Welfare in Economies with Production and Heterogeneous Beliefs*

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Abstract

We introduce the concept of Incomplete Knowledge (IK) efficiency in economies with production and heterogeneous beliefs, in which a social planner has incomplete knowledge about which beliefs are correct. IK-inefficient allocations can be improved upon without taking a stand on which belief, among a whole set of reasonable beliefs, is correct. We relate IK-efficiency to other recently introduced measures in heterogeneous beliefs economies, e.g., belief neutral efficiency. We show that the different welfare measures are equivalent in some important cases, but that in general they differ. Our results highlight the challenges of welfare analysis in production economies with heterogeneous beliefs.

Keywords: *Speculation, heterogeneous beliefs, investments, welfare.*

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1 Introduction

In the aftermath of the financial crisis, welfare analysis in economies with heterogeneous beliefs has received renewed attention in several recent studies (e.g., Brunnermeier et al. 2014; Gilboa et al. 2014; Gayer et al. 2014; Blume et al. 2014). These studies suggest that speculative trades between agents with different beliefs, although increasing the ex ante expected utility of each agent, is inefficient from an ex post perspective, since some agents must necessarily be wrong and end up very poor after speculating away much of their wealth. Speculation in the heterogeneous beliefs economy, instead of creating welfare by allowing agents to hedge risks and smooth consumption across states, then generates volatility of consumption and wealth at the individual level (see Yan 2008, and Fedyk, Heyerdahl-Larsen, and Walden 2013). It also affects asset price dynamics, increasing trading volume and price volatility, and potentially leads to mispricing (see Dumas et al. 2009, Xiong and Yan 2010, Kubler and Schmedders 2012, Simsek 2013, Buss et al. 2016, and Ehling et al. 2016 for recent contributions), effects that may be viewed as negative in their own right. The potential policy implications are, of course, huge, since the friction-free complete market equilibrium — even when it is implementable—may actually be inefficient with this view. To understand when such speculative inefficiencies can be identified is therefore of vital importance.

Brunnermeier et al. 2014 introduce the concept of belief neutral inefficiency, which identifies an allocation as inefficient in the heterogeneous beliefs economy if it can be improved upon for all probability distributions among a whole set of such distributions. They mainly focus on an exchange economy setting, in which their concept is shown to work well. In the production economy, however, additional challenges arise. Specifically, whereas efficiency in the pure exchange economy mainly concerns the distribution of a fixed-size "pie" among the agents in the market, in the production economy these agents may actually disagree about the size of the pie, i.e., about the optimal allocation of productive resources. The assumptions made about the ability of the social planner to identify such an optimal allocation then become important.

In this paper, we propose an efficiency measure for production economies with heterogeneous beliefs. We are specifically interested in what can be said about efficiency without taking a stand on what are the correct beliefs in the economy. We introduce the concept of Incomplete Knowledge (IK) efficiency. Loosely speaking, an allocation, a, is IK-inefficient if there is an alternative allocation that dominates a for some belief among a whole set of reasonable beliefs, and is not dominated by a for any such reasonable belief. This approach is based on the view that requiring a social planner to identify correct beliefs in order to take action is to set the bar too high. Indeed, a crucial role of markets is to allow for the dissemination and aggregation of information, beyond that available to any individual agent, and it is thus unclear how the social planner would obtain such superior knowledge. Our analysis has similarities with, and builds upon, the concepts of belief neutral inefficiency introduced in Brunnermeier et al. (2014), no-betting Pareto dominance in Gilboa et al. (2014), and with unanimity efficiency in Gayer et al. (2014), but also has significant differences. We also discuss the concept of U-efficiency, which is related to unanimity efficiency.

The general implication of our analysis is that the distinction between these welfare measures becomes increasingly important and subtle the further one moves away from the benchmark exchange economy with homogeneous beliefs. In this benchmark economy, all efficiency concepts we consider are equivalent, and also agree with the so-called Arrow efficiency measure that is based on agents' ex ante expected utilities, see Starr (1973).

In the exchange economy with heterogeneous beliefs, which has mainly been the focus of previous studies, IK-efficiency coincides with belief neutral efficiency, as well as with the set of allocations that are not belief neutral inefficient.[1] Arrow optimal allocations, however, are typically inefficient with respect to all of these measures. Moreover, we show that the set of allocations that are U-efficient includes Arrow optima. The concept therefore cannot rule out speculative allocations.

In the most general case, the production economy with heterogeneous beliefs, we show that as long as the production set is not small, the set of belief neutral efficient allocations is typically empty. Thus, in contrast to the exchange economy, belief neutral efficiency is typically too strong a concept to use within the production economy setting. Moreover, U-efficiency is too broad a concept to rule out speculative allocations also in this case. For an important class of standard workhorse production economies, IK-inefficiency and belief neutral inefficiency coincide. Specifically, in economies with convex production and belief sets, that satisfy what we call a strict dominance property, the two efficiency concepts imply each other. Welfare analysis in such economies is thus similar to the exchange economy setting.

Finally, we provide several examples of production economies in which IK-inefficiency and belief neutral inefficiency are distinct. Our main conclusion from this discussion is that both concepts have their merits and, more importantly, that the welfare analysis under heterogeneous beliefs is more challenging as well as subtle in economies for which the concepts differ.

The rest of the paper is organized as follows: In the next section, we develop the IK efficiency measure and compare it with alternative measures, and in Section 3 we provide our main theoretical results. Section 4 introduces several examples. Finally, some concluding remarks are made in Section 5. All proofs are delegated to an Appendix.

2 Efficiency

Consider an economy with $T \geq 1$ time periods, $M \geq 2$ states, and $N \geq 2$ agents. A nonempty compact set, $\mathcal{A} \subset \mathbb{R}_+^{M\times N\times T}$, determines the joint production and allocation of a consumption good among the agents in the economy, and will be denoted by the *set of feasible allocations.* Specifically, for $a \in \mathcal{A}$, $a_{m,n,t}$ represents the allocation of the good in state m to agent n at time t.[2]

[1] In general, there may be a "gap" between belief neutral efficiency and inefficiency, i.e., there may be allocations that are neither belief neutrally inefficient, nor belief neutrally efficient.

[2] Dynamic revelation of information over time can be added to the model by introducing a filtration over the states, and requiring feasible allocations to be adapted to that filtration. The results are identical, so we exclude this extra step for the sake of parsimony.

2.1 Homogeneous beliefs

We first study the case when there is no disagreement about the probabilities for different events, as a benchmark case. These probabilities are represented by a probability vector $q \in S^M$, where S^M is the interior of the unity simplex in $\mathbb{R}^M$,

$$S^M = \left\{ x \in \mathbb{R}^M : x_m > 0, \sum_{m=1}^{M} x_m = 1 \right\},$$

and we also define its closure $\bar{S}^M = \left\{ x \in \mathbb{R}^M : x_m \geq 0, \sum_{m=1}^{M} x_m = 1 \right\}$. Note that we assume that the probability for each state to occur is strictly positive.

Agents are expected utility maximizers. Specifically, agent n's expected utility under allocation a given probability vector q is

$$U^n(a|q) = \sum_{m=1}^{M} U_m^n(a) q_m, \qquad \textit{where} \qquad U_m^n(a) = \sum_{t=1}^{T} u_{m,t}^n(a_{m,n,t}). \tag{1}$$

Here, we assume that each agent-, state-, and time-specific utility function, $u_{m,t}^n : \mathbb{R}_+ \to \mathbb{R}$, is strictly increasing, continuously differentiable, and weakly concave. With each allocation, we associate the utility matrix, $V = \mathcal{V}(a) \in \mathbb{R}^{M \times N}$, through the mapping $V_{m,n} = U_m^n(a), 1 \leq m \leq M, 1 \leq n \leq N$, and define the *utility possibility set* $\mathcal{U} = \mathcal{V}(\mathcal{A}) \subset \mathbb{R}^{M \times N}$ (see Mas-Colell et al. (1995)).

A social planner has a Bergson welfare function over feasible allocations, $U(a|q, \lambda)$, defined by

$$U(a|q, \lambda) = \sum_{n=1}^{N} \lambda^n U^n(a|q), \tag{2}$$

where $\lambda \in S^N$ are Pareto weights in the planner's welfare function. Using the rules of matrix-vector multiplication, and denoting the transpose of the vector q by q^T, it then follows that

$$U(a|q, \lambda) = q^T \mathcal{V}(a) \lambda, \tag{3}$$

i.e., given an allocation, a, the Bergson welfare function is a bilinear mapping from the probability vector, $q \in S^M$, and vector of Pareto weights, $\lambda \in S^N$ to a real number.

If $U(b|q, \lambda) > U(a|q, \lambda)$ for two allocations, $a, b \in \mathcal{A}$, we write $b >_q^\lambda a$, and if $U(b|q, \lambda) \geq U(a|q, \lambda)$, we write $b \geq_q^\lambda a$. It is straightforward to verify that standard Pareto efficiency within this setting can be defined as follows:

Definition 1 (Pareto dominance and efficiency)

(i) Allocation b Pareto dominates *a given q, $b \succ_q a$, if $b >_q^\lambda a$ for all $\lambda \in S^N$.*

(ii) Allocation b is not Pareto dominated by a given q, $a \succeq_q b$, if $a \geq_q^\lambda b$ for some $\lambda \in S^N$.

