

Bounds for path-dependent options

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Received: 5 July 2014 / Accepted: 23 July 2015 / Published online: 19 September 2015
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Abstract We develop new semiparametric bounds on the expected payoffs and prices of European call options and a wide range of path-dependent contingent claims. We first focus on the trinomial financial market model in which, as is well-known, an exact calculation of derivative prices based on no-arbitrage arguments is impossible. We show that the expected payoff of a European call option in the trinomial model with martingale-difference log-returns is bounded from above by the expected payoff of a call option written on an asset with i.i.d. symmetric two-valued log-returns. We further show that the expected payoff of a European call option in the multiperiod trinomial option pricing model is bounded by the expected payoff of a call option in

We thank the Risk Management Institute at the National University of Singapore and the International Center for Finance (ICF) at the School of Management (SOM) at Yale University for support. Ibragimov's work on the final version of the paper was completed when he was visiting Innopolis University and Kazan Federal University-KFU (Kazan, Russia). He gratefully acknowledges partial support from the Russian Ministry of Education and Science (Innopolis University) and the Russian Government Program of Competitive Growth of Kazan Federal University (Higher Institute of Information Technologies and Information Systems, KFU).

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the two-period model with a log-normal asset price. We also obtain bounds on the possible prices of call options in the (incomplete) trinomial model in terms of the parameters of the asset's distribution. Similar bounds also hold for many other contingent claims in the trinomial option pricing model, including those with an arbitrary convex increasing payoff function as well as for path-dependent ones such as Asian options. We further obtain a wide range of new semiparametric moment bounds on the expected payoffs and prices of path-dependent Asian options with an arbitrary distribution of the underlying asset's price. These results are based on recently obtained sharp moment inequalities for sums of multilinear forms and U -statistics and provide their first financial and economic applications in the literature. Similar bounds also hold for many other path-dependent contingent claims.

Keywords Option bounds · Trinomial model · Binomial model · Semiparametric bounds · Option prices · Expected payoffs · Path-dependent contingent claims · Asian options · Moment inequalities

JEL Classification G12 · C02 · C65

1 Introduction and discussion

1.1 Objectives and key results

The present paper develops new semiparametric bounds on the expected payoffs and prices of European call options and a wide range of possibly path-dependent contingent claims. We first focus on the trinomial financial market pricing model in which, as is well-known, calculation of derivative prices based on no-arbitrage arguments is impossible (Theorems 2.1–3.2). Our results show, in particular, that the expected payoff of a European call option in the trinomial model, with log-returns forming a martingale-difference sequence, is bounded from above by the expected payoff of a call option written on an asset with i.i.d. symmetric two-valued log-returns (Theorem 2.2). Thus, the results reduce the problem of derivative pricing in the trinomial model to the binomial case. We further show that the expected payoff of a European call option in the multiperiod trinomial option pricing model is bounded by the expected payoff of a call option in the two-period model with a log-normal asset price. These bounds thus allow one to reduce the problem of pricing options in the trinomial model to the case of two periods and the standard assumption of normal log-returns. Using the fact that risk-averse investors require a higher rate of return on the call option than on its underlying asset (see, e.g., Rodriguez 2003), we also obtain bounds on the possible prices of call options in the (incomplete) trinomial model in terms of the parameters of the asset's distribution (Theorem 2.4).

From our results it follows that semiparametric bounds completely similar to the case of European call options also hold for many other contingent claims in the trinomial option pricing model. In particular, the bounds hold for contingent claims with an arbitrary convex increasing payoff function (Theorem 2.1) as well as for path-dependent ones, such as Asian options written on averages of the underlying asset's

prices (Theorems 3.1, 3.2). The approach can also be applied in the analysis of American option pricing.

The analysis for the trinomial pricing model is based on general characterization results for two-valued martingale difference sequences and multiplicative forms obtained in Sharakhmetov and Ibragimov (2002) (see also de la Peña et al. 2006). These characterization results that we review in Sect. 6 demonstrate, in particular, that martingale-difference sequences consisting of random variables (r.v.'s) each of which takes two values are, in fact, sequences of independent r.v.'s. The results allow one to reduce the study of many problems for three-valued martingales to the case of i.i.d. symmetric Bernoulli r.v.'s and provide the key to the development of semiparametric bounds for the expected payoffs and prices of path-dependent contingent claims in this work.

We further obtain a wide range of new semiparametric bounds on the expected payoffs and prices of general path-dependent Asian options in terms of moments of the underlying asset's distribution. These bounds are based on recently obtained sharp moment inequalities for sums of multilinear forms and U -statistics (see de la Peña et al. 2002, 2003). The moment inequalities provide estimates for moments of sums of U -statistics and multilinear forms in terms of "computable" quantities given by (conditional) moments of the random summands in consideration. The results in the paper provide the first financial and economic applications of the moment estimates in the literature. Semiparametric moment bounds similar to those for Asian options also hold for many other path-dependent contingent claims.

1.2 Related literature

Many approaches to contingent claim pricing have been devised, but ultimately most belong to one of three main methodologies: exact solutions, numerical methods and semiparametric bounds. Numerical approaches, in turn, can be classified into one of four categories: (1) formulas and approximations, including application of transform methods and asymptotic expansion techniques; (2) lattice and finite difference methods; (3) Monte-Carlo simulation; (4) other specialized methods (see the review in Broadie and Detemple 2004, and references therein).

Lattice approaches to contingent claim pricing, first proposed in Parkinson (1977) and Cox et al. (1979), use discrete-time and discrete-space approximations to the underlying asset's price process to compute derivative prices. In the binomial derivative pricing model developed in Cox et al. (1979), the discrete distributions are chosen in such a way that their first and second moments match those of the underlying asset either exactly or in the limit as the discrete time step goes to zero. Ritchken and Trevor (1999) use approximations to log-normal log-returns by a sequence of trivariate or more general discrete r.v.'s for option pricing when the underlying asset's price follows a GARCH process with a massive path-dependence. Discrete approximations based on distributions concentrated on more than two points allow one to match higher moments of the asset's price distribution or include additional parameters providing greater flexibility compared to the binomial model, such as the stretch parameters λ

in [Boyle \(1988\)](#)'s and [Kamrad and Ritchken \(1991\)](#)' trinomial approach to derivative pricing.

As discussed in, e.g., [Kamrad and Ritchken \(1991\)](#), the trinomial approximations are computationally more efficient than the binomial ones in the case of European call options. However, the additional accuracy of the trinomial method for valuing American option prices is almost exactly balanced by the additional computational cost (see [Broadie and Detemple 1996, 2004](#)) and one would expect the same to be the case for path-dependent options, such as Asian options. This emphasizes the importance of the study of bounds on the expected payoffs and prices of (possibly path-dependent) contingent claim in the trinomial model whose values can be calculated efficiently.

