

# Recovery with Unbounded Diffusion Processes\*

Johan Walden<sup>†</sup>

## Abstract

We analyze the problem of recovering the pricing kernel and objective probability distribution from observed option prices, when the state variable is an unbounded diffusion process. We derive necessary and sufficient conditions for recovery. In the general case, these conditions depend on the properties of the diffusion process, but not on the pricing kernel. We also show that when recovery is possible in the unbounded case, approximate recovery is possible from observing option prices on a bounded subdomain, without further specification of boundary conditions. Altogether, our results suggest that recovery is possible for many interesting diffusion processes on unbounded domains.

---

\*I have benefited from very useful discussions with Peter Carr, Hayne Leland, and Steve Ross. Comments and suggestions from Nicolae Gârleanu and Bob Goldstein are also greatly appreciated.

<sup>†</sup>Haas School of Business, University of California at Berkeley, 545 Student Services Building #1900, CA 94720-1900. *E-mail:* [walden@haas.berkeley.edu](mailto:walden@haas.berkeley.edu), *Phone:* +1-510-643-0547. *Fax:* +1-510-643-1420.

In a remarkable paper, Ross (2013a) shows that it is possible to recover the pricing kernel and objective probabilities from prices of contingent claims alone, contrary to what has long been the common belief. The result relies on two insights: first that under so-called translation invariance, observed prices link the pricing kernel across states; second that the fact that the pricing kernel must be positive provides important additional restrictions. The two effects together allow for unique recovery in Ross’s model.

Such information about preferences and risk in the market obtained by recovery would of course be highly valuable to investors, policy makers, and society in general, and it is therefore of fundamental importance to understand under which conditions recovery works. The state space in Ross (2013a) is finite in contrast to many work-horse models in finance, e.g., models in continuous time with diffusion processes. It is an open question if, and if so, when, recovery works in a setting with a larger, unbounded state space. This question is important, since it is a priori unclear which approach, bounded or unbounded, best models financial markets and, from a robustness perspective, results that hold in one setting but not in the other may be viewed with some concern.

Of course, a rationale for not worrying too much about whether the state space is bounded or not is that even if it is unbounded, it may be possible to simply “truncate” the state space far enough out—for very rare events—without affecting the results in the interior more than marginally. Such a rationale is often too simplistic, however. Dubynskiy and Goldstein (2013) provide an example in which the assumptions made at the boundaries have a first-order effect on the solution in the interior, even for states that are very far away from these boundaries. Such dependence on boundary conditions is well-known in the study of the dynamic problems that arise in finance, e.g., parabolic partial differential equations (PDEs) in case of diffusion processes; see John (1991).

In the example in Dubynskiy and Goldstein (2013), the boundary conditions provide important information about the preferences of the representative investor—exactly the information

that the method was designed to recover.<sup>1</sup> Similarly, Carr and Yu (2012) show that for bounded diffusion processes, under appropriate exogenously specified boundary conditions, recovery is possible. Again boundary conditions are needed in their setting.<sup>2</sup> Even if the true state space *is* bounded, truncation may be present, because of a limited number of observable asset prices. The bounds may even be unknown. For example, one may argue that in our world with finite resources, there must be an upper bound on the value of the stock market, GDP, etc. However, it seems virtually impossible to determine whether the correct bound to use for the Dow Jones Industrial Average is at forty-eight thousand, a Million, a Billion, or even higher.

The potential importance of rare events for the recovery problem is related to several fragility results for equilibrium asset pricing models in finance that have been put forward in recent years. We mention a few examples. Barro (2005), building on Rietz (1989), shows that the risk for catastrophic events far out in the tail of the return distribution may have large asset pricing effects, potentially explaining the equity premium puzzle. Parlour et al. (2011) show that adding a very small risk-free consumption stream to an otherwise standard Lucas economy can have drastic effects on stock prices and discount rates, because of the insurance such an asset provides in rare bad states. In a survey article, Ibragimov (2009) discusses how the implications of various models in finance and risk management depend fundamentally on the tail behavior of risk distributions. Kogan et al. (2006) show how a small number of irrational investors in the market can have a disproportionate impact on asset prices by entering into extreme bets on rare events. All these models are therefore fragile with respect to combined assumptions about rare events and agent preferences. For the recovery problem there is a similar potential fragility, namely whether the assumption about boundedness of the state space fundamentally impacts the feasibility of the method.

