\beta + \Delta^2) \sigma^2)^2}.$$

Observe that the above is strictly negative if and only if  $\sigma < \frac{1}{\Delta}$  and  $\beta < \frac{\Delta}{\sigma} - \Delta^2$ . ■

*Proof of Proposition 8:* Following Theorem 1 and (51), we can rewrite agent  $i$ 's demand function as follows:

$$\psi_i(\tilde{x}_i, \tilde{p}) = c_i + \frac{\Delta^2(-\beta s^2 + W_i)}{s^2(\Delta^2 + \beta(\beta + \Delta^2)\sigma^2)} \tilde{X} + \frac{s^2\Delta^2 + (\beta + \Delta^2)\sigma^2 W_i}{s^2(\Delta^2 + \beta(\beta + \Delta^2)\sigma^2)} \tilde{Z} + \frac{\sum_{k \in W(i)} \tilde{\epsilon}_k}{s^2}, \quad (54)$$

where  $c_i$  is a constant scalar. Thus,

$$\begin{aligned} \text{cov}(\psi_i(\tilde{x}_i, \tilde{p}), \psi_j(\tilde{x}_j, \tilde{p})) &= \left( \frac{\Delta^2(-\beta s^2 + W_i)}{s^2(\Delta^2 + \beta(\beta + \Delta^2)\sigma^2)} \right) \left( \frac{\Delta^2(-\beta s^2 + W_j)}{s^2(\Delta^2 + \beta(\beta + \Delta^2)\sigma^2)} \right) \sigma^2 \\ &\quad + \left( \frac{s^2\Delta^2 + (\beta + \Delta^2)\sigma^2 W_i}{s^2(\Delta^2 + \beta(\beta + \Delta^2)\sigma^2)} \right) \left( \frac{s^2\Delta^2 + (\beta + \Delta^2)\sigma^2 W_j}{s^2(\Delta^2 + \beta(\beta + \Delta^2)\sigma^2)} \right) \Delta^2 + W_{ij}. \end{aligned} \quad (55)$$

On the other hand, observe from (54) that the variance of agent  $i$ 's demand,  $\text{var}(\psi_i(\tilde{x}_i, \tilde{p}))$ , does not depend on  $W_{ij}$ . Therefore, following (55) we have

$$\frac{\partial \text{corr}(\psi_i(\tilde{x}_i, \tilde{p}), \psi_j(\tilde{x}_j, \tilde{p}))}{\partial W_{ij}} = \frac{1}{\sqrt{\text{var}(\psi_i(\tilde{x}_i, \tilde{p})) \text{var}(\psi_j(\tilde{x}_j, \tilde{p}))}} > 0.$$

Hence we have the desired result. ■

*Proof of Proposition 9:* (a) From (8) and (51), it follows that  $\psi_i \sim N(0, a_1 + a_2 W_i + a_3 W_i^2)$ , where  $a_1 = \frac{\Delta^6 + \beta^2 \Delta^4 \sigma^2}{a_4^2}$ ,  $a_2 = \frac{1}{s^2} \left( 1 + \frac{2\Delta^6 \sigma^2}{a_4^2} \right)$ ,  $a_3 = \frac{\Delta^2 \sigma^2 (\beta^2 \sigma^2 + \Delta^4 \sigma^2 + \Delta^2 + 2\Delta^2 \beta \sigma^2)}{s^4 a_4^2}$ , and  $a_4 = \beta^2 \sigma^2 + \Delta^2 + \Delta^2 \beta \sigma^2$ . Since,  $E[|\tilde{z}|] = \sqrt{\frac{2A}{\pi}}$  for a general normally distributed random variable,  $z \sim N(0, A)$ , it follows that

$$\psi_i^{\text{unsigned}} = \sqrt{\frac{2(a_1 + a_2 W_i + a_3 W_i^2)}{\pi}}. \quad (56)$$

It immediately follows that this function is increasing and concave, with the given asymptotics. It is also clear that  $E[\psi_i^2] = \text{var}(\psi_i) + E[\psi_i]^2 = \text{var}(\psi_i) = a_1 + a_2 W_i + a_3 W_i^2 = \frac{\pi}{2} (\psi_i^{\text{unsigned}})^2$ , so it is indeed the case that  $\psi_i^{\text{unsigned}} = \sqrt{\frac{2}{\pi} E[\psi_i^2]}$ .

