NUMERICAL ROSS RECOVERY FOR DIFFUSION PROCESSES USING A PDE APPROACH

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Abstract. We develop and analyze a numerical method for solving the Ross recovery problem for a diffusion problem with unbounded support, with a pricing kernel on transition independent form. Asset prices are assumed to only be available on a bounded subinterval $B = [-N, N]$. Theoretical error bounds on the recovered pricing kernel are derived, relating the convergence rate as a function of N to the rate of mean reversion of the diffusion process. Our suggested numerical method for finding the pricing kernel employs finite differences, and we apply Sturm-Liouville theory to make use of inverse iteration on the resulting discretized eigenvalue problem. We numerically verify the derived error bounds on a test bench of three model problems.

1. Introduction. A fundamental question in financial markets is to what extent asset prices may be used to infer the market participants' views about the likelihood for different events, e.g., the risk for a stock market crash. The traditional view has been that because investors are risk averse and therefore require a premium above expected returns when purchasing an asset, only limited inferences may be drawn about such likelihoods. Indeed, what is observed is actually a risk neutral expectation — a combination of a pricing kernel and the true expectation. Recent literature, however, see [20] and also [6, 19, 4, 2, 23, 3, 17, 16, 21, 1, 12, 11, 13, 24, 7, 18], shows that under some circumstances it is actually possible to recover both the pricing kernel and true probabilities from observed prices. Such *Ross recovery*, when possible, provides important information about the market.

Ross recovery is, for example, theoretically possible in complete markets, where the process governing asset payoffs is a diffusion that satisfies certain growth conditions, and the pricing kernel is on *transition independent form*, as discussed in [24]. This setting arises in many work-horse models in finance. The pricing kernel can in this case be recovered as the maximal positive eigenfunction to a second order elliptic differential operator, a problem that is related to Sturm-Liouville theory.

The theoretical conditions needed for perfect recovery are not satisfied in practice. With a diffusion process governing the state space, an infinite number of assets in the market would be needed. In practice, even when derivative markets are included, only a finite number are available. There will therefore be gaps between observed prices and, more importantly, upper and lower bounds on these observations. In this paper we study the numerical challenges that arise in such a context.

We focus on the case when the state space is governed by a diffusion process and the pricing kernel is on transition independent form, but asset prices are only available on a bounded subinterval symmetric around the origin, $B = [-N, N]$, with no given boundary condition. We denote this the *approximate recovery problem.* For simplicity, we assume that all asset prices are available within this interval — i.e., that there are no gaps — and focus on the boundedness of the domain. Standard interpolation techniques can be used to fill in such gaps between observed prices.

Our contribution is two-fold. First, we derive error bounds on the recovered pricing kernel, that depend on the underlying parameters of the model. Briefly, the more mean reverting the process is, the faster is the convergence in N. We numerically

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verify for several test problems that the actual approximation error is in line with these theoretical bounds. Second, we introduce a finite difference method that can be used to efficiently solve the approximate recovery problem. The method addresses the challenge that one must simultaneously solve for the marginal utility function and the largest eigenvalue that admits positivity. Overall, our paper helps make Ross recovery for diffusion processes practically operational.

2. The approximate Ross recovery problem. The Ross recovery problem for a univariate time homogeneous diffusion process can via the Kolmogorov equations be transformed into the following ODE problem on the real line

$$Dz'' + \kappa z' + (\rho - r)z = 0. \tag{2.1}$$

Here, $D(x) = \sigma^2(x)/2$, $\kappa(x)$, and $r(x)$ are known functions that we assume are smooth, and such that σ is uniformly bounded above 0, $\sigma(x) \geq \epsilon > 0$ for all $x \in \mathbb{R}$, and only strictly positive solutions $z(x)$ to (2.1) are considered. The constant ρ is not observed, and neither is the function $z(x)$.

The Ross recovery problem in this context is that of identifying ρ and $z(x)$, given $D(x)$, $\kappa(x)$, and (x). The function $z(x)$ is only unique down to multiplication with an arbitrary positive constant. In what follows, we therefore always normalize $z(x)$ and assume that $z(0) = 1$.

Define the operator $\mathcal{L}_\lambda[z] = Dz'' + \kappa z' + (\lambda - r)z$, and the fundamental ODE for the recovery problem

$$\mathcal{L}_\lambda[z] = 0, \qquad \lambda \in \mathbb{R}. \tag{2.2}$$

The correct z is then a strictly positive solution to the fundamental ODE with the parameter value $\lambda = \rho$. For a given λ and a positive solution w to (2.2), a candidate solution is represented by the pair $\theta = (\lambda, w)$. Define the sets

$$\begin{aligned} W &\stackrel{\text{def}}{=} \{w : w(x) > 0\}, \\ \Theta &\stackrel{\text{def}}{=} \{(\lambda, w) : \mathcal{L}_\lambda[w] = 0 \text{ and } w \in W\}, \\ \Lambda &\stackrel{\text{def}}{=} \{\lambda : (\lambda, w) \in \Theta\}. \end{aligned}$$

Also define the number $\bar{\lambda} = \sup \Lambda \in [\rho, \infty)$. Given that $\bar{\lambda}$ is finite, call solutions on the form $(\bar{\lambda}, w)$ *maximal*. The Ross recovery problem can now be restated as that of identifying a maximal $\theta \in \Theta$.

The functions $\mu(x) = \kappa(x) + 2D(x)\frac{z'(x)}{z(x)}$ and $D(x)$ determine whether Ross recovery is feasible. We define

$$\begin{aligned} Q(x) &= e^{-\int_0^x \frac{\mu(s)}{D(s)} ds}, \\ R(x) &= \int_0^x Q(s) ds, \\ T(x) &= \min\{-R(-x), R(x)\}, \\ Z(x, N) &= \min\left\{\frac{R(N)}{R(x)}, \frac{R(-N)}{R(-x)}\right\}, \end{aligned}$$

and note that $R(x)$ is a smooth, strictly positive, and increasing function, as is $T(x)$ on the positive axis, and that $R(x) = x(1 + O(x))$ for small x.

A necessary and sufficient condition for Ross recovery to be feasible is that $\lim_{x\to\infty} T(x) = \infty$. We have

PROPOSITION 2.1. *If* $\lim_{x\to\infty} T(x) = \infty$*, then there is a unique maximal solution to* (2.2)*,* $(\bar{\lambda}, w) \in \Theta$*, that solves the Ross recovery problem,* $\rho = \bar{\lambda}$ *and* $z = w$*. If* $\lim_{x\to\infty} T(x) < \infty$*, then there are distinct positive solutions to* (2.2)*,* $\theta_1 = (\rho, w_1) \in \Theta$*, and* $\theta_2 = (\rho, w_2) \in \Theta$*, and Ross recovery is therefore not possible.*

We refer to [24] (see also [4] and [19]) for a proof of Proposition 2.1. In light of this result we focus our analysis on the case $\lim_{x\to\infty} T(x) = \infty$ going forward. It follows that in this case $\lim_{N\to\infty} Z(x, N) = \infty$ for all $x \neq 0$.