The present paper provides such bounds for a wide range of general path-dependent contingent claims written on an asset with a three-valued price process. As discussed before, our results allow one to reduce the problem of calculating the prices of contingent claims in the trinomial model with dependent returns to the case of binomial one with i.i.d. assumptions. Furthermore, some of our results provide bounds for the expected payoffs and prices of options in the multiperiod trinomial model in terms of those for contingent claims written on an asset with log-normally distributed price in the two-period model. These results essentially reduce the problem of option pricing in the trinomial model to the case of two periods and the standard assumption of normality of the underlying asset's log-returns.

In addition to its importance in the trinomial model, the bounds approaches to contingent claim pricing have a number of advantages over close analytical solutions and Monte-Carlo techniques that make very strong assumptions concerning the underlying asset's price distribution (see the discussion in [de la Peña et al. 2004](#)). In particular, semiparametric approaches to derivative pricing are more robust than the exact pricing and Monte Carlo methods since they only assume that a specific number of the asset's distributional characteristics, such as moments, are known. These approaches are thus well suited for making inferences on contingent claims prices in the real-world settings, including, heavy-tailed distributions and large price fluctuations typically observed in economic, financial and insurance markets.

The restrictive assumptions needed for deriving exact analytical solutions or applying Monte-Carlo techniques introduce modeling error into the closed-form solution. Unfortunately, these approaches usually do not lend themselves easily to the study of the modeling error nor do they lend to the study of error propagation. Without such error bounds it is difficult to ascertain the validity of a model and its assumptions. The bounds approach gives upper and lower bounds to such errors and thus provide an indirect test of model misspecification for exact or numerical methodologies.¹

Motivated by the appealing properties of the semiparametric bounds approach to derivative pricing, a number of studies in the finance literature have focused on the problems of deriving estimates for the expected payoffs and prices of contingent claims in the last quarter of the century. For instance, [Merton \(1973\)](#) establishes option pricing bounds that requires no knowledge of the underlying asset's price distribution, and only imposes nonsatiation as behavioral assumption. [Perrakis and Ryan \(1984\)](#) and

¹ In fact, [Grundy \(1991\)](#) notes that the problem can be inverted and estimates bounds on the parameters of the assumed distribution can be inferred from the bounds using observed prices.

Perrakis (1986) develop option bounds in discrete time. Although the results in Perrakis and Ryan (1984) and Perrakis (1986) are quite general, they assume that the whole distribution of returns is known.

Lo (1987) and Grundy (1991) extend the option bound results to semi-parametric formulas and thus considerably weaken the necessary assumptions to apply their bounds. Grundy (1991) uses these results to obtain lower bounds on the noncentral moments of the underlying asset's return distribution when option prices are observed. The option bounds in Lo (1987) and Grundy (1991) are closely related to estimates for a firm's expected profit in inventory theory models (see Scarf 1958, 2002). Boyle and Lin (1997) extend Lo's results to contingent claims based on multiple assets. Constantinides and Zariphopoulou (2001) study intertemporal bounds under transaction costs. Frey and Sin (1999) study bounds under a stochastic volatility model. Simon et al. (2000) show that an Asian option can be bounded from above by the price of a portfolio of European options. Rodriguez (2003) shows that many option pricing bounds in the literature can be derived using a single analytical framework and shows, in particular, how the estimates for the expected payoffs of the contingent claims produce corresponding bounds on their prices under the assumption of risk-averse investors. de la Peña et al. (2004) obtain sharp estimates for the expected payoffs and prices of European call options on an asset with an absolutely continuous price in terms of the price density characteristics and also derive bounds on the multiperiod binomial option-pricing model with time-varying moments. The bounds in de la Peña et al. (2004) reduce the multiperiod binomial setup to a two-period setting with a Poisson distribution of the log-returns, which is advantageous from a computational perspective.

The financial applications presented in this paper complement the above literature and provide, essentially, a new approach to pricing path-dependent contingent claims in the discrete state space setting. In addition, they further provide the first financial applications of the probabilistic moment inequalities for sums of U -statistics and multilinear forms available in the literature.

1.3 Organization of the paper

The paper is organized as follows. Section 2 presents the main results of the paper on semiparametric bounds for the expected payoffs and prices of contingent claims in the multiperiod trinomial model with dependent log-returns that. These results allow one to reduce the analysis to the i.i.d. multiperiod case of the binomial model or the two-period case of the derivative pricing model with log-normal returns. Section 3 shows how the approach developed in Sect. 2 can also be used to obtain bounds for path-dependent derivatives and, as an illustration, provide estimates for Asian options in the trinomial option pricing model. Section 4 provides semiparametric bounds for Asian options via moment inequalities for sums of multilinear forms and U -statistics. Finally, some concluding remarks are made in Sect. 5. Appendix 6 reviews the characterization results for two-valued martingale difference sequences and multiplicative forms in Sharakhmetov and Ibragimov (2002) that provide the basis for the analysis in Sects. 2 and 3. Section 5 makes some concluding remarks.

2 Bounds in the trinomial option pricing model

In this section, we present the main results of the paper on semiparametric bounds for the expected payoffs and prices of contingent claims in the trinomial model with dependent log-returns.

Let $\{u_t\}_{t=1}^\infty$ be a sequence of nonnegative numbers and let $\mathfrak{S}_0 = (\Omega, \emptyset) \subseteq \mathfrak{S}_1 \subseteq \dots \subseteq \mathfrak{S}_t \subseteq \dots \subseteq \mathfrak{S}$ be an increasing sequence of σ -algebras on a probability space $(\Omega, \mathfrak{S}, P)$. Throughout the paper, we consider a market consisting of two assets. The first asset is a risky asset with the trinomial price process

$$S_0 = s, \quad S_t = S_{t-1}X_t, \quad t \geq 1, \tag{2.1}$$

where the $(X_t)_{t=1}^\infty$ is an (\mathfrak{S}_t) -adapted sequence of nonnegative r.v.'s representing the asset's gross returns (additional assumptions concerning the dependence structure of X_t 's will be made below). The second asset is a money-market account with a risk-free rate of return r .