(b,c) We have

$$\begin{aligned} E \left[ \sum_i \psi_i^2(W_i) d(i) \right] &= \sum_i E[\psi_i^2(W_i)] d(i) \\ &= \sum_i (a_1 + a_2 W_i + a_3 W_i^2) d(i) = a_1 + a_2 s^2 \beta + a_3 s^4 (\beta^2 + \sigma_\beta^2) \\ &= \beta^2 + \Delta^2 + a_3 \sigma_\beta^2. \end{aligned}$$

Therefore,  $\psi^{market} = \sqrt{\frac{\pi}{2}(\beta^2 + \Delta^2 + a_3\sigma_\beta^2)}$ , and  $\psi^{market}$  is increasing in  $\sigma_\beta$ . Moreover, for small  $\sigma_\beta$ ,  $\psi^{market}$  is increasing in  $\beta$ . Also, it is easy to show that  $\frac{\partial a_3}{\partial \beta} < 0$ , so for large  $\sigma_\beta$ ,  $\psi^{market}$  is decreasing in  $\beta$ .  $\blacksquare$

*Proof of Proposition 10:* The following lemma ensures that the limit of average certainty equivalents is equal to the average certainty equivalent in the large economy.

**Lemma 1.** *If Assumption 1 and the conditions of Theorem 1 are satisfied, and the function  $f : \mathbb{N} \rightarrow \mathbb{R}$  is concave and increasing, then  $\lim_{n \rightarrow \infty} \sum_{i=1}^n d^n(i)f(i) = \sum_{i=1}^{\infty} d(i)f(i)$  with probability one.*

*Proof:* Since  $f$  is concave, it is clear that  $f \leq g$ , where  $g(i) \stackrel{\text{def}}{=} f(1) + (f(2) - f(1))i \stackrel{\text{def}}{=} c_0 + c_1i$ . From (11), and since  $f$  is increasing, it is therefore clear that  $\sum_{i=1}^n d^n(i)f(i) \in [c_0, c_0 + c_1\beta + \epsilon]$ , for arbitrary small  $\epsilon > 0$ , for large  $n$ .

Now, for arbitrary  $m$  and  $\epsilon > 0$ , by Assumption 1, for large enough  $n_0$ , for all  $n \geq n_0$ ,  $|d^n(i) - d(i)| \leq \frac{\epsilon}{m(c_0 + c_1)}$ . Also, for large enough  $m$  and  $n'_0$ , for all  $n \geq n'_0$ ,  $\sum_{i=m+1}^n d^n(i)f(i) \leq \epsilon$ , from (11). Finally, from Assumption 1, for large enough  $m$ ,  $\sum_{i=m+1}^{\infty} d(i)f(i) \leq \epsilon$ .

Thus, for an arbitrary  $\epsilon > 0$ , a large enough  $m$  can be chosen and  $n_0^* = \max(m, n_0, n'_0)$  such that for all  $n \geq n_0^*$ ,

$$\begin{aligned} \left| \sum_{i=1}^n d^n(i)f(i) - \sum_{i=1}^{\infty} d(i)f(i) \right| &\leq \left| \sum_{i=1}^m d^n(i)f(i) - \sum_{i=1}^m d(i)f(i) \right| \\ &+ \left| \sum_{i=m+1}^n d^n(i)f(i) - \sum_{i=m+1}^n d(i)f(i) \right| + \left| \sum_{i=n+1}^{\infty} d(i)f(i) \right| \\ &\leq \epsilon + \epsilon + \epsilon, \end{aligned}$$

and since  $\epsilon > 0$  is arbitrary, convergence follows.