The functions $D(x)$, $\kappa(x)$, and $r(x)$ need to be backed out from the market prices, and in practice such prices are not observed for arbitrarily large x. In other words, the problem needs to be truncated. As discussed in [24], when the functions are only observed on some compact subdomain of $\mathbb{R}$ — for simplicity, we assume on a symmetric interval around the origin, $x \in B \stackrel{\text{def}}{=} [-N, N]$, $0 < N < \infty$ — it is possible to find approximations to ρ and z by solving the problem on this subdomain. Specifically, define

$$\begin{aligned} W^N &\stackrel{\text{def}}{=} \left\{ w^N : w^N(x) > 0 \text{ for } x \in (-N, N) \right\}, \\ \Theta^N &\stackrel{\text{def}}{=} \left\{ \left(\lambda^N, w^N\right) : \mathcal{L}_{\lambda^N}[w^N] = 0 \text{ for } x \in [-N, N] \text{ and } w^N \in W^N \right\}, \\ \Lambda^N &\stackrel{\text{def}}{=} \{\lambda^N : (\lambda^N, w^N) \in \Theta^N\}. \end{aligned}$$

and $\bar{\lambda}^N = \sup \Lambda^N$. Obviously, the set Θ^N shrinks as N grows, and $\Theta \subset \Theta^N$ for all finite N. It is easy to see that Λ^N is closed and bounded above for all $N > 0$. The approximate solution to the Ross recovery problem is now chosen as a maximal element in Θ^N, $\theta^N = (\bar{\lambda}^N, z^N) \in \Theta^N$. We denote the problem of finding such a maximal element, *the approximate recovery problem*, ARP.

It is shown in [24] that θ^N converges to θ as N tends to infinity, but the speed of convergence is not analyzed. Neither is the question of how to jointly solve for $\bar{\lambda}^N$ and z^N in an efficient manner. We mainly focus on these questions in our subsequent analysis.

3. Convergence analysis.

Our main convergence result is the following:

PROPOSITION 3.1. *The approximation errors between the approximate recovered and true solutions,* $\theta^N = (\bar{\lambda}^N, z^N)$ *and* $\theta = (\rho, z)$*, satisfy the following bounds*

$$0 \le \bar{\lambda}^N - \rho \le C_1 \frac{1}{T(N) - \epsilon}, \tag{3.1}$$

$$\left| \frac{z^N(x) - z(x)}{z(x)} \right| \le C_2 \frac{1}{Z(x, N) - 1}. \tag{3.2}$$

Here, the constant C_1 *depends on* μ *and* σ*, and on* $\epsilon > 0$*, and* C_2 *depends on* μ*,* σ*, and* x*, but neither constant depends on* N*.*

Proof. We know that $\Theta \subset \Theta^N$, and for finite N we in general expect there to be solutions outside of Θ, on the form $(\rho + a, z^N) \in \Theta^N$ with $a > 0$. We write any such solution as $z^N(x) = e^{-\int_0^x \frac{\mu(s)}{\sigma^2(s)} ds} g_a(x) z(x)$, where g_a satisfies the following ODE on standard form (see [22])

$$g_a'' + \left(\frac{a}{D} - \frac{1}{4} \left(\frac{\mu}{D} \right)^2 - \frac{1}{2} \frac{d}{dx} \left(\frac{\mu}{D} \right) \right) g_a = 0, \tag{3.3}$$

and we note that positivity of z^N is equivalent to positivity of g_a. In general, there may be multiple positive solutions to (3.3) for a specific a. For $a = 0$, we identify the solution $g_0 = e^{\int_0^x \frac{\mu(s)}{\sigma^2(s)} ds}$, which leads to $z^N = z$.

For $a > 0$, we focus on the solution g_a, such that $g_a(0) = 1$, $g_a'(0) = g_0'(0) = \frac{\mu(0)}{\sigma^2(0)}$. It is easy to see that for any $x > 0$, such that $g_a(x) > 0$, the function $\Gamma(x) \overset{\text{def}}{=} \frac{g_0'(x)}{g_0(x)} - \frac{g_a'(x)}{g_a(x)}$ satisfies the nonlinear equation

$$\Gamma' = \Gamma^2 - \frac{\mu}{D}\Gamma + \frac{a}{D}, \tag{3.4a}$$

$$\Gamma(0) = 0, \tag{3.4b}$$

and that at the smallest $x_0 > 0$ for which $g_a(x_0) = 0$ (there must of course exist such a point if $a > 0$, since no positive solutions exist for $a > 0$; moreover, as shown in the proof of Proposition 2 in [24], there exists at least one positive and one negative such point), $\Gamma(x) \nearrow \infty$ as $x \nearrow x_0$, i.e., Γ blows up at x_0. We want to find a lower bound on x_0, given $a > 0$.

Clearly, x_0 increases toward infinity as a decreases toward 0. We assume that a is sufficiently small so that $x_0 > 1$, and note that $\Gamma(x) > 0$ for all $x > 0$, since if $\Gamma(x)$ approaches 0, the term $\frac{a}{D}$ on the right-hand-side of (3.4a) dominates the others.

Define $\eta = \min_{0 \le x \le 1} \frac{1}{D(x)}$, and $\xi = \max_{0 \le x \le 1} \frac{\mu(x)}{D(x)}$. A standard differential inequality implies that if $\hat{\Gamma}$ solves

$$\hat{\Gamma}' = -\xi\hat{\Gamma} + \eta a, \tag{3.5a}$$

$$\hat{\Gamma}(0) = 0, \tag{3.5b}$$

then $\Gamma \ge \hat{\Gamma}$ for each $0 < x \le 1$. The solution to (3.5a)-(3.5b) when $\xi > 0$ is $\hat{\Gamma}(u) = \frac{\eta(1-e^{-\xi u})}{\xi} a \ge \frac{\eta}{2} au \overset{\text{def}}{=} C_1 au$ for $u \le \min\{1, \frac{1}{\xi}\}$, and thus $\Gamma(u) \ge \hat{\Gamma}(u) \ge C_1 a$, for such u. This inequality also trivially holds when $\xi = 0$ for all u, since $\hat{\Gamma}(u) = \eta au$ in this case.