First, we assume that random log-returns $\log(X_t)$ form an (\mathfrak{S}_t) -martingale-difference sequence and take on three values $u_t, -u_t$ and 0:

$$P(\log(X_t) = u_t) = P(\log(X_t) = -u_t) = p_t, \quad P(\log(X_t) = 0) = 1 - 2p_t, \tag{2.2}$$

$0 \leq p_t \leq 1/2, t = 1, 2, \dots$ (so that, in period t , the price of the asset increases to $S_t = \exp(u_t)S_{t-1}$ with probability p_t , decreases to $S_t = \exp(-u_t)S_{t-1}$ with the same probability or stays the same: $S_t = S_{t-1}$ with probability $1 - 2p_t$). As usual, in what follows, we denote by $E_t, t \geq 0$, the conditional expectation operator $E_t = E(\cdot | \mathfrak{S}_t)$.²

For $t \geq 0$, let $\tilde{S}_\tau, \tau \geq t$, be the price process with $\tilde{S}_t = S_t$ and $\tilde{S}_\tau = \tilde{S}_{\tau-1} \exp(u_\tau \epsilon_\tau), \tau > t$, where, conditionally on $\mathfrak{S}_t, (\epsilon_\tau)_{\tau=t+1}^\infty$ is a sequence of i.i.d. symmetric Bernoulli r.v.'s: $P(\epsilon_\tau = 1 | \mathfrak{S}_t) = P(\epsilon_\tau = -1 | \mathfrak{S}_t) = 1/2$ for $\tau > t$.

The following theorem provides bounds for the time- t expected payoff of contingent claims in the trinomial model with an arbitrary increasing convex payoff functions $\phi : \mathbf{R}_+ \rightarrow \mathbf{R}$. These estimates reduce the problem of derivative pricing in the multiperiod trinomial financial market model to the case of the multiperiod binomial model with i.i.d. returns.

Theorem 2.1 *For any increasing convex function $\phi : \mathbf{R}_+ \rightarrow \mathbf{R}$, the following bound holds: $E_t \phi(S_T) \leq E_t \phi(\tilde{S}_T)$ for all $0 \leq t < T$.*

Proof The theorem follows from Theorem 6.3 in Appendix 6 applied to $n = T - t$ r.v. $Y_\tau = \log(X_\tau), \tau = t + 1, \dots, T$, and the function $f(y_1, \dots, y_{T-t}) = \phi \left(S_t \exp \left(\sum_{k=1}^{T-t} y_k \right) \right)$. □

The choice of the function $\phi(x) = \max(x - K, 0), x \geq 0$, in Theorem 2.1 immediately provides estimates for the time- t expected payoffs of a European call option with strike price $K \geq 0$ on the asset expiring at time T . Furthermore, using the results

² The expectation is taken with respect to the true probability measure P .

in Eaton (1974a), we also obtain bounds for the expected payoff of European call options in the trinomial model in terms of the expected payoff of power options with the payoff function $\phi(x) = [\max(x - K, 0)]^3$ written on an asset with log-normally distributed price. These bounds are similar in spirit to the estimates for linear combinations of i.i.d. symmetric Bernoulli r.v.'s and t -statistics under symmetry in Eaton (1974a) (see also Edelman 1990; and Section 12.G in Marshall et al. 2011) and to the estimates in the binomial model in terms of Poisson r.v.'s obtained in de la Peña et al. (2004). These bounds essentially reduce the problem of option pricing in the trinomial multiperiod model to the problem of pricing in the case of two-periods and the standard assumption of log-normal returns.

Theorem 2.2 *The following bounds hold:*

$$\begin{aligned}
 E_t \max(S_T - K, 0) &\leq E_t \max(\tilde{S}_T - K, 0) \\
 &\leq \left\{ E_t \left[\max \left(S_t e^{Z \sqrt{\sum_{k=t+1}^T u_k^2}} - K, 0 \right) \right]^3 \right\}^{1/3}, \tag{2.3}
 \end{aligned}$$

where, conditionally on \mathfrak{S}_t , Z has the standard normal distribution.

Remark 2.1 Analogues of the bounds in Theorems 2.1 and 2.2 also hold for price processes with asymmetric trivariate distributions of the log-returns. For instance, evidently, the bounds continue to hold in the case of the log-returns X_t with the trivariate distributions $P(\log(X_t) = u_t) = p_t$, $P(\log(X_t) = -d_t) = q_t$, $P(\log(X_t) = 0) = 1 - p_t - q_t$, where $0 < u_t \leq d_t$, $0 \leq p_t \leq q_t \leq 1/2$, $t = 1, 2, \dots$. Similar extensions hold as well for other results in the paper for the trinomial option pricing model. In addition, further generalizations of the bounds in the trinomial model to the asymmetric case may be obtained using symmetrization inequalities for (generalized) moments of sums of r.v.'s (see, for instance, de la Peña and Giné 1999).

Proof As indicated before, the first inequality in (2.3) is an immediate consequence of Theorem 2.1 applied to the increasing convex function $\phi(x) = \max(x - K, 0)$, $x \geq 0$. The second estimate in (2.3) is a consequence of Jensen's inequality and the fact that, as follows from the results in Eaton (1974a), $E_t \left[\max \left(S_t e^{\sum_{k=t+1}^T u_k \epsilon_k} - K, 0 \right) \right]^3 \leq E_t \left[\max \left(S_t e^{Z \sqrt{\sum_{k=t+1}^T u_k^2}} - K, 0 \right) \right]^3$. □

Bounds similar to those given by Theorems 2.1 and 2.2 hold as well for the expected stop-loss for a sum of three-value risks that form a martingale-difference sequence (the proof of the estimates for the expected stop-loss of a sum of risks is completely similar to the argument for Theorems 2.1, 2.2).

Theorem 2.3 *Suppose that the r.v.'s $\{X_t\}_{t=1}^\infty$, form an (\mathfrak{S}_t) -martingale-difference sequence and have distributions*

$$P(X_t = u_t) = P(X_t = -u_t) = p_t, \quad P(X_t = 0) = 1 - 2p_t, \tag{2.4}$$

$0 \leq p_t \leq 1/2, t = 1, 2, \dots$. Further, let $\{\epsilon_t\}_{t=1}^\infty$ be a sequence of i.i.d. symmetric Bernoulli r.v.'s: $P(\epsilon_t = 1) = P(\epsilon_t = -1) = 1/2, t = 1, 2, \dots$, and let Z denote a standard normal r.v. Then the following bound holds for the expectation of any convex function $\phi : \mathbf{R} \rightarrow \mathbf{R}$ of the sum of the risks $\sum_{t=1}^T X_t$:

$$E\phi\left(\sum_{t=1}^T X_t\right) \leq E\phi\left(\sum_{t=1}^T u_t \epsilon_t\right).$$

In particular, the following bounds hold for the expected stop-loss $E \max(\sum_{t=1}^T X_t - K, 0), K \geq 0$:

$$\begin{aligned} E \max\left(\sum_{t=1}^T X_t - K, 0\right) &\leq E \max\left(\sum_{t=1}^T u_t \epsilon_t - K, 0\right) \\ &\leq \left\{ E \left[\max\left(Z \sqrt{\sum_{k=t+1}^T u_k^2} - K, 0\right) \right]^3 \right\}^{1/3}. \end{aligned}$$

Let us now turn to the problem of making inferences on the European call option price in the trinomial model. As is well-known, the trinomial option pricing model is incomplete and allows for an infinite number of equivalent probability measures under which the discounted asset price process is a martingale. The different risk-neutral measures lead to different prices for contingent claims in the model, all of which are consistent with market prices of the underlying assets. Therefore, the standard no-arbitrage pricing approach breaks down.