---

<sup>1</sup>In an alternative formulation, Dubynskiy and Goldstein (2013) solve a problem with an unbounded state space, but at the cost of making additional parametric restrictions on the functional forms, restrictions that the recovery method was designed to avoid.

<sup>2</sup>Carr and Yu (2012) mention the extension of the recovery methodology to unbounded domains as an interesting extension.

We analyze the recovery problem in a representative agent economy where the state evolves in continuous time according to a time homogeneous diffusion process on an unbounded domain. Our main contribution is to derive necessary and sufficient conditions for unique recovery in this setting. We derive properties of the diffusion process that alone determine whether recovery is possible; the form of the pricing kernel, i.e., the marginal utility of the representative agent, is not important. In general, for recovery to be possible, the process cannot be allowed to drift off toward infinity too quickly. A sufficient, but not necessary, condition is that the diffusion process is mean reverting. This complete independence between the possibility of recovery and the functional form of the pricing kernel is a priori quite surprising, and adds to the strength of the recovery method. We also study the recovery problem when additional restrictions on the pricing kernel are imposed. Specifically, when we require marginal utility to be bounded from above and below, the drift of the diffusion process only needs to be restricted in one direction. Altogether, our results show that recovery is possible for a wide class of interesting diffusion processes, but that there are also interesting cases for which it fails.

Our second contribution is to show that even if option prices are only known on a bounded domain, as long as this domain is large enough, approximate recovery is possible on the bounded domain when recovery is possible on the unbounded domain. Specifically, an approximate pricing kernel can be constructed from truncated observations of option prices on a bounded interval, and as the length of this interval grows, the approximation converges pointwise to the true kernel. Importantly, no boundary conditions are needed for this approximation method. Thus, in the case when recovery works on the infinite domain, it is not vital to know boundary conditions for the truncated problem. The result is promising for the use of recovery methods in practice. We show with several examples that the numerical method works well, and also provide Matlab code for the approximation method in an appendix.

As a third contribution, our reformulation of the problem in a setting with diffusion processes allows for additional insight and intuition about how recovery works in a fairly standard

framework. Throughout the paper, we provide examples that underline the theoretical results, and discuss the results extensively to provide further intuition and insight about how they arise.

A restriction on the scope of this paper is that our analysis is mainly theoretical; Although our results are promising for practical applications, these are not our focus. In future work, we plan to address the numerical properties and practical applications of the recovery methods in the diffusion setting, in more detail. Another restriction is that we do not attempt to cover the most general possible specification of agent preferences. Instead, for simplicity, we focus on the case of a representative agent with expected utility. We refer to Ross (2013a) for a discussion about generalizations.

The rest of the paper is organized as follows. In the next section, we give a brief summary of recovery with a finite number of states, as introduced in Ross (2013a). In Section 2, we analyze the recovery problem for a diffusion process on an unbounded domain. In Section 3, we discuss how to back out the prices of Arrow Debreu securities in every state (which is needed for recovery) from observed options prices at a specific state. In Section 4, we show that when recovery is possible in the unbounded state space, approximate recovery is possible when the state space is truncated. Section 5 discusses extensions, and finally Section 6 concludes. Proofs, and the Matlab code for approximate recovery on a bounded domain, are delegated to an Appendix.

## 1 Recovery with finite state space

We summarize the approach in Ross (2013a), which is based on a model in discrete time with a finite number of states. We use the same terminology as Ross, except for the vector of marginal utilities, for which we use  $m$  for instead of  $d$ , because  $d$  is easy to mistake for the derivative operator in the continuous model.