Next, we study the Bernoulli equation

$$Y' = Y^2 - \frac{\mu}{D}Y, \tag{3.6a}$$

$$Y(r) = C_1 a, \tag{3.6b}$$

and note that another differential inequality implies that $\Gamma(x) \ge Y(x)$ for $u \le x \le x_0$, so if Y blows up at x_1, then $x_0 \le x_1$. It is straightforward to verify that the solution to (3.6a-3.6b) satisfies

$$e^{-\int_u^x Y(s) ds} - 1 = -Y(u) e^{\int_0^u \frac{\mu(s)}{D(s)} ds} \left(R(x) - R(u)\right),$$

implying that $Y(x)$ blows up when

$$Y(u) e^{\int_0^u \frac{\mu(s)}{D(s)} ds} \left(R(x) - R(u)\right) \nearrow 1,$$

i.e., for small fixed u, before

$$\frac{C_1}{2} au \left(R(x) - u\right) = 1,$$

corresponding to

$$a = \frac{2}{C_1 u} \times \frac{1}{R(x) - u} \stackrel{\text{def}}{=} C \frac{1}{R(x) - u}. \tag{3.7}$$

By an identical argument for $x < 0$, it follows that $\Gamma(v) \to -\infty$ before v reaches the point x, where

$$a = \frac{2}{C_2 v} \times \frac{1}{-R(-x) - v},$$

and therefore altogether that $\Gamma(v)$ blows up to both negative and positive infinity within the interval

$$v \in B = \left\{ N : |N| \leq \frac{C}{T(N) - \epsilon} \right\},$$

where $C = \max\{2C_1/\epsilon, 2C_2/\epsilon\}$. Thus, g_a has two roots in B, and does therefore not belong to Θ^N. Moreover, the Sturm separation theorem implies that *any* solution to (3.3) (not just the one with the initial conditions studied so far) has at least one root in B, and therefore does not belong to Θ^N either. This leads to inequality (3.1) in the proposition, since any a such that $\rho + a \leq \bar{\lambda}^N$ must be smaller than (3.7).

For inequality (3.2), we note that

$$\left| \frac{z^N(x) - z(x)}{z(x)} \right| = \left| \frac{g_a(x) - g_0(x)}{g_0(x)} \right| = \left| \frac{g_a(x)}{g_0(x)} - 1 \right|,$$

where g_a is a solution to (3.3) that satisfies $g_a(0) = 1$. It follows that $\Gamma(x) = \frac{g_0'(x)}{g_0(x)} - \frac{g_a'(x)}{g_a(x)}$ satisfies

$$\Gamma' = \Gamma^2 - \frac{\mu}{D}\Gamma + \frac{a}{D},$$
$$\Gamma(0) = v,$$

for some v, which we without loss of generality assume is weakly positive, and that Γ is defined on the interval $B = (-N, N)$ if $z^N > 0$ on that interval.

It is easy to verify that

$$\frac{g_a(x)}{g_0(x)} = e^{-\int_0^x \Gamma(s) ds}, \tag{3.8}$$

and moreover, a similar argument as that for a implies that the solution $Y(x)$ to the ODE

$$Y' = Y^2 - \frac{\mu}{D} Y, \tag{3.9a}$$

$$Y(y) = \Gamma(y), \qquad 0 < y < x, \tag{3.9b}$$

satisfies $Y(s) \leq \Gamma(s)$, $y < s < N$, and does therefore not blow up before N.

The solution to (3.9a-3.9b) satisfies

$$e^{-\int_y^N Y(s) ds} = 1 - \Gamma(y) e^{\int_0^y \frac{\mu(s)}{D(s)} ds} (R(N) - R(y)) \geq 0,$$

so

$$\Gamma(y) \leq \frac{e^{-\int_0^y \frac{\mu(s)}{D(s)} ds}}{R(N) - R(x)} = \frac{Q(y)}{R(N) - R(x)}.$$

It follows from (3.8) that

$$\left|\frac{z^N(x)-z(x)}{z(x)}\right| \le 1-e^{-\int_0^x Q(y)dy/(R(N)-R(x))}$$
$$= 1-e^{-1/(Z(x,N)-1)}$$
$$= O\left(\frac{1}{Z(x,N)-1}\right),$$

where the rightmost equality follows from a Taylor expansion of the exponential function and the fact that for large N, $Z(x,N)$ tends to infinity regardless of x.

An identical argument shows that the inequality also holds for $x<0$. We are done. ∎

Proposition 3.1 implies standard convergence orders, depending on the behavior of the function R. For example the following result is an immediate consequence of the proposition when $R(x)$ is a regularly varying function (see [10]).

COROLLARY 3.2. *If $R(x)$ is a regularly varying function, i.e., if $\frac{R(kx)}{R(x)} \sim k^\nu$ for some constant $\nu>0$ for large k, then for a fixed x,*

$$|\bar\lambda^N-\rho| = O(N^{-\nu}), \qquad \left|\frac{z^N(x)-z(x)}{z(x)}\right| = O\left(N^{-\nu}\right). \tag{3.10}$$

We note that $\bar\lambda^N > \rho$, so the approximation error is one-sided with respect to ρ in the corollary.

An important variable with significant economic meaning is the representative agent's pointwise relative risk aversion: $G(x) = -\frac{(1/z)'}{(1/z)} = -\frac{-z'}{z^2(1/z)} = \frac{z'(x)}{z(x)}$, which in our truncated problem is approximated by $G^N(x) = \frac{(z^N)'(x)}{z^N(x)}$. The approximation error of the pointwise relative risk aversion is thus $e^N(x) = \left|G(x)-G^N(x)\right|$. From the definition we expect the convergence rate for $G(x)$ to be the same as that for $z(x)$.

4. A regular Sturm-Liouville problem. It is convenient to reformulate the approximate recovery problem as a regular Sturm-Liouville problem. That this is possible is a consequence of the following result:

PROPOSITION 4.1. *The solution to the approximate recovery problem is zero at both boundaries, $z^N(-N) = z^N(N) = 0$.*

Proof. Assume that $\theta^N = (\bar\lambda^N, z^N)$ solves the approximate recovery problem. If $z^N > 0$ for all $x\in[-N,N]$, then a standard perturbation argument implies that there is an element $(\bar\lambda+\epsilon, \hat z^N)\in\Theta^N$ for some small $\epsilon>0$, and θ^N is therefore not maximal, leading to a contradiction. Thus, z^N is zero in at least one of the boundaries, $x=\pm N$.