The inequalities for the expected payoffs of contingent claims in Theorems 2.1 and 2.2 (under the *true* probability measure), together with estimates for the call return over its lifetime, on the other hand, provide bounds for possible prices of the option. Let R_S and R_C denote, respectively, the required gross returns (over the periods $t + 1, \dots, T$) on the underlying asset with the trinomial price process (2.1), (2.2) and on the European call option with the strike price K on the asset expiring at time $T > t$. The price of the call option at time t is given by $C_t = R_C E_t \max(S_T - K, 0)$ (where the expectation is taken with respect to the true probability measure) and, similarly, the price of the asset satisfies $S_t = R_S E_t S_T$.

As follows from Rodriguez (2003) (see also Theorem 8 in Merton 1973), risk-averse investors require a higher rate of return on the call option than on its underlying asset and, therefore, $R_C < R_S$.³ Combining this with the bounds given by Theorem 2.2, we immediately obtain estimates for the prices of European call options in the trinomial model. Similar bounds also hold for other contingent claims with convex payoff function; they can be obtained using Theorem 2.1 and estimates for the gross return on the

³ As discussed in Jagannathan (1984) and Rodriguez (2003), this result depends critically on the Rothschild and Stiglitz (1970) definition of risk orderings. Grundy (1991) provides an example in which the expected return on the option is less than the risk-free rate; however the condition $dC(S)/dS < 0$ in his example conflicts with theoretical models and empirical findings, see Rodriguez (2003).

contingent claims over their lifetimes. For instance, in the case of independent returns $X_t, R_S = S_t/E_t S_T = \left(\prod_{k=t+1}^T E_t X_k\right)^{-1} = \left\{\prod_{k=t+1}^T [1 + 2(\cosh(u_k) - 1)p_k]\right\}^{-1}$, where $\cosh(x) = (\exp(x) + \exp(-x))/2, x \in \mathbf{R}$, is the hyperbolic cosine. We thus have the following estimates for the call option prices.

Theorem 2.4 *In the case of the trinomial option pricing model with risk-averse investors and independent returns X_t with distribution (2.2), the time- t prices of the European call option on the asset satisfy the following bounds:*

$$\begin{aligned}
 C_t &\leq \left\{ \prod_{k=t+1}^T [1 + 2(\cosh(u_k) - 1)p_k] \right\}^{-1} E_t \max(\tilde{S}_T - K, 0) \\
 &\leq \left\{ \prod_{k=t+1}^T [1 + 2(\cosh(u_k) - 1)p_k] \right\}^{-1} \left\{ E_t \left[\max \left(S_t e^{Z\sqrt{\sum_{k=t+1}^T u_k^2}} - K, 0 \right) \right]^3 \right\}^{1/3},
 \end{aligned}
 \tag{2.5}$$

where, conditionally on \mathfrak{S}_t, Z has the standard normal distribution.

3 Bounds for Asian options: trinomial model

The approach presented in the previous section also allows one to obtain semiparametric bounds for the expected payoffs and prices of path-dependent contingent claims in the trinomial model. As an illustration, in the present section, we derive estimates for the expected payoffs of Asian options written on an asset with the trinomial price process. Similar bounds for other path-dependent contingent claims may also be derived.

Let $0 < t \leq T - n$. Consider an Asian call option with strike price K expiring at time T written on the average of the past n prices of the asset with price process (2.1). The time- t expected payoff of the option is $E_t(A_{n,T})$, where

$$\begin{aligned}
 A_{n,T} &= \max \left[\left(\sum_{k=T-n+1}^T S_k \right) / n - K, 0 \right] \\
 &= \max \left[S_{T-n} \left(\sum_{k=T-n+1}^T X_{T-n+1} \dots X_k \right) / n - K, 0 \right] \\
 &= \max \left[S_t \prod_{j=t+1}^{T-n+1} X_j \left(1 + \sum_{k=T-n+2}^T X_{T-n+2} \dots X_k \right) / n - K, 0 \right].
 \end{aligned}
 \tag{3.1}$$

Using the convexity of the payoff function of the Asian option, the results given in Theorem 6.3 imply, similar to the proof of Theorems 2.1–2.3, the following bounds for the trinomial Asian option pricing model.

Theorem 3.1 *If the log-returns $\log(X_t)$ form an (\mathfrak{S}_t) -martingale-difference sequence and have the distribution (2.2), then the expected payoff of the Asian option satisfies*

$$E_t \max \left[\left(\sum_{k=T-n+1}^T S_k \right) / n - K, 0 \right] \leq E_t \max \left[S_t \left(\sum_{k=T-n+1}^T \exp \left(\sum_{j=t+1}^k u_j \epsilon_j \right) \right) / n - K, 0 \right]. \tag{3.2}$$

Estimate (3.2) becomes even simpler in the case of an Asian option written on an asset whose gross returns form a three-valued martingale-difference sequence (so that the model represents the trinomial financial market with short-selling where the gross returns can take negative values):

Theorem 3.2 *If the returns (X_t) form an (\mathfrak{S}_t) -martingale-difference and have the distribution (2.4), then the expected payoff of the Asian option satisfies the inequality*

$$E_t \max \left[\left(\sum_{k=T-n+1}^T S_k \right) / n - K, 0 \right] \leq E_t \max \left[S_t \left(\sum_{k=T-n+1}^T \left(\prod_{j=t+1}^k u_j \right) \epsilon_k \right) / n - K, 0 \right]. \tag{3.3}$$

Proof Bound (3.3) follows from Theorem 6.3 and the fact that, by Theorem 6.1, the r.v.'s $\eta_k = \epsilon_{t+1} \epsilon_{t+2} \dots \epsilon_k, k = T - n + 1, \dots, T$, are i.i.d. symmetric Bernoulli r.v.'s. □

4 Bounds for Asian options: the general case

Let $0 < t \leq T - n, 1 \leq s < p, a_k, b_k > 0, a_k^p \leq b_k, k = T - n + 1, \dots, T$. Similar to the previous section, consider an Asian call option with strike price K expiring at time T , written on the average of the past n prices of an asset with the (not necessarily identically distributed) gross returns $X_t \geq 0, t = 1, 2, \dots$, such that $E_t X_k^s = a_k^s, E_t X_k^p = b_k, k = T - n + 1, \dots, T$, and price (2.1): $S_0 = s, S_t = S_0 X_1 \dots X_t, t \geq 1$. The time- t expected payoff of the option is $E_t(A_{n,T})$ with $A_{n,T}$ given in (3.1).