(iii) Allocation b is Pareto efficient *given q, if $\forall a \in \mathcal{A} : b \succeq_q a$.*

We denote by E_q the set of all Pareto efficient allocations given probability vector q and note that this set is nonempty. Also, note that an equivalent definition of $b \succ_q a$ is that $b \geq_q^\lambda a$ for all $\lambda \in \bar{S}^N$, with the inequality being strict for at least one such λ. Also, an equivalent definition of allocation b being Pareto inefficient given q is that

$$\exists a \in \mathcal{A}, \forall \lambda \in S^N : a >_q^\lambda b. \tag{4}$$

2.2 Heterogeneous beliefs

Under heterogeneous beliefs, agent n's belief is denoted by q^n. The social planner views a whole (nonempty) set, $\mathcal{Q}_R \subset S^N$ of beliefs as "reasonable." The special case when $q^n = q$ for all n, reduces to the homogeneous beliefs setting, in which case we assume that the planner's reasonable beliefs set is $\mathcal{Q}_R = \{q\}$.

We are agnostic about the choice of $\mathcal{Q}_R$ in the general case. Brunnermeier et al. (2014) suggests using the convex hull of agents' individual beliefs. Specifically, for a set $X \subset \mathbb{R}^K$, define $CH(X) = \left\{\sum_{k=1}^K \rho_k x_k : \rho \in \bar{S}^K, x_k \in X, 1 \leq k \leq K\right\}$. The set suggested in Brunnermeier et al. (2014) is then

$$\mathcal{Q}_R^{CH} = CH\left(\left\{q^1, q^n, \ldots, q^N\right\}\right). \tag{5}$$

This choice may be appropriate in many applications, but we shall argue that other choices may be too. For example, the planner may wish to exclude some agents' beliefs that are obviously incorrect. Moreover, if agents self-report their beliefs, they may for strategic reasons report beliefs that are different than the ones they actually hold, and the social planner may therefore wish to filter out some reported beliefs. Finally, under more complex information structures, nonlinear aggregation of probabilities may be appropriate, rather than the linear aggregation that the convex hull corresponds to. We will discuss this latter point further in Section 4. The only restriction on $\mathcal{Q}_R$ we impose is that the set contains a single element, $\mathcal{Q}_R = \{q\}$ if and only if agents have homogeneous beliefs, i.e., if and only if $q^n = q^{n'}$ for all $1 \leq n, n' \leq N$. Thus, with heterogeneous beliefs, $|\mathcal{Q}_R| \geq 2$.

The novel efficiency concepts we study take into account the welfare associated with all reasonable beliefs. We first introduce the concept of IK-dominance with respect to specific Pareto weights, which we then use to define IK-efficiency:

Definition 2 (IK-dominance) *Allocation b* IK-dominates *a with respect to Pareto weights λ, $b \succ^\lambda a$, if:*

$$(\forall q \in \mathcal{Q}_R : b \geq_q^\lambda a) \text{ and } (\exists q \in \mathcal{Q}_R : b >_q^\lambda a),$$

From Definition 2 it follows that an allocation dominates another if, given Pareto weights, it is never strictly dominated under any reasonable probability, and there exist a probability measure under which it strictly dominates the other allocation. We also define weak IK-dominance, $a \succeq^\lambda b$, if $\neg(b \succ^\lambda a)$.

IK-efficiency is now defined as follows:

Definition 3 (IK-efficiency) *Allocation a is* IK-inefficient *if $\forall\lambda \in S^N, \exists b \in \mathcal{A}$: $b \succ^\lambda a$. Equivalently, a is IK-inefficient if*

$$\forall\lambda \in S^N, \exists b \in \mathcal{A}, \forall q \in \mathcal{Q}_R : b \geq_q^\lambda a, \tag{6}$$

where the inequality is strict for at least one q.

An allocation that is not IK-inefficient is called IK-efficient. We denote the set of IK-efficient allocations by IKE, and the set of IK-inefficient allocations is then $IKE^c = \mathcal{A} \setminus IKE$. An IK-inefficient allocation is thus one for which whatever are the Pareto weights in the welfare function, there exists another allocation that is not dominated by the first regardless of q in the set of reasonable beliefs, and that dominates the first for some reasonable q.

It is straightforward to verify that an equivalent definition for allocation a to be IK efficient is that

$$\exists\lambda \in S^N, \forall b \in \mathcal{A} : \ a \succeq^\lambda b,$$

i.e., that

$$\exists\lambda \in S^N, \forall b \in \mathcal{A} : \ (\exists q \in \mathcal{Q}_R : a >_q^\lambda b, \text{ or } \forall q \in \mathcal{Q}_R : a \geq_q^\lambda b). \tag{7}$$

As discussed in the introduction, an IK-inefficient allocation is operational for a planner who is not able to take a stand on which q is correct among the set of reasonable beliefs, in that whatever the planner's Pareto weights are there is another allocation b that improves upon a without taking a stand on q. The IK concept thus requires the social planner to have a well-defined λ, but not a well-defined q among the set of reasonable beliefs, $\mathcal{Q}_R$.

We wish to compare IK-efficiency with other efficiency concepts. The concepts in Brunnermeier et al. (2014) of belief neutral efficiency and inefficiency are in our setting defined as follows:

Definition 4 (Belief neutral efficiency)

- *Allocation a is* belief neutrally inefficient, *$a \in BNI$, if $\forall q \in \mathcal{Q}_R, \exists b \in \mathcal{A} : b \succ_q a$, i.e., if*

$$\forall q \in \mathcal{Q}_R, \exists b \in \mathcal{A}, \forall\lambda \in S^N : b >_q^\lambda a. \tag{8}$$

- *Allocation a is* belief neutrally efficient, *$a \in BNE$, if $\forall q \in \mathcal{Q}_R, \forall b \in \mathcal{A} : a \succeq_q b$, i.e., if*

$$\forall q \in \mathcal{Q}_R, \forall b \in \mathcal{A}, \exists\lambda \in S^N : a \geq_q^\lambda b. \tag{9}$$

In words, an allocation, a, is belief neutrally inefficient if for every reasonable belief, q, there is another allocation, b, that is strictly better regardless of the Pareto weights, λ. The set of belief neutrally inefficient allocations is in general a strict subset of the complement of the set of belief neutral efficient allocations, $BNI \subsetneq BNE^c$. To avoid the cumbersome terminology of "not belief neutrally inefficient" allocations, we call such allocations "weakly belief neutrally efficient":

Definition 5 *Allocation a is* weakly belief neutrally efficient*, $a \in WBNE$, if $a \notin BNI$, i.e., if*

$$\exists q \in \mathcal{Q}_R, \forall b \in \mathcal{A}, \exists \lambda \in S^N : a \geq_q^\lambda b. \tag{10}$$

We stress that this terminology is not used in Brunnermeier et al. (2014). It follows immediately that equivalent definitions of $WBNE$ and BNE are:

$$WBNE = \cup_{q \in \mathcal{Q}_R} E_q, \tag{11}$$

$$BNE = \cap_{q \in \mathcal{Q}_R} E_q, \tag{12}$$

and thus that $BNE \subset WBNE$.

Note that the roles of the Pareto weights, λ, and probabilities, q, are in some sense dual in the definitions of IK-efficiency and belief neutral efficiency. Specifically, under the IK efficiency concept, the alternative allocation is allowed to vary with λ but not q, whereas under belief neutral efficiency it is allowed to vary with q but not λ.

We also introduce a concept that is related to unanimity Pareto efficiency, discussed in Gayer et al. (2014):

Definition 6 (U-efficiency) *Allocation a is* U-inefficient *if $\exists b \in \mathcal{A}, \forall q \in \mathcal{Q}_R : b \succ_q a$, i.e., if*

$$\exists b \in \mathcal{A}, \forall q \in \mathcal{Q}_R, \forall \lambda \in S^N : b >_q^\lambda a. \tag{13}$$

An allocation, a, that is not U-inefficient is called U-efficient, i.e.,

$$\forall b \in \mathcal{A}, \exists q \in \mathcal{Q}_R, \exists \lambda \in S^N : \ a \geq_q^\lambda b. \tag{14}$$

We denote the set of U-efficient allocations by UE. U-inefficiency is thus a strong form of inefficiency, since it requires the existence of a unique allocation that dominates a current allocation, regardless of both Pareto weights, $\lambda \in \mathcal{S}^N$, and probabilities, $q \in \mathcal{Q}_R$. This is in contrast to IK-inefficiency and belief neutral inefficiency, which both allow the alternative allocation to vary with one of these parameters. It follows that $WBNE \subset UE$, and $IKE \subset UE$.

As mentioned, U-efficiency has similarities with the unanimity efficiency-concept introduced in Gayer et al. (2014), but there are also differences: First, Gayer et al. (2014) focus on agents involved in a transaction, and require all those agents to be strictly better off. Second, they also require each agent to be better off, given his or her own belief, for a reallocation to be identified as an improvement. In a speculative equilibrium outcome in which all agents participate voluntarily, these additional conditions will typically also be satisfied. The aforementioned general relationships between the different efficiency sets are summarized in Figure 1.