Without loss of generality, assume that $z^N(-N)=0$. If $z^N(N)=0$ we are done, so assume that $z^N(x)>0$, for $x\in(-N,N]$ (recall that $z^N(x)>0$ in $x\in(-N,N)$, since $\theta^N\in\Theta^N$). From the theory of second order linear ODEs, it follows that there exists another nontrivial solution, $\tilde z^N$, to the fundamental ODE $\mathcal{L}_{\bar\lambda}[z]=0$. Clearly, it cannot be the case that $\tilde z^N$ is positive for all $x\in[-N,N)$, since then it would follow that $(\bar\lambda^N, z^N+\tilde z^N)\in\Theta^N$, and the same argument as before would imply that θ^N is not maximal. Similarly, $\tilde z^N$ cannot be negative for all $x\in[-N,N)$

So, $\tilde z^N$ has at least one root in $[-N,N)$, and by the Sturm separation theorem, it must have *exactly* one root in $(-N,N)$. Without loss of generality, assume that

$\tilde{z}^N(-N) > 0$ and $\tilde{z}^N(N) \leq 0$. For small enough $\epsilon > 0$, it follows that $\hat{z}^N = z^N + \epsilon \tilde{z}^N$ is strictly positive on $[-N, N]$, and also that $(\bar{\lambda}^N, \hat{z}^N) \in \Theta^N$, again implying that θ^N is not maximal.

So, for θ^N to be maximal it must be that $z^N(-N) = z^N(N) = 0$. We are done.□

Proposition 4.1 immediately leads to an equivalent formulation of the approximate recovery problem as a regular Sturm-Liouville problem:

$$\begin{aligned} -D(w^N)'' - \kappa(w^N)' + rw^N &= \lambda^N w^N, \\ w^N(-N) &= 0, \\ w^N(N) &= 0. \end{aligned} \tag{4.1}$$

Specifically, from Sturm-Liouville theory, it follows that (4.1) has a discrete set of solutions, with associated eigenvalues, $\lambda^N = \lambda_1^N < \lambda_2^N < \lambda_3^N < \ldots < \lambda_n^N < \ldots$, and eigenfunctions, $w_1^N, w_2^N, \ldots w_n^N, \ldots$, where w_n^N has $n-1$ zeros on $(-N, N)$. Hence, the solution to the approximate recovery problem is the smallest eigenvalue and associated eigenfunction to the Sturm-Liouville problem (4.1), $\theta^N = (\bar{\lambda}^N, z^N) = (\lambda_1^N, w_1^N)$.

Our approach in the next section is to use inverse iterations to solve a discretized version of the ODE (4.1), using the finite difference method, and thereby differs from other suggested approaches. As noted in Ross [20], the risk-neutral distribution can be estimated from observed option prices (see also [5, 15, 9]). Ross assumes a discrete, finite, state space, under which exact recovery is possible by solving a recursive system of equations. The approach uses Perron-Frobenius theory for (finite dimensional) nonnegative matrices, and therefore does not address the issue of truncation when the state space is unbounded. Carr and Yu [6] solve the fundamental ODE (2.1) on an interval under the assumption of Robin boundary conditions, allowing for perfect recovery in this case. As noted in [8], however, the specification of the boundary condition has a significant effect on the recovered solution. The truncation issue is therefore present also with their approach.

Park [18], also using a PDE approach, studies the extension of the recovery to transient processes. These processes will violate the condition $\lim_{x\to\infty} T(x) = \infty$ in Proposition 2.1. In this case, recovery is impossible without further knowledge about ρ. When ρ is known and additional conditions are satisfied, recovery is also possible for this case with $T(\infty) < \infty$), see [18]. The numerical analysis of ARP for this case is an interesting potential extension.

Other papers focus directly on the link between risk-neutral and physical probabilities, avoiding the PDE formulation. Jackwerth and Menner [14] apply Ross's discrete model to S&P 500 call option data, and find that recovered distributions are incompatible with physical probabilities. Similarly, Dillschneider and Maurer [7], also using S&P options pricing data and assuming a bounded state space, find empirical evidence that the pricing kernel is misspecified under this approach. Our PDE approach complement these papers, by focusing on the numerical implications of truncating the state space. A general implication of our analysis is that the rate of mean reversion of the physical process determines the numerical challenges associated with empirical recovery. This relation may provide a clue for the poor performance documented in this empirical recovery literature.

5. Analysis of model problems.

We define three model problems that will serve as a test bench.

Model problem 1 — Ornstein-Uhlenbeck process. The Ross recovery problem for a mean reverting Ornstein-Uhlenbeck process with the pricing kernel determined by power preferences is $\mu(x) = -\beta x$, $\beta > 0$, $\sigma(x) = \sigma$, $z(x) = e^{\gamma x}$, $r(x) = \rho - \gamma\beta x - \frac{\gamma^2\sigma^2}{2}$, $\kappa(x) = -\beta x - \gamma\sigma^2$. The pointwise relative risk aversion in this case is thus $G(x) \equiv \gamma$. We choose the parameters $\beta = 0.05$, $\sigma = 0.2$, and $\rho = 0.02$. We use $\gamma = 2$ in the numerical experiments and analyze a problem with $\gamma = 0$.

For this problem, we have $Q(x) = e^{\frac{\beta}{\sigma^2}x^2}$ which gives

$$R(x) = \int_0^x Q(s)ds = \int_0^x e^{\frac{\beta}{\sigma^2}s^2}ds = \frac{\sqrt{\pi}\ \mathrm{erfi}\left(\sqrt{\frac{\beta}{\sigma^2}}\,x\right)}{2\sqrt{\frac{\beta}{\sigma^2}}},$$

and since $\mathrm{erfi}(x) = -\,\mathrm{erfi}(-x)$ we get $T(x) = \frac{\sqrt{\pi}\left|\mathrm{erfi}\left(\sqrt{\frac{\beta}{\sigma^2}}\,x\right)\right|}{2\sqrt{\frac{\beta}{\sigma^2}}}$, and finally $Z(x,N) = \left|\frac{\mathrm{erfi}\left(\sqrt{\frac{\beta}{\sigma^2}}\,N\right)}{\mathrm{erfi}\left(\sqrt{\frac{\beta}{\sigma^2}}\,x\right)}\right|$. This means that $T(N)$ and $Z(x,N)$ grow extremely fast, and we correspondingly expect fast, super-exponential, convergence as N grows.