The r.v.

$$W_{n,T} = \frac{1}{n} S_t \prod_{j=t+1}^{T-n+1} X_j \left(1 + \sum_{k=T-n+2}^T X_{T-n+2} \dots X_k \right) \tag{4.1}$$

which the payoff $A_{n,T}$ of the Asian option in (3.1) depends upon is a particular case of a sum of multilinear forms defined as

$$d_0 + \sum_{c=1}^N \sum_{1 \leq i_1 < \dots < i_c \leq N} d_{i_1, \dots, i_c} Y_{i_1} \dots Y_{i_c}, \tag{4.2}$$

where $d_0 \in \mathbf{R}$, $d_{i_1, \dots, i_c} \in \mathbf{R}$, $1 \leq i_1 < \dots < i_c \leq N$, $c = 1, \dots, N$, are some parameters and Y_1, \dots, Y_N are independent nonnegative r.v.'s. This property allows one to obtain bounds for the expected payoffs of Asian options, combining Grundy (1991)'s results for expected payoffs of European call options in terms of moments of the underlying asset's price with the general sharp inequalities for power moments of sums of multilinear forms and U -statistics obtained in de la Peña et al. (2002, 2003) (see also Ibragimov et al. 2001). For example, the following theorem provides bounds on the day- t expectations of the Asian options in terms of the fixed power moments of the underlying asset's return distribution. Using the inequalities for expectations of general functions of sums of multilinear forms obtained in de la Peña et al. (2002, 2003), one can further obtain estimates similar to those in Theorem 4.1 for the expected payoffs of other path-dependent contingent claims.

Specifically, let $V_{T-n+2}(s, t, a_{T-n+2}, b_{T-n+2}), \dots, V_T(s, t, a_T, b_T)$ be independent r.v.'s with distributions

$$P[V_k(s, p, a_k, b_k) = 0] = 1 - (a_k^t/b_k)^{s/(p-s)},$$

$$P[V_k(s, p, a_k, b_k) = (b_k/a_k^s)^{1/(p-s)}] = (a_k^t/b_k)^{s/(p-s)}, \tag{4.3}$$

$k = T - n + 2, \dots, T$. It is easy to see that the bivariate r.v.'s $V_k(s, p, a, b)$ have the same moments of orders s and p as the gross returns $X_k : E_t[V_k(s, p, a, b)]^s = a_k^s$, $E_t[V_k(s, p, a, b)]^p = b_k$, $k = T - n + 1, \dots, T$. For a r.v. $W \geq 0$, denote by $\|W\|_p = (EW^p)^{1/p}$ the L_p -norm of W .

Further, denote by $F(s, p, a, b)$ and $G(s, p, a, b)$ the functions

$$F(s, p, a, b) = \frac{1}{n} S_t \prod_{j=t+1}^{T-n+1} b_j^{1/p}$$

$$\left\| 1 + \sum_{k=T-n+2}^T V_{T-n+2}(s, p, a_{T-n+1}, b_{T-n+1}) \dots V_k(s, p, a_k, b_k) \right\|_p$$

and

$$G(s, p, a, b) = \frac{1}{n} S_t \prod_{k=t+1}^{T-n+1} b_k^{1/p} \left[\sum_{q=0}^{n-1} \sum_{T-n+2 \leq j_1 < \dots < j_q \leq T} (b_{j_1} - a_{j_1}^p) \dots (b_{j_q} - a_{j_q}^p) \right. \\ \left. \times \left(1 + \sum_{l=q}^{n-1} \sum_{i_1 < \dots < i_{l-q} \in \{T-n+2, \dots, T\} \setminus \{j_1, \dots, j_q\}} a_{i_1} a_{i_2} \dots a_{i_{l-q}} \right) \right]^{1/p}.$$

That is,

$$G^p(s, p, a, b) = \frac{1}{n} S_t b_{t+1} \dots b_{T-n+1} \left[(1 + a_{T-n+2} + a_{T-n+2} a_{T-n+3} + \dots + a_{T-n+2} a_{T-n+3} \dots a_T)^p \right]$$

$$\begin{aligned}
 &+ (b_{T-n+2} - a_{T-n+2}^p) (1 + a_{T-n+3} + a_{T-n+3}a_{T-n+4} + \dots + a_{T-n+3} \dots a_T)^p \\
 &+ \dots + (b_{T-n+3} - a_{T-n+3}^p) (1 + a_{T-n+2} + a_{T-n+2}a_{T-n+4} \\
 &+ \dots + a_{T-n+2}a_{T-n+4} \dots a_T)^p \\
 &+ \dots + (b_T - a_T^p)(a_{T-n+2} \dots a_{T-1})^p \\
 &+ (b_{T-n+2} - a_{T-n+2}^p) (b_{T-n+3} - a_{T-n+3}^p) (1 + a_{T-n+4} \\
 &+ a_{T-n+4}a_{T-n+5} \dots + a_{T-n+4}a_{T-n+5} \dots a_T)^p \\
 &+ (b_{T-n+2} - a_{T-n+2}^p) (b_{T-n+4} - a_{T-n+4}^p) (1 + a_{T-n+3} \\
 &+ a_{T-n+3}a_{T-n+5} \dots + a_{T-n+3}a_{T-n+5} \dots a_T)^p \\
 &+ (b_{T-1} - a_{T-1}^p) (b_T - a_T^p) (a_{T-n+2} + a_{T-n+3} \dots + a_{T-2})^p \\
 &+ \dots + (b_{T-n+2} - a_{T-n+2}^p)(b_{T-n+3} - a_{T-n+3}^p) \dots (b_T - a_T^p)].
 \end{aligned}$$

Evidently, the values of $F(s, p, a, b)$ and $G(s, p, a, b)$ depend only on s, p, S_t , and the values of the underlying asset’s returns’ moments a_k^s and b_k only.

To illustrate the structure of the functions $F(s, t, a, b)$ and $G(s, t, a, b)$, consider the case $n = T = 3$. In this case, the Asian option payoff in (3.1) is

$$\begin{aligned}
 A_{3,3} &= \max \left(\frac{1}{3}(S_1 + S_2 + S_3) - K, 0 \right) \\
 &= \max \left(\frac{1}{3}S_0(X_1 + X_1X_2 + X_1X_2X_3) - K, 0 \right).
 \end{aligned}$$

We have $F(s, p, a, b) = \frac{1}{3}S_0b_1^{1/p}||1 + V_2(s, p, a_2, b_2) + V_2(s, p, a_2, b_2)V_3(s, p, a_3, b_3)||_p$, where $V_2(s, p, a_2, b_2)$ and $V_3(s, p, a_3, b_3)$ are independent r.v.’s with distribution (4.3), and

$$\begin{aligned}
 G(s, p, a, b) &= \frac{1}{3}S_0b_1^{1/p}[(1 + a_2 + a_2a_3)^p + (b_2 - a_2^p) (1 + a_3)^p \\
 &+ (b_3 - a_3^p)a_2^p + (b_2 - a_2^p) (b_3 - a_3^p)]^{1/p}.
 \end{aligned}$$

The function $F(s, p, a, b)$ is thus exactly the L_p -norm $||A_{n,T}||_p = [E(A_{n,T})^p]^{1/p}$ of the Asian option payoff $A_{n,T}$ in (3.1) evaluated at the bivariate r.v.’s $V_k(s, p, a, b)$ $k = T - n + 1, \dots, T$, in (4.3).