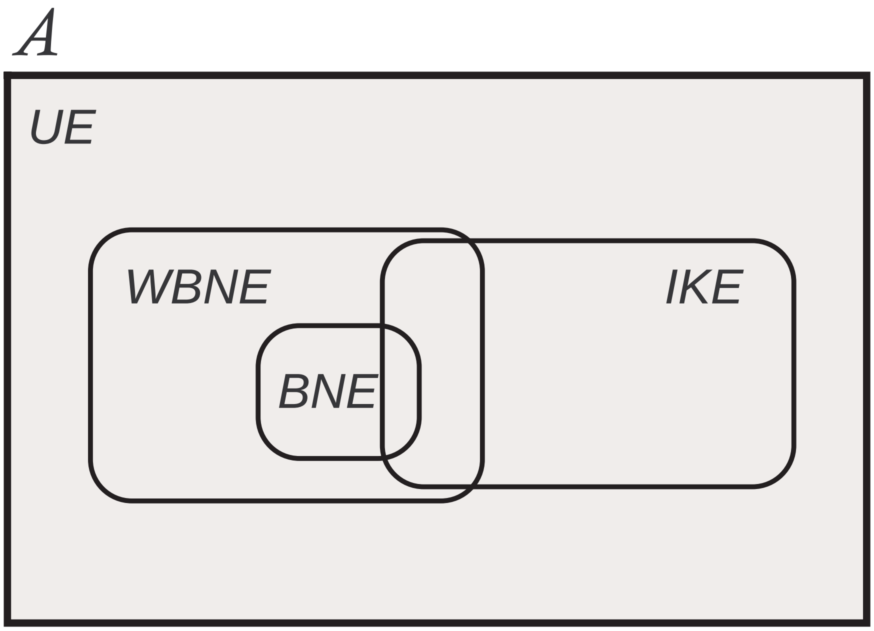

Figure 1: **General relationship between heterogeneous beliefs efficiency concepts.**

Finally, we introduce the concept of an allocation being an *Arrow optimum*. As discussed in Starr (1973), an allocation is an Arrow optimum if it is Pareto efficient with respect to the ex ante expected utilities of agents, based on their own individual beliefs. We therefore also call such an allocation, an *ex ante efficient* allocation. In our setting, the ex ante efficient allocations are identified by replacing the welfare function (2) by

$$U(a|\mathbf{q},\lambda) = \sum_{n=1}^{N} \lambda^n U^n(a|q^n), \tag{15}$$

where $\mathbf{q} = (q^1, q^2, \ldots, q^N) \in \prod_{n=1}^{N} S^M$ represents all agents' beliefs. If $U(b|\mathbf{q},\lambda) > U(a|\mathbf{q},\lambda)$ for two allocations, we then write $b >^{\lambda}_{\mathbf{q}} a$, and if $U(b|\mathbf{q},\lambda) \geq U(a|\mathbf{q},\lambda)$, we write $b \geq^{\lambda}_{\mathbf{q}} a$. The definition of ex ante efficiency (Arrow optimum) is then identical to Definition 1, but with the single probability vector q replaced by the N-tuple of probability vectors, $\mathbf{q}$. The set of Arrow optima, given beliefs $\mathbf{q}$, is denoted by $E^A_{\mathbf{q}}$.

Intuitively, being based on individual agents' ex ante expected utilities, Arrow optima allow for speculative outcomes, in which there may be significant variation of agents' allocations across states, since agents dismiss consumption in states they subjectively believe are highly unlikely. Speculative allocations for which it is objectively known that many agents will starve (since all agents beliefs cannot be correct) may therefore qualify as ex ante efficient, as discussed extensively in Brunnermeier et al. 2014 and Gilboa et al. 2014. For example, two risk-averse agents who have drastically different beliefs about the probability for heads being the outcome of a coin toss may both prefer an allocation where, depending on the outcome of the coin toss, one agents gets all of the good and the other one starves, over one in which they share the good equally in both states. IK-efficiency, belief neutral efficiency, and U-efficiency are all designed to rule out such allocations, by forcing the same probability measure to be used across agents when comparing allocations.

3 Results

We mainly focus on economies that allow for transfers, in which case the first welfare theorem relates Pareto efficient allocations to the social planner's problem of maximizing (2). We define the mapping $\mathcal{P} : \mathbb{R}_+^{M\times N\times T} \to \mathbb{R}_+^{M\times T}$, such that $X_{m,t} = \mathcal{P}(a) = \sum_n a_{m,n,t}$, represents aggregate production in state m at time t for allocation a, and the set $\mathcal{A}_X = \mathcal{P}(\mathcal{A})$.

Definition 7 *An economy is said to allow for transfers if for all $a \in \mathbb{R}_+^{M\times N\times T}$ such that $\mathcal{P}(a) \in \mathcal{A}_X$, $a \in \mathcal{A}$.*

Focusing first on the homogeneous beliefs economy, we have:

Proposition 1 *In the homogeneous beliefs economy with probability vector q:*

(i) In general, $IKE \subset E_q = WBNE = BNE = UE$.

(ii) If the economy allows for transfers, $IKE = E_q = WBNE = BNE = UE$.

Thus, in economies that allow for transfers, the efficiency concepts are all identical in the homogeneous beliefs setting, whereas IKE may be a strict subset of the other efficiency sets without transfers. An example is given in the next section.

We move on to the heterogeneous beliefs economy, in which $q^n \neq q^{n'}$ for at least two agents, and consequently $|\mathcal{Q}_R| \geq 2$. Our first result shows that in general it will not be possible to use U-efficiency to rule out all speculative allocations that are Arrow optimal, whereas the other efficiency concepts typically do rule out such speculative allocations:

Proposition 2 *In the heterogeneous beliefs economy with transfers, such that $q^n \in \mathcal{Q}_R$ for all agents, $1 \leq n \leq N$:*

(i) There is an allocation, $a \in E_{\mathbf{q}}^A$, such that $a \in UE$.

(ii) If all agents' utility functions are strictly concave and aggregate production is the same for any two Arrow optima (i.e., $\mathcal{P}(a) = \mathcal{P}(b)$ for any $a, b \in E_{\mathbf{q}}^A$), then $E_{\mathbf{q}}^A \subset UE$.

(iii) Any Arrow optimal allocation, $a \in E_{\mathbf{q}}^A$, in which two agents who disagree about the relative likelihood of two states to occur are allocated strictly positive amounts of the consumption goods in both those states, is neither IK-efficient, nor weakly belief neutral efficient, $a \notin IKE \cup WBNE$.

Since competitive equilibria in a Walrasian economy, via the welfare theorems, can be identified with the Arrow optima in the economy, the proposition implies that there will be at least one such speculative equilibrium outcome that is identified as U-efficient. Hence, whereas both IK-efficiency and weak belief neutral efficiency rule out all speculative allocations, U-efficiency is not a strong enough criteria to do so.[3] In fact, part (ii) of Proposition 2 shows that under additional assumptions, that are satisfied in several workhorse models in the literature, the set of Arrow equilibrium is a subset of the U-efficient allocations.

We next relate $WBNE$ and IKE. In general, these efficiency concepts differ, but under additional conditions that are satisfied in several work-horse models they will coincide. We introduce the following conditions:

C1. The utility possibility set, $\mathcal{U} \subset R^{M\times N}$, is convex.

C2. The set of reasonable beliefs, $\mathcal{Q}_R$, is convex.

C3. Strict dominance: For all a in $WBNE$

$$\exists \lambda \in \bar{\mathcal{S}}^N, \exists q \in \mathcal{Q}_R, \forall b \neq a : a >_q^\lambda b.$$

The convexity condition for the utility possibility set, C1, is standard (see Mas-Colell et al. (1995)). In an economy that allows for transfers, a sufficient condition for $\mathcal{U}$ to be convex is, e.g., that the aggregate production set, $\mathcal{A}_X$ is convex. Another way of ensuring convexity of $\mathcal{U}$ is by allowing for randomization, see Yaari (1981). Specifically, if the planner uses a randomization device to choose between allocations $a_1, \dots, a_K$ with probabilities $\rho_1, \dots, \rho_K$, the associated utility matrix is $\sum_{k=1}^K \rho_k \mathcal{V}(a_k)$, which consequently belongs to $\mathcal{U}$. This argument for why the utility possibility set is convex is of course more subtle in our setting than in Yaari (1981), because agents need to objectively agree about the probabilities of the randomization device to ensure that new disagreement is not introduced in the randomization process. If, however, such *objective randomization* is possible, convexity of the utility possibility set follows.

The convexity condition for the set of reasonable beliefs, C2, is satisfied under the assumptions made in Brunnermeier et al. (2014), see (5). The strict dominance condition, C3, states that each allocation that is Pareto efficient for some reasonable q is associated with a Pareto weight and reasonable probability vector for which that allocation strictly dominates all other allocations. A sufficient condition for C3 is that the production set is convex and the planner's problem strictly concave, as shown in the following proposition:

Proposition 3 *In an economy that allows for transfers, in which C1 is satisfied and each utility function $u_{m,t}^n$ is strictly concave, C3 is satisfied.*

We now have:

[3] Belief neutral inefficiency of Arrow optima—part (iii) of Proposition 2—actually follows from the analysis in Starr (1973), see Starr's Corollary 3.1 on page 81.

Proposition 4 *In an economy that allows for transfers, in which conditions C1 and C2 are satisfied,* $IKE \subset WBNE$.

Proposition 5 *In an economy that allows for transfers, in which condition C3 is satisfied,* $WBNE \subset IKE$.

Propositions 4 and 5 thus show that in production economies with transfers that satisfy conditions C1-C3, IK-inefficiency and belief neutral inefficiency are equivalent. As already mentioned, and as we shall see, several standard work-horse models fall within this class of economies. It also follows from (11) that both these efficiency sets are nonempty in this case. The previous argument thus links IK-efficiency and weak belief neutral efficiency in the production economy.

As suggested by (11,12), belief neutral efficiency puts substantially stronger restrictions on allocations than weak belief neutral efficiency, and we may therefore expect BNE to be a "small" set in many cases. Indeed, for belief neutral efficiency to hold, allocations must be efficient for *all* reasonable q. In economies for which the optimal aggregate production depends on q, BNE may therefore be empty, as we now explore. Using the formulation (3) for the planner's optimization problem, it follows that q will define a supporting hyperplane to the set $\{V\lambda : V \in \mathcal{U}\}$ for efficient allocations. Now, since smooth boundary points of a convex set in an affine space have unique supporting hyperplanes (see, e.g., Gallier (2011)), it follows that no smooth boundary points in the utility possibility set—and therefore neither in the production set—can be belief neutrally efficient. This argument can be made formal, as long as the utility possibility set is rich enough, as guaranteed by the following condition:

C4. The aggregate production set has nonempty interior, $\mathrm{Int}(\mathcal{A}_X) \neq \emptyset$.