Exact solution with $\gamma = 0$. When $\gamma = 0$, we can derive an exact solution to the bounded approximation problem. Equation (4.1) with $\sigma(x) = \sigma$, $\mu(x) = -\beta x$, $r = \rho$ is given by

$$\begin{array}{rcl} -\frac{\sigma^2}{2}(z^N)'' + \beta x(z^N)' + \rho z^N &=& \bar{\lambda}^N z, \\ z^N(0) &=& 1, \\ z^N(N) &=& 0, \end{array} \tag{5.1}$$

with analytical solution

$$z^N(x) = c_1\frac{e^{\frac{\beta}{2\sigma^2}x^2}\,wM\left(-\frac{2\bar{\lambda}^N-2\rho+\beta}{4\beta}, -\frac{1}{4}, -\frac{\beta}{\sigma^2}x^2\right)}{\sqrt{x}} + c_2\frac{e^{\frac{\beta}{2\sigma^2}x^2}\,wW\left(-\frac{2\bar{\lambda}^N-2\rho+\beta}{4\beta}, -\frac{1}{4}, -\frac{\beta}{\sigma^2}x^2\right)}{\sqrt{x}}. \tag{5.2}$$

Here wM and wW are Whittaker functions defined by

$$\begin{array}{rcl} wM(k,m,y) &=& y^{1/2+m}e^{-y/2}M(m-k+\frac{1}{2}, 1+2m, y), \\ wW(k,m,y) &=& y^{1/2+m}e^{-y/2}U(m-k+\frac{1}{2}, 1+2m, y), \end{array} \tag{5.3}$$

where M and U are Kummer's functions defined by

$$\begin{array}{rcl} M(a,b,y) &=& \sum_{n=0}^{\infty}\frac{a_{[n]}}{n!b_{[n]}}y^n, \\ U(a,b,y) &=& \frac{\Gamma(1-b)}{\Gamma(a-b+1)}M(a,b,y) + \frac{\Gamma(b-1)}{\Gamma(a)}y^{1-b}M(a-b+1, 2-b, y), \end{array} \tag{5.4}$$

and $x_{[n]}$ denotes the Pochhammer symbol defined by

$$x_{[n]} = x(x+1)(x+2)\cdots(x+n-1). \tag{5.5}$$

Inserting (5.4) and (5.3) in (5.2) gives

$$\begin{array}{rcl} z^N(x) &=& e^{\frac{\beta}{\sigma^2}x^2}(-\frac{\beta}{\sigma^2})^{1/4}\Big(c_1 M\left(\frac{\bar{\lambda}^N-\rho+\beta}{2\beta}, \frac{1}{2}, -\frac{\beta}{\sigma^2}x^2\right) + \\ && c_2\left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{\bar{\lambda}^N-\rho+2\beta}{2\beta})}M\left(\frac{\bar{\lambda}^N-\rho+\beta}{2\beta}, \frac{1}{2}, -\frac{\beta}{\sigma^2}x^2\right) + \right. \\ && \left. i\cdot\frac{\Gamma(-\frac{1}{2})}{\Gamma(\frac{\bar{\lambda}^N-\rho+\beta}{2\beta})}\left(\frac{\beta}{\sigma^2}x^2\right)^{1/2}M\left(\frac{\bar{\lambda}^N-\rho+2\beta}{2\beta}, \frac{3}{2}, -\frac{\beta}{\sigma^2}x^2\right)\right)\Big) \end{array} \tag{5.6}$$

The unknown coefficients c_1 and c_2 as well as $\bar{\lambda}^N$ are determined by the boundary conditions

$$\begin{array}{rcl} z^N(0) & = & 1, \\ z^N(N) & = & 0, \end{array} \tag{5.7}$$

together with the fact that we are looking for real-valued solutions $(\bar{\lambda}^N, z^N)$. The latter condition gives that $c_2 = 0$ since $M(a, b, y)$ is real for all values of a, b, and y. Since

$$\lim_{x\to 0} M\left(\frac{\bar{\lambda}^N - \rho + \beta}{2\beta}, \frac{1}{2}, -\frac{\beta}{\sigma^2}x^2\right) = 1$$

we get from the boundary condition in $x = 0$ that $(-\frac{\beta}{\sigma^2})^{1/4} c_1 = 1$. Thus, the solution is given by

$$z^N(x) = e^{\frac{\beta}{\sigma^2}x^2} M\left(\tfrac{\bar{\lambda}^N - \rho + \beta}{2\beta}, \tfrac{1}{2}, -\tfrac{\beta}{\sigma^2}x^2\right). \tag{5.8}$$

Finally, $\bar{\lambda}^N$ is determined from the boundary condition in $x = N$, i.e.

$$M\left(\frac{\bar{\lambda}^N - \rho + \beta}{2\beta}, \frac{1}{2}, -\frac{\beta}{\sigma^2}N^2\right) = 0, \tag{5.9}$$

is solved for $\bar{\lambda}^N$.

We verify the high convergence rate of $\bar{\lambda}^N$ to ρ in the following proposition

Proposition 5.1. *Equation (5.9) implies the following error bound:*

$$|\bar{\lambda}^N - \rho| \leq CNe^{-\frac{\beta}{\sigma^2}N^2},$$

where the constant C does not depend on N.

Proof. Specifically, define $\delta = \frac{\bar{\lambda}^N - \rho}{2\beta}$, and $\tau^2 = \frac{\beta}{\sigma^2}N^2$. We then have that

$$M\left(\delta + \frac{1}{2}, \frac{1}{2}, -\tau^2\right) = 0, \tag{5.10}$$

which determines the relation $\delta(\tau)$. It follows from standard properties of the Kummer function that there is exactly one strictly positive solution to (5.10) (in τ) for $\delta > 0$, and that

$$\begin{aligned} M\left(\delta + \frac{1}{2}, \frac{1}{2}, -\tau^2\right) &= \frac{\tau^{\frac{1}{2}}}{\Gamma(1/2 + \delta)} \int_0^\infty e^{-t} t^{\delta - \frac{1}{4}} \sqrt{\frac{2}{\pi}} \frac{\cos(2\sqrt{t}\tau)}{\sqrt{2\sqrt{t}\tau}}\, dt \\ &= \frac{1}{\Gamma(1/2 + \delta)} \int_0^\infty e^{-\frac{x^2}{4\tau^2}} \left(\frac{x}{2\tau}\right)^{2\delta} \sqrt{\frac{1}{\pi}} \cos(x) \frac{1}{2\tau}\, dx \end{aligned}$$

where we substituted $\sqrt{t}$ for $\frac{x}{2\tau}$ in the integration, so, defining

$$H(\tau, \delta, a, b) = \int_a^b e^{-\frac{x^2}{4\tau^2}} x^{2\delta} \cos(x)\, dx,$$

and $H(\tau, \delta) = H(\tau, \delta, 0, \infty)$, $\tau(\delta)$ satisfies $H(\tau(\delta), \delta) = 0$. Now, $H(\tau, 0) = \sqrt{\pi}\tau e^{-\tau^2} > 0$, and moreover $H(0+, \delta) > 0$, so if $H(\hat{\tau}, \delta) < 0$, then $\tau < \hat{\tau}$.

Define the function $V(x) = e^{-\frac{x^2}{4\hat{\tau}^2}}(x^{2\delta} - 1)$, which is increasing on $x \in [0, x^*]$, and decreasing on $x \in [x^*, \infty)$, where a Taylor expansion around $\delta = 0$ reveals that $\hat{\tau} = \frac{1}{\sqrt{2}} x^* \sqrt{\log(x^*)}(1 + o(\delta))$, so that $x^*(\hat{\tau})$ increases slightly slower than linearly in $\hat{\tau}$, $x^* \leq \hat{\tau}$ for sufficiently large $\hat{\tau}$. For a given $\hat{\tau}$, define the integer m, such that $\pi/2 + 2m\pi < x^* \leq \pi/2 + (2m+2)\pi$.