The structure of the function $G(s, p, a, b)$ is similar to the following representation for the sum of multilinear forms $\sum_{k=T-n+2}^T X_{T-n+2} \dots X_k$ in (4.2):

$$\begin{aligned}
 \sum_{k=T-n+2}^T X_{T-n+2} \dots X_k &= \sum_{q=0}^{n-1} \sum_{T-n+2 \leq j_1 < \dots < j_q \leq T} X_{j_1} \dots X_{j_q} \\
 &\times \left(1 + \sum_{l=q}^{n-1} \sum_{i_1 < \dots < i_{l-q} \in \{T-n+2, \dots, T\} \setminus \{j_1, \dots, j_q\}} X_{i_1} X_{i_2} \dots X_{i_{l-q}} \right).
 \end{aligned} \tag{4.4}$$

In the function $G^p(s, p, a, b)$, the random terms $X_{j_r}, r = 1, \dots, q$, are replaced by the function of their moments $b_{j_r} - a_{j_r}^p$ and the multilinear form factors in (4.4) are evaluated at the degenerate r.v.'s $X_{i_1} = a_{i_1}, \dots, X_{i_{l-q}} = a_{i_{l-q}}$ (a.s.).

We have

Theorem 4.1 *Let $1 < p < 2, 0 < s \leq p - 1$ or $p \geq 2, 0 < s \leq 1$. If $K \leq \frac{p-1}{p}G(s, p, a, b)$, then*

$$E_t(A_{n,T}) \leq F(s, p, a, b) - K. \tag{4.5}$$

If $K > \frac{p-1}{p}F(s, p, a, b)$, then

$$E_t(A_{n,T}) \leq F^p(s, t, a, b) \frac{1}{p} \left(\frac{p-1}{pK} \right)^{p-1}. \tag{4.6}$$

Let $1 < p < 2, 1 \leq s < p$ or $t \geq 2, p - 1 \leq s < p$. If $K \leq \frac{p-1}{p}F(s, p, a, b)$, then $E_t(A_{n,T}) \leq G(s, p, a, b) - K$. If $K > \frac{p-1}{p}G(s, p, a, b)$, then $E_t(A_{n,T}) \leq G^p(s, p, a, b) \frac{1}{p} \left(\frac{p-1}{pK} \right)^{p-1}$.

Proof From Proposition 2 in Grundy (1991) it follows that the Asian option's day-t expected payoff $E_t(A_{n,T}) = E \left[\max \left(\frac{1}{n}W_{n,T} - K, 0 \right) \right]$ with $A_{n,T}$ and $W_{n,T}$ in (3.1) and (4.1) satisfies

$$E_t(A_{n,T}) \leq \|W_{n,T}\|_p - K, \tag{4.7}$$

if $K \leq \frac{p-1}{p}\|W_{n,T}\|_p$ and

$$E_t(A_{n,T}) \leq E(W_{n,T})^p \frac{1}{p} \left(\frac{p-1}{pK} \right)^{p-1} \tag{4.8}$$

for $K > \frac{p-1}{p}\|W_{n,T}\|_p$.

Further, from Theorem 3.3 in de la Peña et al. (2002) (see also de la Peña et al. 2003) we obtain that the following sharp inequalities hold for the moments $E(W_{n,T})^p$ in (4.7) and (4.8). If $1 < p < 2, 0 < s \leq p - 1$ or $p \geq 2, 0 < s \leq 1$, then

$$G^p(s, p, a, b) \leq E(W_{n,T})^p \leq F^p(s, p, a, b). \tag{4.9}$$

If $1 < p < 2, 1 \leq s < p$ or $t \geq 2, p - 1 \leq s < p$, then

$$F^p(s, p, a, b) \leq E(W_{n,T})^p \leq G^p(s, p, a, b). \tag{4.10}$$

Bounds (4.7)–(4.10) imply the conclusion of the theorem. □

Remark 4.1 The bounds presented in Theorem 4.1 hold for arbitrary two fixed moments of the returns of order higher than one; the moments fixed do not have to be, e.g., the mean and variance of the returns. In particular, the bounds hold for assets with heavy-tailed distributions typically observed in economic, financial and insurance markets (see, for instance, the reviews and results in Gabaix 2008; Ibragimov 2009; Ibragimov et al. 2015; Ibragimov and Prokhorov 2016) including the infinite fourth moment and even the infinite variance case.

In the case of the fixed first moment $s = 1$ and the returns X_k satisfying $E_t X_k = a_k$ and $E_t X_k^p = b_k, k = T - n + 1, \dots, T$, the r.v.'s $V_{T-n+2}(1, t, a_{T-n+2}, b_{T-n+2}), \dots, V_T(1, t, a_T, b_T)$ in (4.3) have distributions

$$P[V_k(1, p, a_k, b_k) = 0] = 1 - (a_k^t/b_k)^{1/(p-1)},$$

$$P\left[V_k(1, p, a_k, b_k) = (b_k/a_k)^{1/(p-1)}\right] = (a_k^t/b_k)^{1/(p-1)},$$

$k = T - n + 2, \dots, T$. In this case, Theorem (4.1) implies the bounds in the following corollary.

Corollary 4.1 *If $p \geq 2$ and $K \leq \frac{p-1}{p}G(1, p, a, b)$, then*

$$E_t(A_{n,T}) \leq F(1, p, a, b) - K. \tag{4.11}$$

If $1 < p < 2$ and $K \leq \frac{p-1}{p}F(1, p, a, b)$, then $E_t(A_{n,T}) \leq G(1, p, a, b) - K$.

Remark 4.2 The moment bounds in (4.5)-(4.11) are functions of the bivariate price process with the returns $V_k(s, t, a, b)$ in (4.3). This is because, as follows from de la Peña et al. (2002, 2003), the maxima of moments of sums of multilinear forms (4.2) in r.v.'s Y_k with fixed two moments is achieved for r.v.'s Y_k taking on at most three values. As follows from Hoeffding (1955) and Karr (1983), an analogue of this result holds for expectations of arbitrary continuous functions of r.v.'s with an arbitrary number of fixed (generalized) moments. Namely, consider the problem of determining the extrema of the expectation $Ef(X)$ over the r.v.'s X with fixed $Eh_s(X) = c_s$, where $f, h_s : \mathbf{R} \rightarrow \mathbf{R}, s = 1, \dots, m$, are some continuous functions and $c_s \in \mathbf{R}$ are some constants. According to Hoeffding (1955) and Karr (1983), in the above problem, it suffices to consider the r.v.'s X that take on at most $m + 1$ values.