We then have:

Proposition 6 *In a heterogeneous beliefs economy that allows for transfers and satisfies C1 and C4,* BNE *contains no allocations that are associated with smooth boundary points of* $\mathcal{A}_X$.

The following corollary follows immediately:

Corollary 1 *In a heterogeneous beliefs economy that allows for transfers and satisfies C1 and C4, with a smooth aggregate production set, there are no belief neutrally efficient allocations,* $BNE = \emptyset$.

Only boundary points in the aggregate production set that are so "pointy" that they allow for a whole set of supporting beliefs may therefore be belief neutrally efficient. An extreme example is the exchange economy, in which the production set is a singleton, and in which the belief neutral and IK-efficiency concepts coincide:

Definition 8 *An* exchange economy *is an economy that allows for transfers, in which aggregate production is a singleton,* $\mathcal{A}_X = \{X\}$.

Proposition 7 *In the exchange economy,*

(a) $E_q = E_{q'}$ *for all* $q, q' \in S^M$,

(b) $IKE = WBNE = BNE$.

The exchange economy is thus one work-horse model in which a robust definition of efficiency under heterogeneous beliefs is possible.[4]

4 Examples

We provide several examples that highlight the properties of IK-efficiency and relate the measure to other efficiency measures.

4.1 An economy with non-convex utility possibility set

There are two risk averse expected utility maximizing agents, three mutually exclusive production technologies, one date, and two states. The expected utility of agent $n \in \{1, 2\}$ is

$$U^n = q\sqrt{c_1^n} + (1-q)\sqrt{c_2^n}, \tag{16}$$

where c_m^n is the consumption of agent n in state $m \in \{1, 2\}$, and q is the probability for state 1. The first production technology is risk free and generates a total output of 2 units of the consumption good in either state. The second technology is risky, generating 18 units in state 1 and 0 units in state 2, as is the third technology which generates 18 units in state 2 and 0 in state 1. The economy neither permits transfers, nor objective randomization over events.

There are 4 possible allocations, captured by the set $\mathcal{A} = \{a_1, a_2, a_3, a_4\}$. In allocations $a_1 - a_3$, the consumption good is divided equally between the two agents, after choosing each of the three production technologies. In allocation a_4, the risk-free production technology is used, as in allocation a_1, but agent 1 receives 1.9 in state 1 and agent two receives 0.1, whereas agent 2 receives 1.9 in state 2 and agent 1 receives 0.1. The consumption by the two agents for different allocations and states is shown in Table 1. Allocation a_4 thus allows for speculation.

The two agents disagree about the probability for state 1 to occur, q. Agent 1 believes that the probability is $q^1 = 0.9$, whereas agent 2 believes it is $q^2 = 0.1$. The planner, not knowing which beliefs are correct, views any probability in the interval $\mathcal{Q}_R = [0.1, 0.9]$ as reasonable.

It is easy to verify that the only ex ante (Arrow) inefficient allocation is a_1, which both agents agree is dominated by a_4, based on their different beliefs. But both agents also agree that the welfare improvement is purely speculative, and that whatever the true q is, any allocation in which individual consumption shares vary across states can be improved upon by risk sharing. Thus, a_4 is inefficient whenever q is forced to be the same across agents. This is the speculative inefficiency that is captured by the novel measures.

[4] Note that since an exchange economy always fails condition C4, Proposition 7(b) and Corollary 1 do not contradict each other.

Allocation,	a_1		a_2		a_3		a_4	
Agent	1	2	1	2	1	2	1	2
State 1	1	1	9	9	0	0	1.9	0.1
State 2	1	1	0	0	9	9	0.1	1.9
Ex ante utility	1	1	2.7	0.3	0.3	2.7	1.27	1.27

Table 1: **Four allocations in economy with two agents and two states. Allocation a_1 and a_4 are based on investments in the risk-free technology, with equal (a_1) and unequal (a_4) sharing between agents in the two states. Allocation a_2 and a_3 both have equal sharing, but invest in risky technologies that pay off in state 1 and 2, respectively.**

In Figure 2, the right panel compares allocation a_1 with a_4. The horizontal (black) line represents the (same) utility of the two agents under the risk free allocation a_1, whereas the sloped (blue) lines represent the utilities of the two agents under the risky allocation a_4. For low and high q's, one of the agents is better off under a_4 than under a_1, whereas the other is worse off, and for q close to $\frac{1}{2}$ both agents are worse off. Now, the reason one agent is better off for extreme q's is exactly because of speculative redistributions. Regardless of q, allocation a_4 is therefore inferior with any measure that forces the same q to be used for the two agents. What changes with q under allocation a_4 is which one of the two agents who has the speculative advantage. A planner with well-defined Pareto weights can therefore always improve upon a_4.

It follows from the efficiency definitions in the previous section that a_1, a_2, and a_3 are all IK-efficient ($IKE = \{a_1, a_2, a_3\}$), that a_1 and a_4 are belief neutrally inefficient ($WBNE = \{a_2, a_3\}$), that there are no belief neutrally efficient allocations ($BNE = \emptyset$), and that that all four allocations are U-efficient ($UE = \{a_1, a_2, a_3, a_4\}$). Also, it follows that conditions C2 and C3 are satisfied, whereas C1 is not. These results are robust to allowing for transfers between agents, as discussed in the appendix.

The results are consistent with our analysis in the previous section, in that $BNE \subset WBNE \subset IKE \subset UE$, with each inclusion being strict. Specifically, since the set of feasible allocations is not convex, it is possible for an allocation to be IK-efficient but not weakly belief neutrally efficient (allocation a_1 in this example), since the conditions for Proposition 4 are not satisfied.

We argue that it is reasonable to view a_1 as efficient in this example. As shown in the left panel of Figure 2, a_1 is dominated by some other allocation for every $q \in \mathcal{Q}_R$, and is thereby belief neutrally inefficient. However, since the planner with incomplete knowledge about q will not be able to determine which allocation of a_2 and a_3 to choose over a_1, the allocation may still be reasonable for the planner with incomplete knowledge about q.

If objective randomization is possible, a_1 no longer remains IK-efficient, because a randomization of a_2 and a_3 with equal probability leads to expected utility of 1.5 for both agents in both states, regardless of q. In this case, it therefore follows that $WBNE = IKE = \{a_2, a_3\}$, in line with Propositions 4 and 5, since the utility possibility set is convex, i.e., condition C1 is satisfied when

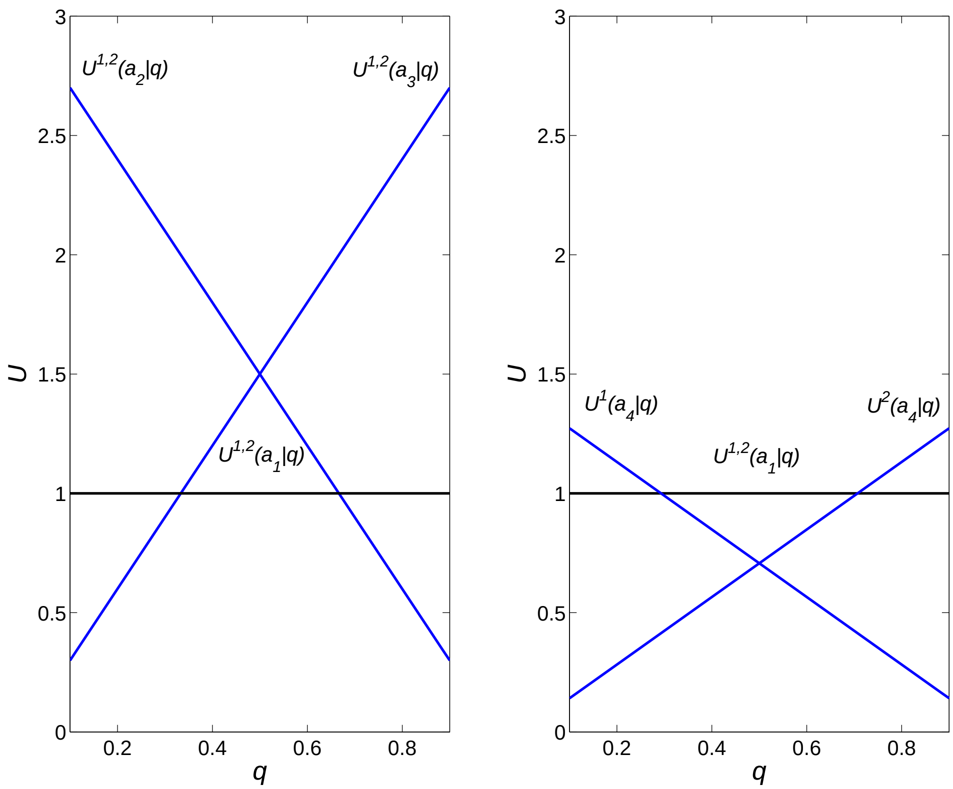

Figure 2: **Expected utilities of agents as a function of q. Left panel: Comparing utilities of allocations a_1, a_2 and a_3. Right panel: Comparing utilities of allocations a_1 and a_4.**

objective randomization is possible.

4.2 An economy with non-convex reasonable belief set

The previous example explored an economy in which the utility possibility set was not convex, and as a consequence belief neutral inefficiency was distinct from IK-inefficiency. The following example provides a case in point that when the reasonable beliefs set is nonconvex, the two concepts may also differ.