The expected utility in the large economy of an agent with  $W$  connections is

$$\begin{aligned} U(W) &= E \left[ -e^{-\psi_i(\bar{x}_i, \bar{p})(\bar{X} - \bar{p})} \right] \\ &= \frac{1}{\sqrt{8\pi^3\sigma^2\Delta^2W/s^2}} \int \int \int -e^{-\psi_i(X+\eta_i, p)(X+\eta_i-p) - \frac{X^2}{2\sigma^2} - \frac{Z^2}{2\Delta^2} - \frac{\eta_i^2}{2W/s^2}} dX dZ d\eta_i \\ &= -\frac{s(\beta^2\sigma^2 + \Delta^2 + \Delta^2\beta\sigma^2)}{\sqrt{(\Delta^2 + (\beta + \Delta^2)^2\sigma^2)(\beta^2s^2\sigma^2 + \Delta^2\sigma^2 + \Delta^2\sigma^2W)}}, \end{aligned}$$

where the last equality follows by using (12-15,51). Since  $U(W) = -e^{-CE(W)}$ , condition (a) immediately follows.

Moreover, since the function  $CE(W)$  is increasing and concave in  $W$ , from Lemma 1, it is clear that the average certainty equivalent is as defined in (b).  $\blacksquare$

*Proof of Proposition 11:* (a) This follows immediately from Jensen's inequality, since  $CE(W)$  is a strictly convex function of  $W \geq 1$ .

(b) We first note that the "two-point distribution," for which a fraction  $\beta - \lfloor \beta \rfloor$  of the agents has  $\lfloor \beta \rfloor + 1$  connections and the rest has  $\lfloor \beta \rfloor$  connections, has connectedness  $(\beta - \lfloor \beta \rfloor)(\lfloor \beta \rfloor + 1) + (1 - \beta + \lfloor \beta \rfloor)\lfloor \beta \rfloor = \beta$ , so the two-point distribution is indeed a candidate for an optimal distribution. Clearly, this is the only two-point distribution with support on  $\{n, n + 1\}$  that has connectedness  $\beta$ , and for  $\beta \notin \mathbb{N}$ , there is no one-point distribution with connectedness  $\beta$ . We define  $n = \lfloor \beta \rfloor$ ,  $q_n = 1 - \beta + \lfloor \beta \rfloor$ ,  $q_{n+1} = \beta - \lfloor \beta \rfloor$ .

We introduce some new notation. We wish to study a larger space of distributions than the ones with support on the natural numbers. Therefore, we introduce the space of discrete distributions with finite first moment,  $D = \{\sum_{i=0}^{\infty} r_i \delta_{x_i}\}$ , where  $r_i \geq 0$ , and  $0 \leq x_i$  for all  $i$ ,  $0 < \sum_i r_i < \infty$  and  $\sum_i r_i x_i < \infty$ .<sup>1</sup> The subset,  $D^1 \subset D$ , in addition satisfies  $\sum_i r_i = 1$ .

The c.d.f. of a distribution in  $D$  is a monotone function,  $F_d : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , defined as  $F_d(x) = \sum_{i \geq 0} r_i \theta(x - x_i)$ , where  $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ . Here,  $\theta$  is the Heaviside step function. Clearly,  $F_d$  is bounded:  $\sup_{x \geq 0} F_d(x) = \sum_i r_i < \infty$ . We use the Lévy metric to separate distributions in  $D$ ,  $\mathcal{D}(d_1, d_2) = \inf\{\epsilon > 0 : F_{d_1}(x - \epsilon) - \epsilon \leq F_{d_2}(x) \leq F_{d_1}(x + \epsilon) \text{ for all } x \in \mathbb{R}_+\}$ . We thus identify  $d_1 = d_2$  iff  $\mathcal{D}(d_1, d_2) = 0$ .

For  $d \in D$ , we define the operation of addition and multiplication:  $d_1 = \sum_i r_i^1 \delta_{x_i^1}$ ,  $d_2 = \sum_i r_i^2 \delta_{x_i^2}$  leads to  $d_1 + d_2 = \sum_i r_i^1 \delta_{x_i^1} + \sum_i r_i^2 \delta_{x_i^2}$  and  $\alpha d_1 = \sum_i \alpha r_i^1 \delta_{x_i^1}$ , for  $\alpha > 0$ . The two-point distribution can then be expressed as  $\hat{d} = q_n \delta_n + q_{n+1} \delta_{n+1}$ .