There are  $N$  states, and a stochastic, irreducible, aperiodic, matrix,  $F$ , such that  $F_{ij}$  denotes

the probability of moving from state  $i$  to  $j$ . Since  $F$  is stochastic,

$$F\mathbf{1} = \mathbf{1}, \tag{1}$$

where  $\mathbf{1}$  is an  $N$ -vector of ones. There is a representative agent, with discount factor  $\delta < 1$ , and marginal utility  $m_i > 0$  in state  $i$ . We define the vector  $\mathbf{m} = (m_1, \dots, m_N)^T$ , and its reciprocal  $\mathbf{z} = (1/m_1, \dots, 1/m_N)^T$ . Let  $P_{ij}$  denote the time-0 price in state  $i$  of an Arrow-Debreu (AD) security that pays a dollar at time 1 if the state is  $j$ . In a Walrasian complete market equilibrium, the price can then be expressed as

$$P_{ij} = \delta \frac{m_j}{m_i} F_{ij},$$

or in matrix form

$$P = \delta M^{-1} F M, \tag{2}$$

where  $M$  is the diagonal matrix,  $M = \text{diag}(\mathbf{m})$ . From (2), it follows that

$$F = \delta^{-1} M P M^{-1}, \tag{3}$$

which when plugged into (1) yields

$$\delta^{-1} P M^{-1} \mathbf{1} = M^{-1} \mathbf{1}, \text{ i.e.,} \tag{4}$$

$$P \mathbf{z} = \delta \mathbf{z}. \tag{5}$$

From the Perron-Frobenius theorem, there is a unique strictly positive pair  $\delta$  and  $\mathbf{z}$  that solves the eigenvector problem (5),<sup>3</sup> and then via (3),  $F$  can be recovered.

We next move to the case with unbounded diffusion processes.

---

<sup>3</sup>Uniqueness ensured, because  $F$  is irreducible and aperiodic.

## 2 Recovery for unbounded diffusion process

The state evolves according to a time homogeneous diffusion process:

$$dX_t = \mu(X_t)dt + \sigma(X_t)d\omega, \quad t \geq 0. \quad (6)$$

It will be convenient to define the function

$$D(x) = \frac{\sigma^2(x)}{2}.$$

We make the technical assumptions  $\mu$  and  $\sigma$  are continuously differentiable, and that there are constants,  $C_1$ ,  $C_2$ ,  $C_3$ , and  $\alpha$ , such that  $|\mu(x)| \leq C_1(1 + |x|)$ ,  $0 < C_2 \leq \sigma(x) \leq C_3(1 + |x|)$ , to ensure that the process is well defined and with positive probability covers the whole of the real line,  $\mathbb{R}$ .

Associated with the diffusion process is a filtered probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ ,  $t \geq 0$ , satisfying the usual assumptions. We define the transition density function  $f^t(x, y) = \frac{\partial F^t}{\partial y}$ , where  $F^t(x, y) = \mathbb{P}(X_t \leq y | X_0 = x)$ , and it then follows that  $f^t$  satisfies the Fokker-Planck equation

$$\begin{aligned} \frac{\partial f^t}{\partial t} &= \mathcal{L}^* f^t, \\ f^0(x, y) &= \delta_x(y), \end{aligned} \quad (7)$$

where the operator  $\mathcal{L}^*$  is defined as

$$\mathcal{L}^* f = -\frac{d}{dy} (\mu(y)f(y)) + \frac{d^2}{dy^2} \left( \frac{\sigma^2(y)}{2} f(y) \right). \quad (8)$$

Here,  $\delta_x(y)$  is the Dirac distribution centered at  $x$ , defined by the conditions  $\delta_x(y) = 0$ ,  $x \neq y$ , and  $\int \delta_x(y)dy = 1$ . The fact that  $\sigma \geq C_2 > 0$  implies that  $f^t(x, y) > 0$  for all  $t > 0$ ,  $x \in \mathbb{R}$ , and

$y \in \mathbb{R}$ , i.e., that for any  $x_0$ ,  $y$ , and  $t$ , the probability density of  $X_t$  at  $y$  is strictly positive. The function  $f$ , as a function of  $x$  and  $y$  corresponds to the matrix  $F$  in the discrete case.