Consider a one-date economy with two agents and three production technologies that depend on the outcome of three tosses of a coin. If the outcome of the coin tosses is three tails, production technology a_2 delivers one unit of utility to both agents, otherwise 0. If the outcome is three heads, production technology a_3 delivers one unit of utility, otherwise 0. If the outcome is neither 3 heads, nor 3 tails, production technology a_4 delivers one unit of utility, otherwise 0. Two agents agree that the three tosses are independent and identically distributed, but not on the probability, p, for heads in each toss, believing it is p^1 and p^2, respectively, where we assume that $p^1 < p^2$. The economy allows for objective randomization, and is also robust to allowing for transfers.

The above probability structure may be viewed as a stylized model for a process where the outcome depends on a multiplicative chain reaction. Such processes arise in biology (in epidemiology, for example, the reproductive ratio represents the number of individuals infected by a single individual, which determines whether a virus spreads) and physics (in nuclear physics, for example, the radioactive decay rate of an atom's nucleus determines the success of fission).

Since agents agree on the i.i.d. nature of the coin tosses but disagree on p, one can argue that it is natural to include all probability vectors for the states

$$\{\text{All heads}, \text{All tails}, \text{Both heads and tails}\}$$

on the form $(q_1, q_2, q_3) = (p^3, (1-p)^3, 1-p^3-(1-p)^3)$, for $p \in [p^1, p^2]$. Note that this corresponds to a view that either agent can be correct in his or her belief about the probability for a tail, and that probabilities in-between the agents' individual beliefs are also reasonable. The corresponding set $\mathcal{Q}_R$ is obviously not the convex hull of the two agents beliefs, which is

$$CH\left(\left\{((p^1)^3, (1-p^1)^3, 1-(p^1)^3-(1-p^1)^3), ((p^2)^3, (1-p^2)^3, 1-(p^2)^3-(1-p^2)^3)\right\}\right),$$

and it is not even convex.

The convex hull approach to defining $\mathcal{Q}_R$ corresponds to the planner determining that there is some probability that agent 1 is correct, and that otherwise agent 2 is correct, but neglecting the possibility that some p in-between their beliefs for heads in an individual coin toss may actually be correct. The convex hull approach then leads to a mixture of the two probability vectors.[5]

The utilities associated with the three technologies are shown in Figure 3 below (blue lines), as a function of p. In addition to the three risky technologies there is a risk-free technology a_1 that delivers a utility of 0.45 to all agents. We also show the utility of a randomization with equal probabilities for the three risky production technologies, as represented by the dotted (red) line.

Similarly to the previous example, one can argue that it is reasonable for the planner to view a_1 as efficient. Indeed, it is easy to check that a_1 is IK-efficient, since it is above any other technology (including all randomizations) for some p. However, a_1 is belief neutral inefficient, since it is below some technology for each p. This is consistent with Proposition 5, since condition C2 is not satisfied.

4.3 Economies in which $IKE \subsetneq WBNE$

We provide two examples of economies in which there are allocations that are weakly belief neutral efficient but not IK-efficient. The first example is a homogeneous beliefs economy with one state and date, that does not allow for transfers. Thus part (i) of Proposition 1 holds, but not part (ii). There are three allocations and two agents. The first allocation, a_1, provides utility of 3 to agent 1 and 0 to agent 2, the second allocation, a_2, 0 to agent 1, and 3 to agent 2, whereas the third

[5] The problem described here of choosing an appropriate set, $\mathcal{Q}_R$, is more generally related to that of defining appropriate sets in dynamic multiple priors models, as, e.g., analyzed in Epstein and Schneider (2003). Although not the focus of this paper, we believe that similar restrictions on conditional probabilities as those introduced to ensure dynamic consistency may be a fruitful approach also in the heterogeneous beliefs setting.

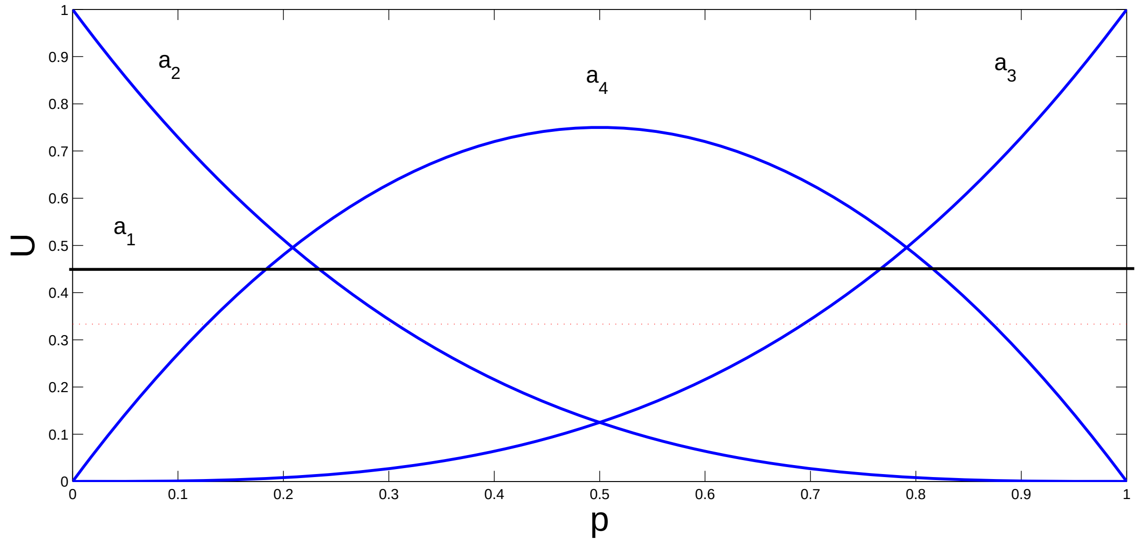

Figure 3: *Relationship between different efficiency concepts.*

allocation, a_3 provides utility of 1 to both agents. There is thus no uncertainty in this example. The utility possibility set is therefore $\mathcal{U} = \{(3,0),(0,3),(1,1)\}$.

The different production technologies could, for example, represent different prospective locations of a new airport. Allocation a_3 could represent a location that is very remote from both agents so travel time is long, whereas a_1 and a_2 represent locations that are too close to one of the agents and therefore give raise to noise pollution, but are at an optimal distance of the other agent.

All three allocations are obviously Pareto efficient and consequently $WBNE = \{a_1, a_2, a_3\}$. However, for Pareto weights $\lambda^1 > \frac{1}{3}$, $a_1 >_q^\lambda a_3$, and for Pareto weights $\lambda^1 < \frac{2}{3}$, $a_2 >_q^\lambda a_3$, so it is possible to dominate a_3 regardless of Pareto weights, and a_3 is therefore IK-inefficient, $IKE = \{a_1, a_2\}$. The issue is a manifestation of the well-known fact that without transfers, Bergson optimality will not identify all Pareto efficient allocations (see Chapter 8.4 in Kreps (2013)). Specifically, without transfers, limited inferences can be drawn about the efficiency of an allocation from the cardinal properties of the expected utility specification, since marginal utilities may not line up across states. If transfers are allowed, the issue disappears, because only the aggregate amount of the consumption good, $(a_i)_{1,1,1} + (a_i)_{1,2,1}$, matters in the ranking of the projects, ruling out a_3, and possibly one of a_1 and a_2 from being efficient. Thus, the IK-efficiency concept suffers from the same weaknesses as the Bergson welfare measure in economies without transfers.

Note that if objective randomization is allowed, the two concepts are equivalent. For instance, a policy that randomizes between a_1 and a_2 with equal probability dominates a_3 both with respect to Pareto efficiency and IK-efficiency. Thus, by letting a coin flip determine the position of the airport, a_3 is ruled out as IK-inefficient.

Another example for which there is an allocation that is weakly belief neutrally efficient, but not IK-efficient is given by considering only allocations a_2 and a_3 in the example in Section 4.1, with reasonable belief set, $\mathcal{Q}_R = \{(q_1, 1-q_1) : q_1 \in [0.2, 0.5]\}$. Both allocations are efficient in the homogeneous beliefs economy with $q_1 = \frac{1}{2}$, and both allocations are therefore in $WBNE$. However, a_3 is IK-inefficient, since it is dominated by a_2 for all q in $\mathcal{Q}_R$ except $q = (0.5, 0.5)$, and does not dominate a_2 for any q in $\mathcal{Q}_R$. One can argue that excluding a_3 from the set of efficient allocations in this example is indeed appropriate, but by defining $WBNE = \cup_{q \in \mathcal{Q}} E_q$ there is no possibility to exclude an allocation that belongs E_q for some $q \in \mathcal{Q}_R$ because of its inferiority for some other $q' \in \mathcal{Q}_R$.

4.4 A work-horse production economy

Consider a simple production economy with two dates, $t = 1, 2$, that allows for transfers. There are M states, and $N > 1$ agents with heterogeneous beliefs. The true state of the world is revealed at $t = 2$, so we require that $a_{m,n,1} = a_{m',n,1}$ for $1 \leq m, m' \leq M$, for all $a \in \mathcal{A}$. Moreover, we assume that agents have strictly concave utility, i.e., that the functions $u^n_{m,t}$ are strictly concave for all n and m, $t \in \{1, 2\}$, and that condition Q2 is satisfied, i.e., that the set of reasonable beliefs is convex.

There is one unit of a divisible and perishable good that can either be consumed at $t = 1$ or invested in a linear production technology. Each unit invested yields a random, strictly positive, amount, $R \in S^M$, at time $t = 2$, at which point it can be consumed. This leads to:

Definition 9 *An allocation is feasible, $a \in \mathcal{A}$, with aggregate investment, $I \in [0, 1]$, if*

(i) $I = 1 - \sum_{n=1}^N a_{1,n,1}$,

(ii) $\sum_{n=1}^N a_{m,n,2} \leq IR_m$, $m = 1, \dots, M$.