The support of a distribution  $d = \sum r_x \delta_{x_i}$  in  $D$  is now  $\text{supp}[d] = \{x_i : r_i > 0\}$ . A subset of  $D$  is the set of distributions with support on the integers,  $D_{\mathbb{N}} = \{d \in D : \text{supp}[d] \subset \mathbb{N}\}$ . For this space, we can without loss of generality assume that the  $x$ 's are ordered,  $x_i = i$ . The expectation of a distribution is  $E[d] = \sum_i r_i x_i$  and the total mass is  $S(d) = \sum_i r_i$ . Both the total mass and expectations operators are linear. Another subset of  $D$ , given  $\beta > 0$ , is  $D_{\beta} = \{d \in D : E[d] = \beta\}$ .

Given a strictly concave, function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ , we define the operator  $V_f : D \rightarrow D$ , such that  $V_f(d) = \sum_i r_i \delta_{f(x_i)}$ . The function  $f(x) = CE(x)$ , is, of course, strictly concave  $\mathbb{R}_+$ .

<sup>1</sup>Distribution here is in the sense of a functional on the space of infinitely differentiable functions with compact support,  $C_0^{\infty}$  (see Hörmander [1]), and  $\delta_x$  is the Dirac distribution, defined by  $\delta_x(f) = f(x)$  for  $f \in C_0^{\infty}$ .



Clearly,  $V_f$  is a linear operator,  $V_f(d_1 + d_2) = V_f(d_1) + V_f(d_2)$ .

The second part of the theorem, which we wish to prove, now states that for all  $d \in D^1 \cap D_{\mathbb{N}} \cap D_{\beta}$ , with  $\beta \notin \mathbb{N}$ , if  $d \neq \hat{d}$ , it is the case that  $E[V_f(d)] > E[V_f(\hat{d})]$ . It turns out that the inequality holds for any strictly concave function on  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ . To prove this, we use Jensen's inequality, which in our notation reads:

**Lemma 2. (Jensen):** *For any  $d \in D$ , with support on more than one point, and for a strictly concave function,  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ , the following inequality holds:*

$$E[V_f(d)] < S(d)E[V_f(\delta_{E[d]/S(d)})] = E[V_f(S(d)\delta_{E[d]/S(d)})].$$

Now, let us take a candidate function for an optimal solution,  $d \neq \hat{d}$ , such that  $d \in D^1 \cap D_{\mathbb{N}} \cap D_{\beta}$ . Clearly, since  $\hat{d}$  is the only two-point distribution in  $D^1 \cap D_{\mathbb{N}} \cap D_{\beta}$ , and there is no one-point distribution in  $D^1 \cap D_{\mathbb{N}} \cap D_{\beta}$ , the support of  $d$  is at least on three points.

Also, since  $q_n + q_{n+1} = 1$ , and  $d \in D^1$ , it must either be the case that  $r_n < q_n$ , or  $r_{n+1} < q_{n+1}$ , or both. We will now decompose  $d$  into three parts, depending on which situation holds: First, let's assume that  $r_{n+1} \geq q_{n+1}$ . If, in addition,  $r_{n+1} > q_{n+1}$ , then it must be that  $r_n < q_n$ , and  $r_i > 0$  for at least one  $i < n$ . Otherwise, it could not be that  $E[d] = \beta$ . In this case, we define  $d_1 = \sum_{i < n} r_i \delta_i$ ,  $d_2 = r_n \delta_n + q_{n+1} \delta_{n+1}$  and  $d_3 = (r_{n+1} - q_{n+1}) \delta_{n+1} + \sum_{i > n+1} r_i \delta_i$ . If, on the other hand,  $r_{n+1} = q_{n+1}$ , then there must be an  $i < n$  such that  $r_i > 0$  and also a  $j > n + 1$  such that  $r_j > 0$ , since otherwise it would not be possible to have  $E[d] = \beta$ . In this case, we define,  $d_1 = \sum_{i < n} r_i \delta_i$ ,  $d_2 = r_n \delta_n + q_{n+1} \delta_{n+1}$  and  $d_3 = \sum_{i > n+1} r_i \delta_i$ . Exactly the same technique can be applied in the case of  $r_n \geq q_n$  and  $r_{n+1} < q_{n+1}$ .