We now have

$$
\begin{aligned}
H(\hat{\tau}, \delta) &= H(\hat{\tau}, 0) + H(\hat{\tau}, \delta) - H(\hat{\tau}, 0) \\
&= \sqrt{\pi}\hat{\tau} e^{-\hat{\tau}^2} + \int_0^\infty V(x)\cos(x)dx.
\end{aligned}
$$

The first term is positive and quickly decreasing in $\hat{\tau}$, and if we show that the second term is negative and dominates the first term, then it follows that $H(\hat{\tau}, \delta) \leq 0$, and therefore that $\tau \leq \hat{\tau}$.

We decompose the second term into

$$
\begin{aligned}
\int_0^\infty V(x)\cos(x)dx = \int_0^{3\pi/2} V(x)\cos(x)dx + \int_{3\pi/2}^{\pi/2+2m\pi} V(x)\cos(x)dx \\
+ \int_{\pi/2+2m\pi}^{7\pi/2+2m\pi} V(x)\cos(x)dx + \int_{7\pi/2+2m\pi}^{\infty} V(x)\cos(x)dx.
\end{aligned}
$$

Note that for the second and fourth terms we have

$$
\int_{3\pi/2}^{\pi/2+2m\pi} V(x)\cos(x)dx \leq 0,
$$

since $V(x)$ is increasing on $[3\pi/2, \pi/2 + 2m\pi]$, and

$$
\int_{7\pi/2+2m\pi}^{\infty} V(x)\cos(x)dx \leq 0,
$$

since $V(x)$ is decreasing on $[7\pi/2 + 2m\pi, \infty)$.

For the first term, we get

$$
\int_0^{3\pi/2} V(x)\cos(x)dx = e^{-\frac{x^2}{4\hat{\tau}^2}}(x^{2\delta} - 1)\cos(x)dx \leq
$$

$$
\frac{1}{2}\int_0^1 (x^{2\delta} - 1)dx + \frac{1}{2}\int_1^{\pi/2} (x^{2\delta} - 1)dx = \frac{\pi}{2}\left(\frac{\left(\frac{\pi}{2}\right)^{2\delta}}{1 + 2\delta} - 1\right) =
$$

$$
-\pi(1 - \log(\pi/2))\delta + O(\delta^2) \approx -1.72\delta + O(\delta^2).
$$

Finally, a Taylor expansion around $\delta = 0$ of the third term yields

$$
\int_{\pi/2+2m\pi}^{7\pi/2+2m\pi} V(x)\cos(x)dx \leq \frac{6\pi}{\log(\log(\sqrt{\log(1/\delta)}))} \times \delta + O(\delta^2) \leq \delta,
$$

for sufficiently small δ.

Altogether, we therefore arrive at

$$
H(\hat{\tau}, \delta) \leq \sqrt{\pi}\hat{\tau} e^{-\hat{\tau}^2} - 0.7\delta \leq 0,
$$

when $\hat{\tau}$ is large enough such that $\frac{\sqrt{\pi}}{0.7}\hat{\tau} e^{-\hat{\tau}^2} \leq \delta$, and since $\tau^2 = \frac{\beta}{\sigma^2} N^2 \leq \hat{\tau}^2$, the result follows. $\square$

Model problem 2 — Algebraic convergence. The functions $\mu(x) = \frac{\beta}{2}(1-\alpha)x$, $\alpha > 0$, $\beta > 0$, and $\sigma(x) = \sqrt{\beta(1+x^2)}$ lead to $Q(x) = (1+x^2)^{\frac{\alpha-1}{2}}$, which implies $R(x) = xF([\frac{1}{2}, -\frac{\alpha-1}{2}], \frac{3}{2}, -x^2)$ where $F(a,b,y) = \sum_{k=0}^{\infty} \left(\frac{(a_1)_{[k]}(a_2)_{[k]}\cdots(a_p)_{[k]}}{(b_1)_{[k]}(b_2)_{[k]}\cdots(b_q)_{[k]}}\right)\left(\frac{y^k}{k!}\right)$ is the generalized hypergeometric function, where $a = (a_1, \dots, a_p)$, $b = (b_1, \dots, b_q)$, and $x_{[n]}$ denotes the Pochhammer symbol defined in (5.5). This gives $T(N) = NF([\frac{1}{2}, -\frac{\alpha-1}{2}], \frac{3}{2}, -N^2)$ and finally $Z(x,N) = \frac{NF([\frac{1}{2}, -\frac{\alpha-1}{2}], \frac{3}{2}, -N^2)}{xF([\frac{1}{2}, -\frac{\alpha-1}{2}], \frac{3}{2}, -x^2)}$. Since $Q(x) \approx x^{\alpha-1}$ for large x we get $R(x) \approx \frac{x^\alpha}{\alpha}$, i.e. the convergence behavior for large N is approximately given by $T(N) = \frac{N^\alpha}{\alpha}$ and $Z(x,N) = \left(\frac{N}{x}\right)^\alpha$. Hence, from Corollary 3.2 we expect the convergence to be of order α.

We have $z(x) = e^{\gamma x}$, $\gamma = 2$, $r(x) = \rho + \gamma\frac{\beta}{2}(1-\alpha)x - \frac{\gamma^2}{2}\beta(1+x^2)$, $\kappa(x) = \frac{\beta}{2}(1-\alpha)x - \gamma\beta(1+x^2)$, and we choose $\beta = 0.04$, $\rho = 0.02$, and $\gamma = 2$.

Model problem 3 — Higher dimensions. In higher dimensions, the Ross recovery problem leads to an elliptic PDE. An example is given in [24] for the two-dimensional diffusion process

$$\begin{aligned} dX &= -\beta Y dt + \sigma_x dw_X, \\ dY &= -Y dt + \sigma_y dw_Y, \end{aligned}$$

with $z(x,y) = e^{\gamma x}$, which leads to the PDE:

$$\frac{\sigma_x^2}{2} z_{xx} + \frac{\sigma_y^2}{2} z_{yy} - y z_y - (\beta y + \gamma\sigma_x^2) z_x + (\lambda - r)\, z = 0, \tag{5.11}$$

with $r(x,y) = \rho - \gamma\beta y - \gamma^2\frac{\sigma_x^2}{2}$. The parameters are $\gamma = 2$, $\rho = 0.02$, $\sigma_x = 0.2$, $\sigma_y = 0.5$, $\beta = 0.1$. As discussed in [24], the solution technique yields a unique solution in this case too.