It follows that the aggregate production set, $\mathcal{A}_X \subset \mathbb{R}^{2\times 2}$ is convex, and therefore that condition C1 is satisfied. From Propositions 3-5, it then follows that IK-efficiency and weak belief neutral efficiency coincide in this economy,

$$IKE = WBNE.$$

The concept of efficiency under heterogeneous beliefs can thus be extended in a robust manner from the exchange economy to this work-horse production economy.

Next, we study belief neutral efficiency. Consider the special but important case when all agents have separable power utility across states and time: $u^n_{m,t}(c) = \rho^t \frac{c^{1-\gamma}}{1-\gamma}$, for all n, m, and t, with $\gamma > 0$, $\rho > 0$, and with logarithmic utility as the special case when $\gamma = 1$.

Note that $\mathcal{A}_X$ has empty interior, since there is only one production technology, so condition C4 is not satisfied and Corollary 1 can consequently not be used. However, BNE *will* still typically be empty in this economy. Specifically, it is is well known that for any homogeneous q, a representative agent formulation of the social planner's problem exists, in which the representative agent also has power utility, regardless of the Pareto weights, $\lambda \in \mathcal{S}^N$. Optimal investments, I^* therefore typically depend on q, but not on λ, $I^* = I^*(q)$. As long as there is more than one optimal investment level, $I^*(q) \neq I^*(q')$, $q, q' \in \mathcal{Q}_R$, it follows that BNE is empty. Moreover, it also follows that the conditions of Proposition 2 (ii) and (iii) are satisfied, so all Arrow optima are U-efficient, as well as IK-inefficient and belief neutrally inefficient.

To illustrate, let us assume that there are two states with returns $R_1 = 0.9$ and $R_2 = 1.2$. Moreover, assume that there are two agents with beliefs given by $q_1^1 = 0.1$ and $q_1^2 = 0.9$, that the set of reasonable beliefs is the convex hull of the two agents' beliefs, i.e., $Q_R = \{(q_1, 1 - q_1) : 0.1 \leq q_1 \leq 0.9\}$, and that the discount factor is $\rho = 1$.

For an Arrow optimum, $a \in E^A_{(q^1,q^2)}$, with Pareto weights λ^1, λ^2, the first order conditions of optimality then imply that

$$q_m^1 \lambda^1 \left(a_{m,1,2}\right)^{-\gamma} = q_m^2 \lambda^2 \left(a_{m,2,2}^2\right)^{-\gamma}, \; m \in \{1,2\} \qquad \text{and} \qquad \lambda^1 \left(a_{1,1,1}\right)^{-\gamma} = \lambda^2 \left(a_{1,2,1}\right)^{-\gamma},$$

and consequently agent 1 consumes relatively more in state 2 than in state 1, being relatively optimistic about state two. The outcome is therefore speculative. Note that the allocation is U-efficient, since there is no other allocation that is better for both agents for all reasonable beliefs.

Aggregate time 1 consumption is $1-I$, and therefore the economy's consumption (to investment) ratio is $\frac{1-I}{I}$. For a given q, it is easy to derive the (λ-independent) optimal consumption ratio as

$$\frac{1 - I^*(q)}{I^*(q)} = E_0 \left[\tilde{R}^{1-\gamma} \middle| q \right]^{-\frac{1}{\gamma}}. \tag{17}$$

Figure 4 shows the optimal consumption ratio for three different values of risk aversion, $\gamma =$

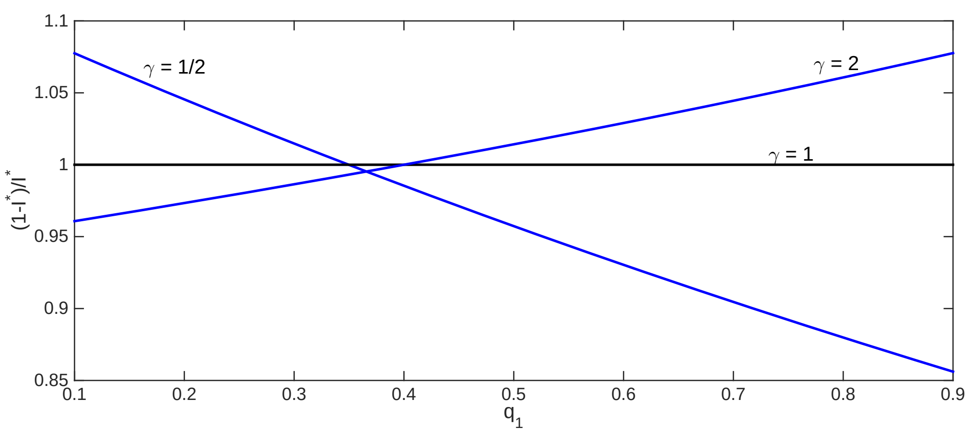

Figure 4: **Optimal consumption ratio, as a function of q_1, for different coefficients of risk aversion.**

$\left\{\frac{1}{2}, 1, 2\right\}$, when varying q in Q_R. When agents have logarithmic utility, corresponding to $\gamma = 1$, the ratio does not vary with q. This corresponds to the well-known property of logarithmic utility that the investment opportunity set does not affect the consumption ratio, see Merton (1969). In this knife edge case, $|I^*| = \frac{1}{2}$, and $BNE = WBNE = IKE$. For risk aversion greater than unity, the optimal consumption ratio increases in q_1 while for risk aversion less than unity the ratio decreases in q_1. Whenever $\gamma \neq 1$, BNE is therefore empty, in line with the discussion above.

Altogether, this work-horse economy thus exemplifies that U-efficiency is too broad a concept to rule out speculative allocations, that belief neutral efficiency is such a strong concept that it may rule out all allocations, and that a robust efficiency concept that includes both weak belief neutral efficiency and IK-efficiency can be introduced in interesting cases of production economies.

5 Concluding Remarks

Speculation in times of significant differences in opinions can have substantial impact on market outcomes, increasing trading volume and price volatility, and may also generate significant wealth inequality. Recently introduced heterogeneous beliefs welfare measures have mainly been applied to the exchange economy environment. In this paper we analyze Incomplete Knowledge (IK) efficiency, a measure designed for production economies with heterogeneous beliefs, in which the social planner does not take a stand on which beliefs are correct.

The general implications of our analysis are three-fold: First, in the exchange economy, several different heterogeneous beliefs welfare measures are equivalent, and the welfare analysis is therefore robust. Second, under additional assumptions of feasibility of transfers, convex belief and production sets, and so-called strict dominance, similar robustness as in the exchange economy also holds in the production economy setting. Third, when these additional assumptions are not satisfied, equivalence as well as robustness breaks down, and consequently, the measure that is most appropriate needs to be determined on a case-by-case basis, as we show in several examples.

Altogether, our study highlights the challenges of extending the heterogeneous beliefs welfare measure to the production economy setting.

A Example 4.1 with transfers

When allowing for transfers in the example, $\mathcal{A} = \{b_1^{\tau_1,\tau_2}, b_2^\tau, b_3^\tau\}$ for $\tau_1, \tau_2, \tau \in [0,1]$, where the allocations are defined as in Table 2. It follows that $a_1 = b_1^{0.5,0.5}$, $a_2 = b_2^{0.5}$, $a_3 = b_3^{0.5}$, and $a_4 = b_1^{0.95,0.05}$.

Allocation,	$b_1^{\tau_1,\tau_2}$		b_2^τ		b_3^τ	
Agent	1	2	1	2	1	2
State 1	$2(1-\tau_1)$	$2\tau_1$	$18(1-\tau)$	18τ	0	0
State 2	$2(1-\tau_2)$	$2\tau_2$	0	0	$18(1-\tau)$	18τ

Table 2: **Economy with three mutually exclusive technologies, two agents and two states with transfers.**

Let us first focus on E_q for a specific $q \in \mathcal{Q}_R$. For $b_1^{\tau_1,\tau_2}$ with $\tau_1 \neq \tau_2$, there is speculation since individual consumption is state dependent even though aggregate output is not. No such speculative allocation can be in E_q for any q, i.e., $b_1^{\tau_1,\tau_2} \notin \cup_{q\in\mathcal{Q}_R} E_q = WBNE$. The same argument as without transfers implies that for any q, and τ, $b_1^{\tau,\tau}$ is dominated by either b_2^τ or b_3^τ, which in turn are undominated. So $WBNE = \{b_2^\tau, b_3^\tau : \tau \in [0,1]\}$. It also follows immediately that $b_2^\tau \notin E_{(q_1,1-q_1)}$ for large q_1, and $b_3^\tau \notin E_{(q_1,1-q_1)}$ for small q_1, so $BNE = \cap_{q\in\mathcal{Q}_R} E_q = \emptyset$.

For IK-efficiency, among the $b_1^{\tau_1,\tau_2}$ allocations, for a given λ, the allocation $b_1^{\tau,\tau}$ where τ is chosen such that $(1-\lambda)u'(2(1-\tau)) = \lambda u'(2\tau)$, i.e., such that $\frac{1-\tau}{\tau} = \left(\frac{1-\lambda}{\lambda}\right)^2$, dominates all other allocations, regardless of q. Now, any $b_1^{\tau,\tau}$ will dominate any of b_2^τ and b_3^τ for specific $q \in \mathcal{Q}_R$, and therefore $IKE = \{b_1^{\tau,\tau}, b_2^\tau, b_3^\tau : \tau \in [0,1]\}$.

For U-efficiency, we know that $IKE \subset UE$, but there are also other $b_1^{\tau_1,\tau_2}$, allocations with $\tau_1 \neq \tau_2$ that are U-efficient. From (14) it follows that for $b_1^{\tau_1,\tau_2}$ to be U-efficient, it is sufficient for any candidate of a dominant allocation, c, to find a q and a Pareto weight such that $b_1^{\tau_1,\tau_2} \geq_q^\lambda c$. It is easy to see that b_2^τ candidates will always be dominated for some $q \in \mathcal{Q}_R$ and $\lambda \in S^N$, as will b_3^τ candidates.