The instantaneous flow of a single consumption flow at time  $t$  is  $g(X_t)dt$ , where  $g$  is a strictly increasing, twice continuously differentiable, function. A price taking representative agent seeks to maximize expected utility of a consumption flow,  $c_t$ :

$$U = E \left[ \int_0^T e^{-\rho t} u(c_t) dt \right], \quad (9)$$

for some  $0 < T \leq \infty$ . Here, the constant  $\rho > 0$  is the agent's personal discount rate, and  $u$  is a strictly increasing, three times continuously differentiable function, such that  $|U| < \infty$ , when  $c_t = g(X_t)$ .

A complete financial market of Arrow-Debreu (AD) securities exists (e.g., implemented through dynamic trading of a finite number of assets). The time 0 price of an AD security that pays off  $\delta_y(X_t)$  at time  $t$ , given that  $X_0 = x$ , is defined as  $p^t(x, y)$ . Absence of arbitrage then implies that the time 0 price of a simple contingent claim with time  $t$  payoff  $\Phi(X_t)$  is  $P = \int_{-\infty}^{\infty} p^t(x, y) \Phi(y) dy$ .

A standard argument implies that the Walrasian equilibrium prices of the AD securities are

$$p^t(x, y) = e^{-\rho t} \frac{m(y)}{m(x)} f^t(x, y), \quad (10)$$

where  $m(x) = u'(g(x))$  is strictly positive and twice continuously differentiable. This corresponds to (2) in the finite case. It will be convenient to define the functions  $q(x) = \frac{m'(x)}{m(x)}$  and  $z(x) = \frac{1}{m(x)}$ . Clearly,  $q$  is closely related to the representative agent's relative risk-aversion coefficient at  $x$ ,  $\gamma(x)$ , since  $-q = -\frac{m'(x)}{m(x)} = -g'(x) \frac{u''(g(x))}{u'(g(x))} = \frac{g'(x)}{g(x)} \gamma(x)$ . The extra factor  $\frac{g'(x)}{g(x)}$  arises because  $g(x)$ , rather than  $x$  represents, units of the consumption good (which in turn allows us to cover both arithmetic and geometric consumption processes within a unified framework). We note that  $m(x)$  is only unique up to multiplication with an arbitrary positive constant, given



the equivalence of two utility functions that are positive affine transformations of each other. However,  $q$  is unique, since any constant will occur both in the dominator and numerator of  $q$  and therefore cancel out.

The function

$$\Lambda_t = e^{-\rho t} \frac{m(X_t)}{m(X_0)} \quad (11)$$

is the pricing kernel in the economy, leading to the standard pricing formula

$$P = E \left[ \frac{\Lambda_t}{\Lambda_0} \Phi(X_t) \right], \quad (12)$$

for the time 0 price of a simple contingent claim with time  $t$  payoff  $\Phi(X_t)$ . In the terminology of Ross (2013a), the specific kernel is transition independent, being the product of a constant discount rate depreciation factor, and the fraction of a function evaluated at  $X_t$  and  $X_0$ , respectively.