6. Numerical method.

We discretize (4.1) using centered finite differences on a uniform grid $x_j = -N + (j-1)\cdot\Delta x$, $j = 1, \dots, 2M_x + 1$, $\Delta x = N/M_x$:

$$\begin{aligned} -D_j \frac{w_{j+1}^N - 2w_j^N + w_{j-1}^N}{\Delta x^2} - \kappa_j \frac{w_{j+1}^N - w_{j-1}^N}{2\Delta x} + r_j w_j^N &= \lambda^N w_j^N \\ j = 2, \dots, 2M_x, & \\ w_1^N &= 0, \\ w_{2M_x+1}^N &= 0, \end{aligned} \tag{6.1}$$

where $w_j^N \approx w^N(x_j)$. Equation (6.1) can be written as

$$A w^N = \lambda^N w^N, \tag{6.2}$$

where $z^N = \left(z_2^N, \dots, z_{2M_x}^N\right)^T$.

For the two-dimensional problem (5.11) we also introduce $y_k = -N + (k-1)\cdot\Delta y$,

$k = 1, \ldots, 2M_y + 1$, $\Delta y = N/M_y$ and discretize

$$(6.3)\quad \begin{gathered} -\frac{\sigma_x^2}{2} \cdot \frac{w_{j+1,k}^N - 2w_{j,k}^N + w_{j-1,k}^N}{\Delta x^2} - \frac{\sigma_y^2}{2} \cdot \frac{w_{j,k+1}^N - 2w_{j,k}^N + w_{j,k-1}^N}{\Delta y^2} + \\ y_k \cdot \frac{w_{j,k+1}^N - w_{j,k-1}^N}{2\Delta y} + (\beta y_k + \gamma\sigma_x^2) \cdot \frac{w_{j+1,k}^N - w_{j-1,k}^N}{2\Delta x} + r w_{j,k}^N = \lambda^N w_{j,k}^N, \\ j = 2, \ldots, 2M_x,\ k = 2, \ldots, 2M_y \\ w_{j,k}^N = 0, \\ j = 1,\ j = 2M_x + 1,\ k = 1,\ k = 2M_y + 1, \end{gathered}$$

where $w_{j,k}^N \approx w^N(x_j, y_k)$. Now (6.3) can be written as (6.2) with

$$w^N = \left(w_{2,2}^N, \ldots, w_{2M_x,2}^N, w_{2,3}^N, \ldots, w_{2M_x,2M_y}^N\right)^T.$$

We want to find the smallest eigenvalue $\bar{\lambda}^N$ to (6.2). Since $\bar{\lambda}^N > \rho > 0$, this is also the eigenvalue to (6.2) with the smallest magnitude. Hence, we can compute $\theta^N = (\bar{\lambda}^N, z^N)$ using inverse iteration.

Algorithm 6.1 Inverse Iteration

- Choose $z^{N,0}$.
- for $\ell = 0, \ldots,$ until convergence
  - Solve $Aq^{\ell+1} = z^{N,\ell}$,
  - $z^{N,\ell+1} = \dfrac{q^{\ell+1}}{\|q^{\ell+1}\|}$,
  - $\bar{\lambda}^{N,\ell+1} = \dfrac{(z^{N,\ell+1})^T A z^{N,\ell+1}}{(z^{N,\ell+1})^T z^{N,\ell+1}}$,
- end for

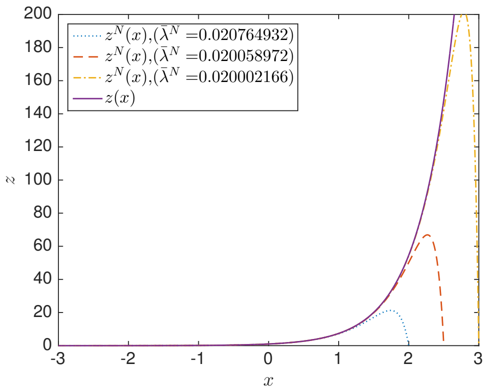
The matrix A is LU-factorized prior to the loop over ℓ and the factors are then used in the solution of $Aq^{\ell+1} = LUq^{\ell+1} = z^\ell$. As convergence criterion we use

$$(6.4)\quad |\bar{\lambda}^{N,\ell+1} - \bar{\lambda}^{N,\ell}| \leq \varepsilon.$$

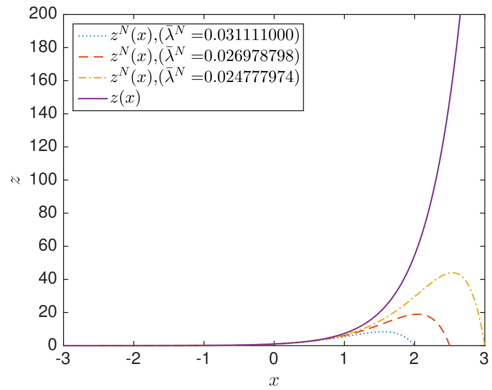
In all numerical experiments in Section 7 we use $\varepsilon = 10^{-6}$.

Since the solution z^N only is determined up to a multiplicative constant, we scale the obtained solution z^N from Algorithm 6.1 such that the final solution fulfills $z^N(0) = z(0)$.

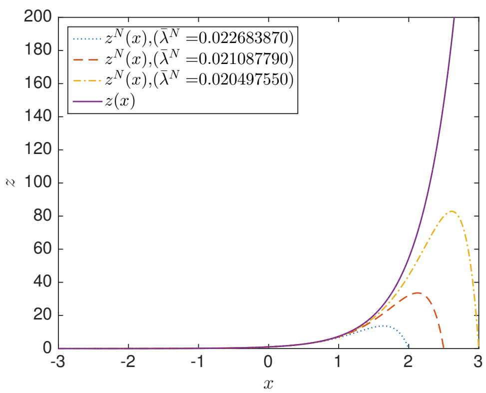
7. Numerical Results. In Figure 1 we show $z^N(x)$ for Model problems 1, 2 (with $\alpha = 2$ and $\alpha = 5$), and 3 with $\gamma = 2$, together with $z(x)$. The corresponding values of $\bar{\lambda}^N$ are given in the legends. We use the numerical method described in Section 6 to compute $z^N(x)$. For Model problems 1 and 2 we use discretization parameter $\Delta x = 10^{-3}$ and for Model problem 4 we use discretization parameters $\Delta x = 2 \cdot 10^{-2}$ and $\Delta y = 2 \cdot 10^{-2}$.