For candidate allocations, $c = b_1^{\tau_1,\tau_2}$, for reasonable belief sets $\mathcal{Q}_R = \{(q, 1-q) : p \leq q \leq 1-p\}$ where $p < 0.5$ (in our example, $p = 0.1$), the set (τ_1, τ_2) such that $b_1^{\tau_1,\tau_2} \in UE$,= includes all allocations such that $\tau_1 = \tau_2$, but excludes some extreme allocations such that one of τ_1 and τ_2 is close to 0 and the other is close to 1, as shown in Figure 5, for $p = 0.1, 0.25, 0.4$. The figure also shows the allocation $a_4 = b_1^{0.95,0.05}$, which when $p = 0.1$ (i.e., when $\mathcal{Q}_R = \{(q, 1-q) : 0.1 \leq q \leq 0.9\}$) consequently is U-efficient.

Thus, the efficiency properties of a_1, a_2, a_3, and a_4 with respect to IKE, $WBNE$, BNE, and UE are identical as in the original example without transfers.

B Proofs

Proof of Proposition 1:

(i) If a is IK-efficient, from (7) it follows that with homogeneous beliefs $\exists\lambda \in S^N, \forall b \in \mathcal{A}, a \geq_q^\lambda b$, which of course is stronger than (9), since λ is allowed to depend on b in (9), so $IKE \subset WBNE$. Also, $E_q = BNE = WBNE$ trivially follows from (11,12). Finally, (8) is obviously equivalent to (13) when $\mathcal{Q}_R = \{q\}$, so $BNE = UE$.

(ii) From (i), it is sufficient to show that $E_q \subset IKE$, i.e., that $a \notin IKE \Rightarrow a \notin E_q$. From (6), it follows that if a is IK-inefficient, $\forall\lambda \in S^N, \exists b \in \mathcal{A} : b >_q^\lambda a$. However, such an a can clearly not be the solution to the planner's problem for any $\lambda \in S^N$ (recall that $\lambda \in \partial S^N$ can be excluded in economies that allow for transfers), and therefore cannot be in E_q. ∎

Proof of Proposition 2:

(i) First note that UE is non-empty. This follows from the fact that $WBNE \subset UE$, with $WBNE = \cup_{q\in\mathcal{Q}_R} E_q$, and that E_q is non-empty. Assume that $a \in E_{\mathbf{q}}^A$. Then either $a \in UE$ and the result follows,

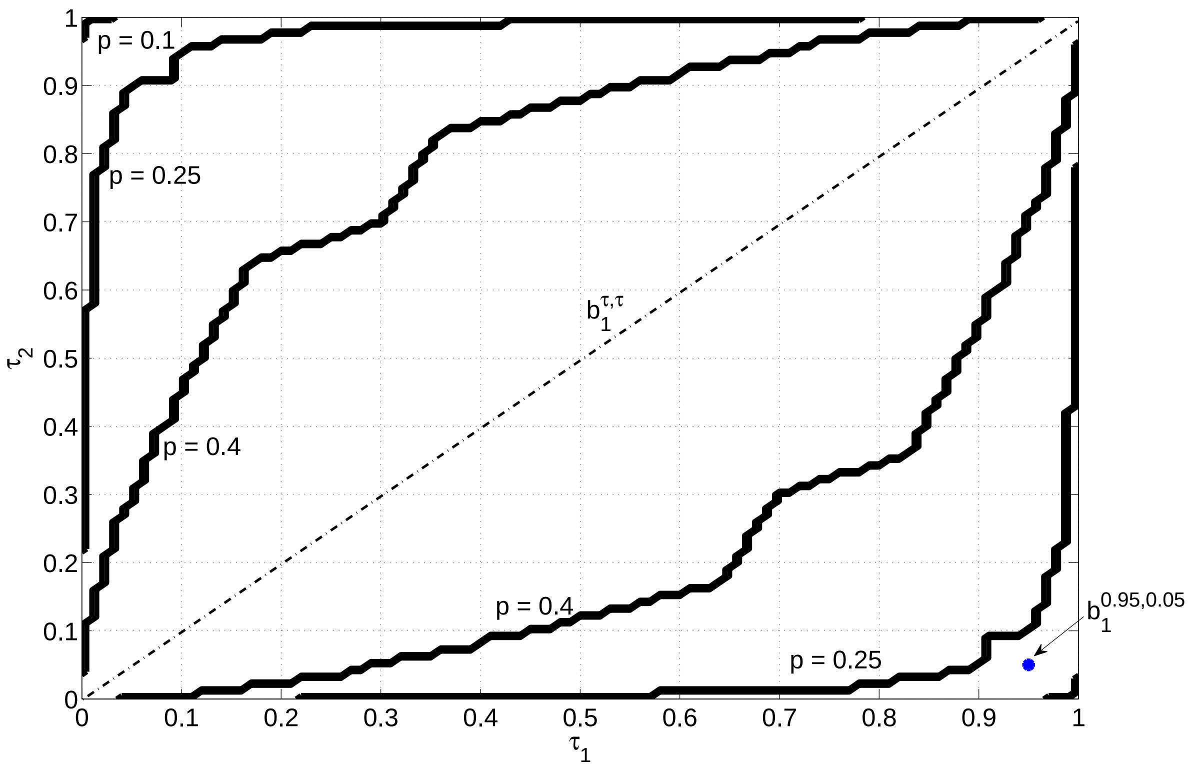

Figure 5: **Region of τ_1, τ_2, such that allocation $b_1^{\tau_1,\tau_2}$ is U-efficient, for different choices if $\mathcal{Q}_R = \{(q, 1-q) : p \leq q \leq 1-p\}$, $p = 0.1, 0.25, 0.4$.**

or $a \notin UE$. If a is not in UE, then since UE is non-empty there exists a $b \in UE$ such that b dominates a. It follows that $b >_q^\lambda a$ for all $q \in \mathcal{Q}_R$ and $\lambda \in S^N$, which implies that $U^n(b|q) \geq U^n(a|q)$ for all n and $q \in \mathcal{Q}_R$, with the inequality being strict for at least one n (for each q). Consequently $U(b|\mathbf{q}, \lambda) \geq U(a|\mathbf{q}, \lambda)$, so $b \in E_{\mathbf{q}}^A$.

(ii) Following similar arguments as above, assume that $a \in E_{\mathbf{q}}^A$, with associated λ in (15), but that $a \notin UE$. Then there is an allocation $b \in \mathcal{A}$ such that $b >_q^\lambda a$ for all $q \in \mathcal{Q}_R$ and $\lambda \in S^N$, which implies that $U^n(b|q) \geq U^n(a|q)$ for all n and $q \in \mathcal{Q}_R$, with the inequality being strict for at least one n (for each q). If there is an agent n for which $U^n(b|q^n) > U^n(a|q^n)$, then it immediately follows that $a \notin E_{\mathbf{q}}^A$ which leads to a contradiction. If there is no agent for which $U^n(b|q^n) > U^n(a|q^n)$, i.e., $U^n(b|q^n) = U^n(a|q^n)$ for all n, then we can consider the allocation $b' = \frac{1}{2}b + \frac{1}{2}a$. It follows that $\mathcal{P}(b') = \mathcal{P}(b) = \mathcal{P}(a)$, and as transfers are allowed $b' \in \mathcal{A}$. By strict concavity of the utility function, $U^n(b'|q^n) > \frac{1}{2}U^n(b|q^n) + \frac{1}{2}U^n(a|q^n) = U^n(a|q^n)$ which implies that $a \notin E_{\mathbf{q}}^A$, again leading to a contradiction.

(iii) Without loss of generality, we assume that agents 1 and 2 disagree about the relatively likelihood of states 1 and 2 such that $\frac{q_1^1}{q_2^1} > \frac{q_1^2}{q_2^2}$. Since redistribution is possible, the social planner's first order conditions imply that

$$\frac{q_1^1 u_{1,t}^{1'}(a_{1,1,t})}{q_2^1 u_{2,t}^{1'}(a_{2,1,t})} = \frac{q_1^2 u_{1,t}^{2'}(a_{1,2,t})}{q_2^2 u_{2,t}^{2'}(a_{2,2,t})},$$

(see (19) in the proof of Proposition 4). However, for any $q \in \mathcal{Q}_R$, the planner's first order conditions are

$$\frac{u_{1,t}^{1'}(a_{1,1,t})}{u_{2,t}^{1'}(a_{2,1,t})} = \frac{u_{1,t}^{2'}(a_{1,2,t})}{u_{2,t}^{2'}(a_{2,2,t})},$$

so $a \notin E_q$ for any such q, and thus $a \notin WBNE$. Now, an identical argument as will be used in Proposition 7 (ii), shows that if such improving redistributions are possible without changing the aggregate production, i.e., keeping X the same, then $a \notin IKE$.

∎

Proof of Proposition 3:

Since the economy allows for transfers, and since $u_{m,t}^n$ are differentiable on $[0,\infty)$ and thus $(u_{t,m}^n)'(0)$ is finite for all n, m, and t, for small enough strictly positive λ^n it will not be optimal to allocate the good to agent n in any state. It follows that $E_q = \cup_{\lambda \in S^N} E_{q,\lambda}$ in this case, where $E_{q,\lambda} = \arg\max_{a\in\mathcal{A}} U(a|q,\lambda)$, i.e., that Pareto weights on ∂S^N are not needed in determining efficiency. Thus, from (11) it follows that $WBNE = \cup_{q\in\mathcal{Q}_R} \cup_{\lambda\in S^N} E_{q,\lambda}$. Strict dominance is therefore equivalent to $|E_{q,\lambda}| = 1$ for all $q \in \mathcal{Q}_R$, and $\lambda \in S^N$.