Assume that the prices of all AD securities are known, i.e., that the function  $p^t(x, y)$  is known for all  $t > 0$ ,  $x \in \mathbb{R}$  and  $y \in \mathbb{R}$ . It is well-known that we can draw inferences about the underlying parameters,  $\rho$ ,  $\mu(x)$ ,  $\sigma(x)$ , and  $m(x)$  from  $p$ . Specifically, using Itô's lemma and differential notation, given that  $X_0 = x$ , the price of an asset that pays 1 at  $dt$  is  $P^r = 1 - rdt$  where

$$r(x) = \rho - q(x)\mu(x) - (q'(x) + q(x)^2)D(x) \quad (13)$$

is the short risk-free rate, which in general is a function of  $x$ . Technically,

$$r(x) = \lim_{t \searrow 0} \frac{1 - \int p^t(x, y) dy}{t}. \quad (14)$$

Thus, the short rate at any  $x$  can be recovered from knowledge of  $p$ . Similarly, the price of the

AD security  $p^{\Delta t}(x, x + \Delta t)$  is approximately  $\frac{1}{\sqrt{2\pi\sigma(x)^2\Delta t}}$ , so we can back out

$$D(x) = \lim_{t \searrow 0} \frac{1}{4\pi t} \times \left( \frac{1}{p^t(x, x + t)} \right)^2. \quad (15)$$

Finally, consider

$$\kappa(x) \stackrel{\text{def}}{=} \mu(x) + 2q(x)D(x). \quad (16)$$

The price of a security that pays off  $X_{dt}$  at  $dt$ , given that  $X_0 = x$ , is

$$\begin{aligned} P^x &= E \left[ e^{-\rho dt} \frac{m(X_{dt})}{m(x)} X_{dt} \right] \\ &= E \left[ x + \frac{d(X e^{-\rho t} m(X))}{m(x)} \right] \\ &= x - \rho x dt + xq(x)\mu(x)dt + x(q'(x) + q(x)^2)D(x)dt + \mu(x)dt + 2q(x)D(x)dt \\ &= x(1 - r(x)dt) + \kappa(x)dt. \end{aligned}$$

In risk neutral terminology,  $\kappa(x)$  is the drift of the state variable,  $x$ , in the risk neutral measure.

We can then back out  $\kappa(x)$  as

$$\kappa(x) = r(x)x + \lim_{t \searrow 0} \frac{\int p^{dt}(x, y)y dy - x}{dt}. \quad (17)$$

To summarize, if the prices of AD securities are observable for all  $t > 0$ ,  $x$ , and  $y$ , then  $r$ ,  $D$ , and  $\kappa$  are known, from (14, 15, 17).

For any given  $x$ , (13,16) provide two equations for the three unknown  $\rho$ ,  $\mu(x)$ , and  $m(x)$ , and it may therefore seem as if there is one degree of freedom at each point  $x$ . For example, such pointwise indeterminacy arises in a one-factor time homogeneous term structure model, where an unknown risk-premium process  $\lambda(X_t)$  is introduced, and  $\lambda(x)$  may be quite arbitrary. An insight in Ross (2013a) is that a pricing kernel on the form (11) leads to strong constraints on how the marginal utility can change with  $x$ , and that we therefore have much more information

compared with the one-factor term structure model.<sup>4</sup> We can see this in our context, by plugging (16) into (13), to get

$$q' = \frac{\rho - r}{D} - \frac{\kappa}{D}q + q^2. \quad (18)$$

Equation (18) is a Riccati ordinary differential equation (ODE) for  $q$ . It is a quadratic first order equation, which means that if  $q$  is known at  $x$ , then  $q$  can be calculated in a neighborhood of  $x$ . Thus, the equation provides a law of motion for  $q$ .

We can also use that  $z = \frac{1}{m}$ ,  $z' = -\frac{q}{m}$ , and  $z'' = -\frac{1}{m}(q' - q^2)$ , to rewrite (18) as a second order linear ODE in  $z$ :

$$z'' + \frac{\kappa}{D}z' + \frac{\lambda - r}{D}z = 0, \quad (19)$$

where  $\lambda = \rho$ . The two formulations, (18) and (19), are equivalent.<sup>5</sup> We call (19) the fundamental ODE for the recovery problem of a diffusion process.