Remark 4.3 Similar to Theorem 2.4, the results in Theorems 3.1, 3.2 and 4.1 and Corollary 4.1, together with the lower estimates for the call return over its lifetime provide bounds on Asian option prices. For instance, since, in the notation preceding Theorem 3.1, $R_C < R_S = S_t/E_t S_T = \left(\prod_{k=t+1}^T E_t X_k\right)^{-1}$, from Corollary 4.1 it follows that the following bounds hold for the time- t Asian option price $C_t : C_t < \left(\prod_{k=t+1}^T a_k\right)^{-1} (F(1, p, a, b) - K)$ if $p \geq 2$ and $K \leq \frac{p-1}{p}G(1, p, a, b)$; and $C_t < \left(\prod_{k=t+1}^T a_k\right)^{-1} (G(1, p, a, b) - K)$ if $1 < p < 2$ and $K \leq \frac{p-1}{p}F(1, p, a, b)$, where, as before, $a_k = E_t X_k, k = t + 1, \dots, T$. Similar to Grundy (1991), these results,

together with the observed option prices, provide restrictions on the feasible values of moments of the underlying asset's returns under the true probability measure.

5 Concluding remarks

This paper presents a number of new semiparametric bounds on the expected payoffs and prices of European call options and path-dependent contingent claims.

The first set of the obtained bounds reduces the problem of option pricing in the trinomial model with dependent returns to the i.i.d. binomial case and the two-period model with the standard assumption of a log-normal asset price. Similar results also hold for Asian options written on averages of the underlying asset's prices and other path-dependent contingent claims.

The second set of our results provides semiparametric bounds on the expected payoffs and prices of Asian options in terms of moments of the underlying asset's returns. The bounds hold for an arbitrary distribution of the underlying asset's price. They provide the first financial and economic applications of the recently obtained moment inequalities for sums of multilinear forms and U -statistics in the literature. Analogues of the results also hold for many other path-dependent contingent claims.

The advantage of the semiparametric bounds approach to option pricing is that it does not require the knowledge of the entire distribution of the underlying asset's price. The bounds are easy to calculate and depend only on several parameters of the asset's price distribution, such as moments or the values taken by the asset's returns. Therefore, they provide an appealing alternative to exact option pricing and computationally expensive numerical pricing methods.

Further research on the topic may focus on extensions of the bounds to the case of probability weighting functions and the related prospect theory framework (see [Barberis and Huang 2008](#)). In particular, it would be of interest to obtain bound analogues of the results in [Barberis and Huang \(2008\)](#) for securities with skewed bivariate payoff distributions in the case of skewed asymmetric trivariate payoffs (see [Remark 2.1](#)).⁴ Further generalizations and applications of the results obtained in the paper may also focus on discrete price processes taking on more than three values and derivations of bounds in the continuous case using discrete approximations.

6 Appendix: Probabilistic foundations for the analysis

Let $(\Omega, \mathfrak{F}, P)$ be a probability space equipped with a filtration $\mathfrak{F}_0 = (\Omega, \emptyset) \subseteq \mathfrak{F}_1 \subseteq \dots \subseteq \mathfrak{F}_t \subseteq \dots \subseteq \mathfrak{F}$. Further, let $(a_t)_{t=1}^\infty$ and $(b_t)_{t=1}^\infty$ be arbitrary sequences of real numbers such that $a_t \neq b_t$ for all t .

The key to the analysis in [Sects. 2 and 3](#) is provided by the following theorems. These theorems are consequences of more general results obtained in [Sharakhmetov](#)

⁴ The analysis of preferences over payoff distributions and the effects of skewness and other higher moments (e.g., kurtosis) in this framework may also relate to applications of majorization theory (see [Marshall et al. 2011](#)) and heavy-tailed distributions (see, among others [Embrechts et al. 1997](#); [Gabaix 2008](#); [Ibragimov 2009](#); [Ibragimov et al. 2015](#); [Ibragimov and Prokhorov 2016](#)).

and Ibragimov (2002) that show that r.v.'s taking $k + 1$ values form a multiplicative system of order k if and only if they are jointly independent (see also de la Peña et al. 2006). These results imply, in particular, that r.v.'s each taking two values form a martingale-difference sequence if and only if they are jointly independent.

To illustrate the main ideas of the proof, we first consider the case of r.v.'s taking values ± 1 .

Theorem 6.1 *If r.v.'s $U_t, t = 1, 2, \dots$, form a martingale-difference sequence with respect to a filtration $(\mathfrak{F}_t)_t$ and are such that $P(U_t = 1) = P(U_t = -1) = \frac{1}{2}$ for all t , then they are jointly independent.*

For completeness, the proof of the theorem is provided below.

Proof It is easy to see that, under the assumptions of the theorem, one has that, for all $1 \leq l_1 < l_2 < \dots < l_k, k = 2, 3, \dots$,

$$EU_{l_1} \dots U_{l_{k-1}} U_{l_k} = E(U_{l_1} \dots U_{l_{k-1}} E(U_{l_k} | \mathfrak{F}_{l_k-1})) = E(U_{l_1} \dots U_{l_{k-1}} \times 0) = 0. \tag{6.1}$$

It is easy to see that, for $x_t \in \{-1, 1\}, I(X_t = x_t) = (1 + x_t U_t)/2$. Consequently, for all $1 \leq j_1 < j_2 < \dots < j_m, m = 2, 3, \dots$, and any $x_{j_k} \in \{-1, 1\}, k = 1, 2, \dots, m$, we have

$$\begin{aligned} &P(U_{j_1} = x_{j_1}, U_{j_2} = x_{j_2}, \dots, U_{j_m} = x_{j_m}) \\ &= EI(U_{j_1} = x_{j_1})I(U_{j_2} = x_{j_2}) \dots I(U_{j_m} = x_{j_m}) \\ &= \frac{1}{2^m} E(1 + x_{j_1} U_{j_1})(1 + x_{j_2} U_{j_2}) \dots (1 + x_{j_m} U_{j_m}) \\ &= \frac{1}{2^m} \left(1 + \sum_{c=2}^m \sum_{i_1 < \dots < i_c \in \{j_1, j_2, \dots, j_m\}} EU_{i_1} \dots U_{i_c} \right) = \frac{1}{2^m} \\ &= P(U_{j_1} = x_{j_1})P(U_{j_2} = x_{j_2}) \dots P(U_{j_m} = x_{j_m}) \end{aligned}$$

by (6.1). □

The proof of the analogue of the result in the case of r.v.'s each of which takes arbitrary two values is completely similar and the following more general result holds.