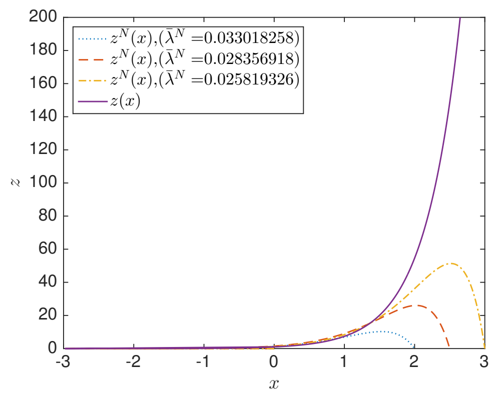
(a) Model problem 1.



(b) Model problem 2 with $\alpha = 2$.



(c) Model problem 2 with $\alpha = 5$.



(d) Model problem 3, we plot $z^N(x, 0)$.

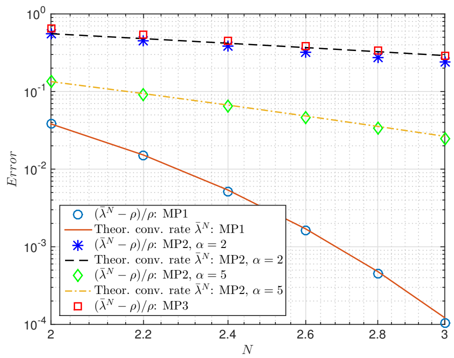
Figure 1: Plots of $z^N(x)$ for $N = 2$, 2.5, and 3 together with $z(x) = e^{\gamma x}$ for $\gamma = 2$. The corresponding values of $\bar{\lambda}^N$ are given in the legends.

From Figure 1 it is clear that for Model problem 1 the convergence is very fast in N. For Model problem 2 we have faster convergence in N for larger α as expected from the theory.

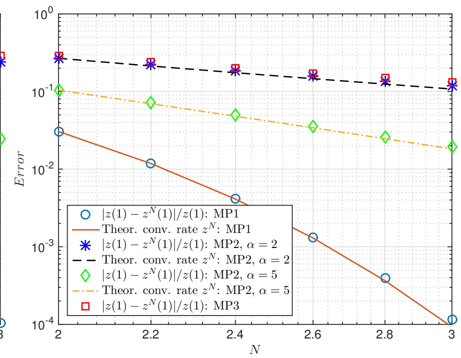
In Figure 2 we display $\frac{\bar{\lambda}^N - \rho}{\rho}$, $\frac{|z^N(1) - z(1)|}{z(1)}$ and $\frac{|G^N(1) - G(1)|}{G(1)}$ as a function of N for Model problems 1, 2 (with $\alpha = 2$ and $\alpha = 5$), and 3 using $\gamma = 2$. We have used the discretization parameters in Table 1 that are chosen such that the discretization error in (6.1) is negligible compared to the approximation errors between the approximate recovered and true solutions. In the same figure we show the theoretical error as a function of N originating from $N = 2$.

MP 1	$\Delta x = 10^{-5}$
MP 2	$\Delta x = 10^{-4}$
MP 3	$\Delta x = 2 \cdot 10^{-2}$, $\Delta y = 10^{-2}$

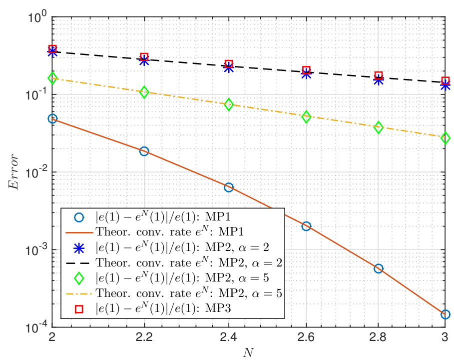
Table 1: Discretization parameters.



(a) Convergence of $\bar{\lambda}^N$ for MP 1, 2 (with $\alpha = 2$ and $\alpha = 5$), and 3.



(b) Convergence of z^N for MP 1, 2 (with $\alpha = 2$ and $\alpha = 5$), and 3.



(c) Convergence of G^N for MP 1, 2 (with $\alpha = 2$ and $\alpha = 5$), and 3.

Figure 2: Convergence plots for MP 1, 2 (with $\alpha = 2$ and $\alpha = 5$), and 3.

In Table 2, we compare theoretical and actual convergence rates.

	Theoretical (upper limit of) damping of relative error between $N = 2$ and $N = 3$			Actual damping of relative error in computed solution between $N = 2$ and $N = 3$		
	$\bar{\lambda}^N$	z^N	G^N	$\bar{\lambda}^N$	z^N	G^N
MP1	3.18e-3	3.03e-3	3.03e-3	2.73e-3	3.79e-3	3.00e-3
MP2, $\alpha = 2$	5.23e-1	4.02e-1	4.02e-1	4.30e-1	4.39e-1	3.71e-1
MP2, $\alpha = 5$	1.97e-1	1.75e-1	1.75e-1	1.85e-1	1.87e-1	1.71e-1

Table 2: Convergence rate.

From Figure 2 and Table 2 we conclude that the convergence behavior for Model problems 1–2 agrees very well with the theory. Finally, we note that the two-dimensional Model problem 3 converges approximately as Model problem 2 with $\alpha = 2$.

In Table 3 we display the number of iterations in Algorithm 6.1 to reach the convergence criterion (6.4) for $\varepsilon = 10^{-6}$. We have used the discretization parameters in Table 1.

N	MP1	MP2, $\alpha = 2$	MP2, $\alpha = 5$	MP3
2.0	10	11	9	15
2.2	11	11	9	16
2.4	11	11	9	17
2.6	11	11	9	18
2.8	11	11	9	20
3.0	11	12	9	21

Table 3: Number of iterations.

From Table 3 we conclude that the number of iterations to reach convergence is relatively low and does not increase much with problem size.

8. Conclusions. In this paper we have considered Ross recovery when the state space is governed by a diffusion process and the pricing kernel is on transition independent form, but asset prices are only available on a bounded subinterval $B = [-N, N]$, symmetric around the origin, with no known boundary condition. We denote this the *approximate recovery problem*, ARP. We show that the solution to this problem is zero at both boundaries $x = -N$ and $x = N$ and hence can be written as a regular Sturm-Liouville problem. We derive theoretical error bounds on the recovered pricing kernel, that depend on the underlying parameters of the model, and show that the more mean reverting the process is, the faster is the convergence with increasing N.

We introduce a finite difference method to the ARP which results in a discretized eigenvalue problem. Using Sturm-Liouville theory we get that the solution of the ARP is given by the smallest eigenvalue of A in (6.2) and its associated eigenfunction. Hence, we can use inverse iteration to obtain the solution. A test bench of three model problems are defined and analyzed. Employing our introduced numerical method to these problems verify the theoretical error bounds. Moreover, a numerically accurate solution is obtained in relatively few iterations in Algorithm 6.1. Our approach helps make Ross recovery for diffusion processes practically operational.

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