We can get some intuition for the important coupling of  $q$ —and thereby marginal utility,  $m$ —across states that is expressed by (18). Let us suppose that, miraculously,  $q(x)$  is known at  $x = X_0$ . Now, consider the portfolio of selling  $q(x)$  shares of the asset that pays  $X_{dt}$  at  $dt$ , and purchasing  $q(x)x + 1$  of a short bond that pays 1 at  $dt$ . It is easy to verify that the payoff of such a portfolio at  $dt$  is  $1 - q(x)dX$ . More importantly, the price at time 0 of the portfolio is

$$(q(x)x + 1)P^r - q(x)P^x = 1 + (D(x)(q'(x) - q(x)^2) - \rho)dt. \quad (20)$$

Now, assume that we also have a good idea what the agent's personal discount rate,  $\rho$ , is. Since we know  $D(x)$ , and  $q(x)$ , we can then back out  $q'(x)$ , which in turn implies that we know  $q(x + dx) = q(x) + q'(x)dx$ . The same argument can now be repeated, allowing us to calculate  $q(x + 2dx)$ , and in extension  $q(x)$  for all  $x$ . Thus, because of this coupling of marginal utilities

---

<sup>4</sup>Of course, in the presence of interest rate derivative contracts that allow interest rate state prices to be backed out, this information is also present in the one-factor term structure model.

<sup>5</sup>Specifically, for any positive solution to (19),  $q = -\frac{z'}{z}$  is a solution to (18), and for any solution to (18),  $z = ce^{-\int q dx}$  is a positive solution to (19), where  $c$  is an arbitrary positive constant.

across states, the number of degrees of freedom for  $q$  is at most 2 (one degree in  $\rho$ , and one in  $q(X_0)$ , respectively), rather than one degree for each value of  $x$ , as in the one-factor term structure model.

Following the insight in Ross (2013a), we shall see how positivity can be used to decrease the number of degrees of freedom even further. First, however, we further study the similarities between the eigenfunction formulation of the problem in Ross (2013a) and the ODE formulation in our setting.

## 2.1 Relationship to integral equation formulation

There is a close relationship between the fundamental ODE and the eigenvalue problem in Ross (2013a), (5). In the diffusion process setting, the eigenvalue problem turns into a linear integral equation. Specifically, we have

$$\int f^t(x, y)dy = 1, \quad \forall x, \quad (21)$$

which is the continuous version of (1). We rewrite this on operator form as

$$f^t[1] = 1, \quad \text{where} \quad f[s](x) \stackrel{\text{def}}{=} \int f(x, y)s(y)dy,$$

for an arbitrary function  $s(y)$ .

From (10), we have  $f^t(x, y) = e^{\rho t}m(x)^{-1}p^t(x, y)m(y)$ , similar to (3), which when plugged into (21) yields  $\int e^{\rho t}m(x)p(x, y)m(y)^{-1}dy = 1$ , or

$$\int p^t(x, y)m(y)^{-1}dy = e^{-\rho t}m^{-1}(x).$$

or, for  $z = \frac{1}{m}$ ,

$$p^t[z](x) \stackrel{\text{def}}{=} \int p^t(x, y)z(y)dy = e^{-\rho t}z(x),$$

similar to (5). On operator form, this reads

$$p^t[z] = e^{-\rho t} z, \quad (22)$$

which is an integral equation eigenfunction problem. This is the continuous time diffusion process equivalent of the eigenvalue problem in Ross (2013a).