Finally, in the case of  $r_n < q_n$  and  $r_{n+1} < q_{n+1}$ , there must, again, be an  $i < n$  such that  $r_i > 0$  and a  $j > n + 1$ , such that  $r_j > 0$ , otherwise  $E[d] = \beta$  would not be possible. In this case, we decompose  $d_1 = \sum_{i < n} r_i \delta_i$ ,  $d_2 = r_n \delta_n + q_{n+1} \delta_{n+1}$  and  $d_3 = (r_{n+1} - q_{n+1}) \delta_{n+1} + \sum_{i > n+1} r_i \delta_i$ .

These decompositions imply that

$$\begin{aligned} E[V_f(d)] &= E[V_f(d_1)] + E[V_f(d_2)] + E[V_f(d_3)] \\ &\leq S(d_1)E[V_f(\delta_{E[d_1]/S(d_1)})] + E[V_f(d_2)] + S(d_3)E[V_f(\delta_{E[d_3]/S(d_3)})] \\ &= E[V_f(S(d_1)\delta_{E[d_1]/S(d_1)} + d_2 + S(d_3)\delta_{E[d_3]/S(d_3)})] \\ &= E[V_f(d_m)], \end{aligned}$$

where  $d_m = d_L + d_2 + d_R$ ,  $d_L = S(d_1)\delta_{E[d_1]/S(d_1)}$  and  $d_R = S(d_3)\delta_{E[d_3]/S(d_3)}$ . Clearly,  $d_m \in$

$D^1 \cap D_\beta$ .

Now, if  $r_{n+1} \geq q_{n+1}$ , since  $d \in D^1$ , it must be that  $S(d_1) + S(d_3) = q_n - r_n$ , and since  $E[d_L + d_2 + d_R] = \beta = E[q_n \delta_n + q_{n+1} \delta_{n+1}]$  it must be that  $E[d_L + d_R] = (q_n - r_n)E[\delta_n] = E[(S(d_1) + S(d_2))\delta_n] = E[d_a]$ , where  $d_a = (S(d_1) + S(d_2))\delta_n$ . Moreover, since  $d_a + d_2$  has support on  $\{n, n+1\}$  and  $E[d_a + d_2] = \beta$ , it is clear that  $d_a + d_2 = \hat{d}$ .

From Jensen's inequality, it is furthermore clear that  $E[V_f(d_L + d_R)] < E[V_f(d_a)]$ , and therefore  $E[V_f(d_m)] = E[V_f(d_L + d_R + d_2)] < E[V_f(d_a + d_2)] = E[V_f(\hat{d})]$ . Thus, all in all,  $E[V_f(d)] \leq E[V_f(d_m)] < E[V_f(\hat{d})]$ . A similar argument can be applied if  $r_n \geq q_n$ .

Finally, in the case in which  $r_n < q_n$  and  $r_{n+1} < q_{n+1}$ , we define  $\alpha = E[d_1]/S(d_1)$  and  $\beta = E[d_3]/S(d_3)$ . Obviously,  $\alpha < n < n+1 < \beta$ . Now, we can define  $g_1 = \frac{\beta-n}{\beta-\alpha}(q_n - r_n)\delta_\alpha + \frac{n-\alpha}{\beta-\alpha}(q_n - r_n)\delta_\beta$  and  $g_2 = \frac{\beta-n-1}{\beta-\alpha}(q_{n+1} - r_{n+1})\delta_\alpha + \frac{n+1-\alpha}{\beta-\alpha}(q_n - r_n)\delta_\beta$ . Clearly,  $g_1 \in D$  and  $g_2 \in D$  and, moreover,  $g_1 + g_2 + d_2 = d_1 + d_2 + d_3 = d$ . Also, Jensen's inequality implies that  $E[V_f(g_1)] < E[V_f((q_n - r_n)\delta_n)]$  and  $E[V_f(g_2)] < E[V_f((q_{n+1} - r_{n+1})\delta_{n+1})]$ , so  $E[V_f(d)] = E[V_f(g_1 + g_2 + d_2)] < E[V_f((q_n - r_n)\delta_n + (q_{n+1} - r_{n+1})\delta_{n+1} + d_2)] = E[V_f(\hat{d})]$ . We are done.