Theorem 6.2 *If r.v.'s $X_t, t = 1, 2, \dots$, form a martingale-difference sequence with respect to a filtration $(\mathfrak{F}_t)_t$ and each of them takes two (not necessarily the same for all t) values $\{a_t, b_t\}$, then they are jointly independent.*

Proof Let the r.v. X_t take the values a_t and $b_t, a_t \neq b_t$, with probabilities $P(X_t = a_t) = p_t$ and $P(X_t = b_t) = q_t$, respectively. It not difficult to check that, for $x_t \in \{a_t, b_t\}$,

$$\begin{aligned}
 I(X_t = x_t) &= P(X_t = x_t) \left(1 + \frac{(X_t - a_t p_t - b_t q_t)(x_t - a_t p_t - b_t q_t)}{(a_t - b_t)^2 p_t q_t} \right) \\
 &= P(X_t = x_t) \left(1 + \frac{(X_t - EX_t)(x_t - EX_t)}{(a_t - b_t)^2 p_t q_t} \right) \\
 &= P(X_t = x_t) \left(1 + \frac{X_t x_t}{(a_t - b_t)^2 p_t q_t} \right) = P(X_t = x_t) \left(1 + \frac{X_t x_t}{\text{Var}(X_t)} \right),
 \end{aligned}$$

where $EX_t = a_t p_t + b_t q_t = 0$ and $\text{Var}(X_t) = (b_t - a_t)^2 p_t q_t$ are the mean and the variance of X_t . Since the r.v.'s X_t satisfy property (6.1) with U_{l_j} replaced by X_{l_j} , $j = 1, \dots, k$, similar to the proof of Theorem 6.1 we have that, for all $1 \leq j_1 < j_2 < \dots < j_m$, $m = 2, 3, \dots$, and any $x_{j_k} \in \{a_{j_k}, b_{j_k}\}$, $k = 1, 2, \dots, m$,

$$\begin{aligned}
 &P(X_{j_1} = x_{j_1}, X_{j_2} = x_{j_2}, \dots, X_{j_m} = x_{j_m}) \\
 &= EI(X_{j_1} = x_{j_1})I(X_{j_2} = x_{j_2}) \dots I(X_{j_m} = x_{j_m}) \\
 &= \prod_{s=1}^m P(X_{j_s} = x_{j_s}) E \left(1 + \frac{X_{j_1} x_{j_1}}{\text{Var}(X_{j_1})} \right) \dots \left(1 + \frac{X_{j_m} x_{j_m}}{\text{Var}(X_{j_m})} \right) \\
 &= \prod_{s=1}^m P(X_{j_s} = x_{j_s}) \\
 &\quad \times \left(1 + \sum_{c=2}^m \sum_{i_1 < \dots < i_c \in \{j_1, j_2, \dots, j_m\}} EX_{i_1} \dots X_{i_c} x_{i_1} \dots x_{i_c} / (\text{Var}(X_{i_1}) \dots \text{Var}(X_{i_c})) \right) \\
 &= P(X_{j_1} = x_{j_1}) P(X_{j_2} = x_{j_2}) \dots P(X_{j_m} = x_{j_m}).
 \end{aligned}$$

□

Let $X_t, t = 1, 2, \dots$, be an (\mathfrak{F}_t) -martingale-difference sequence consisting of r.v.'s each of which takes three values $\{-a_t, 0, a_t\}$. Denote by $\epsilon_t, t = 1, 2, \dots$, a sequence of i.i.d. symmetric Bernoulli r.v.'s independent of $(X_t)_{t=1}^\infty$. The following theorem provides an upper bound for the expectation of arbitrary convex function of X_t in terms of the expectation of the same function of the r.v.'s ϵ_t .

Theorem 6.3 *If $f : \mathbf{R}^n \rightarrow \mathbf{R}$ is a function convex in each of its arguments, then the following inequality holds:*

$$Ef(X_1, \dots, X_n) \leq Ef(a_1 \epsilon_1, \dots, a_n \epsilon_n). \tag{6.2}$$

Proof Let $\tilde{\mathfrak{S}}_0 = \mathfrak{S}_n$. For $t = 1, 2, \dots, n$, denote by $\tilde{\mathfrak{S}}_t$ the σ -algebra spanned by the r.v.'s $X_1, X_2, \dots, X_n, \epsilon_1, \dots, \epsilon_t$. Further, let, for $t = 0, 1, \dots, n$, E_t stand for the conditional expectation operator $E(\cdot | \tilde{\mathfrak{S}}_t)$ and let $\eta_t, t = 1, \dots, n$, denote the r.v.'s $\eta_t = X_t + \epsilon_t I(X_t = 0)$.

Using conditional Jensen's inequality, we have

$$\begin{aligned} Ef(X_1, X_2, \dots, X_n) &= Ef(X_1 + E_0[\epsilon_1 I(X_1 = 0)], X_2, \dots, X_n) \\ &\leq E[E_0 f(X_1 + \epsilon_1 I(X_1 = 0), X_2, \dots, X_n)] \\ &= Ef(\eta_1, X_2, \dots, X_n). \end{aligned} \quad (6.3)$$

Similarly, for $t = 2, \dots, n$,

$$\begin{aligned} Ef(\eta_1, \eta_2, \dots, \eta_{t-1}, X_t, X_{t+1}, \dots, X_n) \\ &= Ef(\eta_1, \eta_2, \dots, \eta_{t-1}, X_t + E_{t-1}[\epsilon_t I(X_t = 0)], X_{t+1}, \dots, X_n) \\ &\leq E[E_{t-1} f(\eta_1, \eta_2, \dots, \eta_{t-1}, X_t + \epsilon_t I(X_t = 0), X_{t+1}, \dots, X_n)] \\ &= Ef(\eta_1, \eta_2, \dots, \eta_{t-1}, \eta_t, X_{t+1}, \dots, X_n). \end{aligned} \quad (6.4)$$

From equations (6.3) and (6.4) by induction it follows that

$$Ef(X_1, X_2, \dots, X_n) \leq Ef(\eta_1, \eta_2, \dots, \eta_n). \quad (6.5)$$

It is easy to see that the r.v.'s $\eta_t, t = 1, 2, \dots, n$, form a martingale-difference sequence with respect to the sequence of σ -algebras $\tilde{\mathfrak{F}}_0 \subseteq \tilde{\mathfrak{F}}_1 \subseteq \dots \subseteq \tilde{\mathfrak{F}}_t \subseteq \dots$, and each of them takes two values $\{-a_t, a_t\}$. Therefore, from Theorems 6.1 and 6.2 we get that $\eta_t, t = 1, 2, \dots, n$, are jointly independent and, therefore, the random vector $(\eta_1, \eta_2, \dots, \eta_n)$ has the same distribution as $(a_1 \epsilon_1, a_2 \epsilon_2, \dots, a_n \epsilon_n)$. This and (6.5) implies estimate (6.2). \square

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