Now, for $a \in WBNE$, and associated $q \in \mathcal{Q}_R$, $\lambda \in S^N$, assume that $E_{q,\lambda}$ contains another allocation $b \neq a$, and therefore that $U(a|q,\lambda) = U(b|q,\lambda)$. Since the production set is convex, $c = \frac{1}{2}a + \frac{1}{2}b \in \mathcal{A}$. Moreover, from (3) and the strict concavity of agents' utilities, it follows that $U(\cdot|q,\lambda)$ is strictly concave over allocations, and thus

$$U(c|q,\lambda) > \frac{1}{2}U(a|q,\lambda) + \frac{1}{2}U(b|q,\lambda) = U(a|q,\lambda),$$

contradicting the assumption that $a \in E_{q,\lambda}$. So, no such $b \neq a \in E_{q,\lambda}$ exists, $|E_{q,\lambda}| = 1$, and the result thus follows.

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Proof of Proposition 4:

The planner's optimization problem, given q and λ is to maximize

$$\max_{X\in\mathcal{A}_X} \max_{a|X} \sum_n \lambda^n \sum_t \sum_m u_{m,t}^n(a_{m,n,t})q_m = \max_{V\in\mathcal{U}} q^T V \lambda. \tag{18}$$

Note that from the first welfare theorem it follows that if we define $E_{q,\lambda} = \{a_{q,\lambda}\}$ as the solutions to the planner's problem given $q \in \mathcal{Q}_R$ and $\lambda \in \mathcal{S}^N$, and $E_\lambda = \cup_{q\in\mathcal{Q}_R} E_{q,\lambda}$, then $E_Q = \cup_{\lambda\in\mathcal{S}^N} E_\lambda$, since transfers are allowed.

The F.O.C., given X is that

$$\lambda^n (u_{m,t}^n)'(a_{m,n,t}) = \rho^{m,t} \tag{19}$$

across all m, and t, for all agents for which $a_{m,n,t} > 0$, and

$$\lambda^n (u_{m,t}^n)'(0) \leq \rho^{mt} \tag{20}$$

for all agents such that $a_{m,n,t} = 0$, where $\rho^{m,t} > 0$ are the Lagrange multipliers that make total consumption equal to total production in each state.

Let us assume that $a \notin E_Q$. Then, it could either be (i) that (19,20) are not satisfied across states for any $\lambda \in S^N$, or (ii) that an optimal X is not chosen, or both (i) and (ii).

If $a \notin E_Q$ because of (i), then regardless of $\lambda \in S^N$, there is some state and time for which (3,4) are not satisfied for two agents, at least one of which has a strictly positive consumption, which we w.l.o.g. assume are agents 1 and 2. But this then means that a redistribution of consumption between the two agents will improve welfare, regardless of $q \in \mathcal{Q}_R$. Indeed, it is possible to strictly increase

$$\lambda^1 u_{m,t}^1(a_{m,1,t})q_m + \lambda^2 u_{m,t}^2(a_{m,2,t})q_m \tag{21}$$

regardless of $q_m > 0$, since the FOC is not satisfied in the state t, m. But this then implies that for all $q \in \mathcal{Q}_R$, (6) in the paper is satisfied with strict inequality, so the allocation is IK-inefficient.

If (i) is satisfied for some nonempty set $\Lambda \subset S^N$, but $a \notin E_Q$ because of (ii), then for any $\lambda' \notin \Lambda$, a redistribution such that (19,20) holds, with the same X, leads to a strict improvement regardless of $q \in \mathcal{Q}_R$, along the same lines as the previous point when (i) failed. Therefore, (6) holds for any such $\lambda' \notin \Lambda$. It remains to be shown that (6) also holds for $\lambda \in \Lambda$ for which (19,20) are satisfied, given that X is not optimal for any $q \in \mathcal{Q}_R$.

For such a λ, define the mapping $\mathcal{F}_\lambda : \mathcal{U} \to \mathbb{R}^M$, by $\mathcal{F}_\lambda(V) = V\lambda$, and the set $F_\lambda = \mathcal{F}_\lambda(\mathcal{U})$, which is a closed, convex, and bounded subset of $\mathbb{R}^M$, because of Q2. Now, defining $f_a = \mathcal{F}_\lambda(\mathcal{R}(a))$, since $a \notin E_Q$ it follows from (8) that

$$\max_{q \in \mathcal{Q}_R} \min_{f \in F_\lambda} q^T(f_a - f) = s < 0,$$

where $s < 0$ follows from the fact that the optimum is realized for some f^*, q^* (since both $\mathcal{Q}_R$ and F_λ are compact).

Sion's minmax theorem then implies that

$$\min_{f \in F_\lambda} \max_{q \in \mathcal{Q}_R} q^T(f_a - f) = s,$$

where the same, f^*, q^* can be chosen for the maxmin and minmax problems, and thus that for all $q \in \mathcal{Q}_R$,

$$q^T(f_a - f^*) \leq s < 0.$$

This, in turn, implies that (6) holds for any such $\lambda \in \Lambda$.

The allocation a is therefore IK-inefficient, so indeed $a \notin E_Q \Rightarrow a \notin IKE$.

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Proof of Proposition 5: Consider $a \in E_Q$, and associated $q \in \mathcal{Q}_R$, $\lambda \in S^N$ (∂S^N excluded since transfers are allowed). Then, $\forall b \neq a$, $a >_q^\lambda b$, immediately implying (7) for λ and q, not even depending on b.

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Proof of Proposition 6:

Assume that $a \in BNE$, then $a \in E_q$ and $a \in E_{q'}$ for $q \in \mathcal{Q}_R$, $q' \in \mathcal{Q}_R$, $q \neq q'$, and with associated Pareto weights $\lambda \in S^N$, $\lambda' \in S^N$ (see proof of Proposition 4), and with associated $X \in \mathcal{A}_X$. Given that (19,20) do not depend on q, the allocation to individual agents can be written $W(X, \lambda)$, and therefore, $W(X, \lambda) = W(X, \lambda')$ (otherwise a would not be in both $E_{q,\lambda}$ and $E_{q',\lambda'}$). Moreover, (19,20) imply that λ and λ' are basically unique, down to possible differences on weights for agents that are not allocated the good in any state. Indeed, there is a $\hat{\lambda} \in S^N$ that puts extremely low weight on any such agents, such that $\lambda_i/\lambda_j = \lambda'_i/\lambda'_j = \hat{\lambda}_i/\hat{\lambda}_j$ for all agents i, j who receive strictly positive allocation in any state, and such that a is in both $E_{q,\hat{\lambda}}$ and $E_{q',\hat{\lambda}}$.

But this then means that $f_a = \mathcal{F}_{\hat{\lambda}}(\mathcal{R}(a))$, as defined in the proof of Proposition 4, is a point in the convex set $F_{\hat{\lambda}}$ that has both the hyperplanes defined by q and q' as supporting hyperplanes. It follows (see, e.g., Gallier (2011)) that f_a must be a non-smooth point on $\partial F_{\hat{\lambda}}$ (as defined on the whole of $\mathbb{R}^M$). Indeed, $F_{\hat{\lambda}}$ does not lie in any proper affine subspace of $\mathbb{R}^M$ (because $\mathcal{A}_X$ has nonempty interior), and must therefore be a nonsmooth point to have multiple supporting hyperplanes (see Definition 3.3.3 in Gallier (2011), page 108). Finally, since $\mathcal{F}_{\hat{\lambda}} : \mathbb{R}^{M \times N} \to \mathbb{R}^M$, $\mathcal{V} : \mathbb{R}^{T \times M \times N} \to \mathbb{R}^{M \times N}$ and $\mathcal{P} : \mathbb{R}^{T \times M \times N} \to \mathbb{R}^{M \times N}$ are all smooth mappings, it follows that $\mathcal{P}(a)$ is also nonsmooth in $\mathcal{A}_X$.

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Proof of Proposition 7:

(i) The planner's problem in the exchange economy is simplified since only one X is feasible. The F.O.C., w.r.t. λ, (19,20), are q independent, and the result therefore follows immediately.

(ii) Since $\mathcal{Q}_R = \{q\}$, it follows immediately that $E_Q = \cap_q E_q = \cup_q E_q$, so $BNE = E_Q = WBNE$. Now, Q3 is satisfied in the exchange economy, because efficient allocations are characterized by (19,20), and X is

fixed, which implies that whether an allocation is considered efficient does not dependent of q. Thus for all $a \in E_Q$,

$$\exists \lambda \in S^N, \forall b \in \mathcal{A}, \forall q \in S^N : a >_q^\lambda b. \tag{22}$$

It is easy to show that if transfers are allowed, then the definition of Q3 can equivalently be stated using $\lambda \in S^N$, implying that (22) is stronger than Q3 (the argument is identical as in the proof of Proposition 3: for any $\lambda \in \partial S^N$, there is a close enough $\lambda \in S^N$ that leads to an identical allocation, that can be chosen instead). It therefore follows that $WBNE \subset IKE$ from Proposition 5.

For $a \notin WBNE$, we follow the same argument as in the proof of Proposition 4. Given that $\mathcal{A}_X = \{X\}$ the reason why $a \notin WBNE$ must be that equality of Pareto weighted marginal utilities across agents and states fails. An identical argument as in the proof of Proposition 4 (which in that part does not depend on convexity of either $\mathcal{Q}_R$ or $\mathcal{U}$) implies that a is IK-inefficient, so $BNI \subset IKI$, i.e., $IKE \subset WBNE$.

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