(c) From (a,b,18,19) it follows that  $\overline{CE}^*(\beta)$  is of the form  $\overline{CE}^*(\beta) = \frac{1}{2} \log(v(\beta))$ , where  $v(\beta) = \frac{(\Delta^2 + (\beta + \Delta^2)^2 \sigma^2)(\beta^2 s^2 \sigma^2 + \Delta^2 s^2 + \beta \Delta^2 \sigma^2)}{s^2(\beta^2 \sigma^2 + \Delta^2 + \Delta^2 \beta \sigma^2)^2}$ . It immediately follows that  $v'(\beta)$  is of the form  $-v_2(\beta)(c_4 \beta^4 + c_3 \beta^3 + c_2 \beta^2 + c_1 \beta + c_0)$ , where  $v_2(\beta) > 0$  for all  $\beta > 0$ ,  $c_4 > 0$ ,  $c_3 > 0$  and  $c_2 > 0$ , and where  $c_1 = \Delta^2 + 4s^2 - 3$  and  $c_0 = 2\Delta^2 s^2 \sigma^2 - \Delta^2 \sigma^2 - 1$ . Moreover, since  $c_4 > 0$ , it follows that  $v'(\beta) < 0$  for large  $\beta$ .

From Descartes' rule of signs, it follows that the maximum number of roots to  $v'(\beta) = 0$  is two, and there can only be two roots if  $c_1 < 0$  and  $c_0 > 0$ . The condition  $c_0 > 0$  is equivalent to  $2s^2 - 1 > \frac{1}{\Delta^2 \sigma^2}$ , which in particular implies that  $s^2 > \frac{1}{2}$ . Similarly,  $c_1 < 0$  iff  $3 - 4s^2 > \Delta^2 \sigma^2$ , which in particular implies that  $s^2 < \frac{3}{4}$ . Multiplying these two conditions, we get that a necessary condition for the two roots to be possible is that  $(3 - 4s^2)(2s^2 - 1) > 1$ , for  $s \in (1/2, 3/4)$ , but it is easy to check that  $(3 - 4s^2)(2s^2 - 1)$  is in fact less than one in this region. Therefore, it can not be the case that  $c_1 < 0$  and  $c_0 > 0$  at the same time, and there can be at most one root to the equation  $v'(\beta) = 0$ . Since  $v'(\beta) < 0$  for large  $\beta$ , it must therefore be the case that  $v(\beta)$  is either decreasing for all  $\beta$ , or initially increasing and then decreasing, with a unique maximum. It is easy to check numerically that both cases are in fact possible. We are done.  $\blacksquare$

*Proof of Proposition 12:* Since  $f$  is weakly convex and  $CE$  is concave,  $CE - f$  is a concave function of  $W$ , and an identical argument as in the proofs of Proposition 11 (a),(b) can be

made to show that a degenerate (i.e., uniform) network is optimal. Now, since  $\overline{CE}^*$ , as defined in Proposition 11, is decreasing for large  $\beta$ , it follows that  $\overline{CE}^* - f$  is decreasing for large  $\beta$ , so the optimal  $\beta$  must be interior. ■

## References

- [1] L. Hörmander, The Analysis of Linear Partial Differential Operators, I-IV, Springer Verlag, 1983.
- [2] D.J. Watts, Small Worlds, Princeton University Press, 1999.