For small  $\Delta t$ , the Fokker-Planck equation implies that

$$f^{\Delta t}(x, y) \approx \frac{1}{\sqrt{2\pi\sigma(x)^2\Delta t}} e^{-\frac{(x-y-\mu(x)\Delta t)^2}{2\sigma(x)^2\Delta t}},$$

which implies that for a smooth function,  $s(y)$ , that is bounded by  $C_\epsilon e^{\epsilon y^2}$  for large  $y$  and any  $\epsilon > 0$ ,

$$f^{\Delta t}[s](x) = s(x) + \Delta t \left( \frac{\sigma^2(x)}{2} s''(x) + \mu(x) s'(x) \right) + h.o.t.,$$

where ‘‘h.o.t.’’ denotes higher order terms in  $\Delta t$ . We define the infinitesimal operator  $\mathcal{L}s \stackrel{\text{def}}{=} \frac{\sigma^2(x)}{2} s''(x) + \mu(x) s'(x)$ , so that  $\mathcal{L}$  is the adjoint of  $\mathcal{L}^*$ , and we can then write the relation as

$$f^{dt}[s] - s = dt \times \mathcal{L}s. \quad (23)$$

Thus, an eigenfunction to  $f^{dt}$  must satisfy  $\lambda s = \mathcal{L}s$ . Clearly,  $s \equiv 1$  is such a function, with  $\lambda = 0$ , leading to  $f^{dt}[1] = 1$ , in line with (21).

Using (10), we get that for an arbitrary function,  $v$ ,

$$\begin{aligned} p^{dt}[v](x) &= (1 - \rho dt) m(x)^{-1} \int f^{dt}(x, y) m(y) v(y) dy \\ &= (1 - \rho dt) m(x)^{-1} f^{dt}[mv](x), \end{aligned}$$

which, using (23), leads to

$$\begin{aligned}
p^{dt}[v](x) &= (1 - \rho dt)m(x)^{-1}(1 + dt \times \mathcal{L})[mv](x) \\
&= (1 - \rho dt)v(x) + dt \times m(x)^{-1}\mathcal{L}[mv](x) \\
&= v(x) - \rho v(x)dt + dt \times \mathcal{Q}[v](x),
\end{aligned}$$

where  $\mathcal{Q}[v](x) \stackrel{\text{def}}{=} (m^{-1}\mathcal{L}[mv])(x)$ .

We rewrite

$$\begin{aligned}
\mathcal{Q}[v](x) &= \frac{1}{m(x)} \left( \mu(x) \frac{d}{dx}[m(x)v(x)] + D(x) \frac{d^2}{dx^2}[m(x)v(x)] \right) \\
&= \frac{1}{m(x)} (\mu(x)(m'(x)v(x) + m(x)v'(x)) + D(x)(m''(x)v(x) + 2m'(x)v'(x) + m(x)v''(x))) \\
&= (\rho - r)v + \kappa v' + Dv'',
\end{aligned}$$

and for  $z = \frac{1}{m}$  we then have

$$0 = m(x)^{-1}\mathcal{L}[1] = \mathcal{Q}[z](x) = (\rho - r)z + \kappa z' + Dz''.$$

The fundamental ODE (19) is thus the differential form of the integral equation eigenvector problem (22).

## 2.2 Recovering pricing kernel from fundamental ODE

We now address the problem of under which conditions there is sufficient information to uniquely recover  $m(x)$  and  $\rho$  from the fundamental ODE. Here, uniqueness of  $m$  is defined up to scaling with an arbitrary positive constant, in line with our previous discussion of invariance under positive affine transformations of the utility function. In this case, we say that recovery is possible.

We define the operator  $\mathcal{W}[s|\lambda] = \frac{d^2s}{dx^2} + \frac{\kappa}{D} \frac{ds}{dx} + \frac{\lambda-r}{D} s$ , and can for general  $\lambda$  solve

$$\mathcal{W}[s|\lambda] = 0. \tag{24}$$

Of course, from (19),  $\mathcal{W}[\frac{1}{m}|\rho] = 0$ .

Under general conditions, given  $\rho$ , the solution to (19) is on the form  $c_1 z_1(x) + c_2 z_2(x)$ , for arbitrary constants,  $c_1$ , and  $c_2$ . But, since  $z$  is only unique up to multiplication by a finite constant, there is effectively only one degree of freedom:  $z = cz_1 + (1 - c)z_2$ . Thus, in general, (19) has only two degrees of freedom, one degree in  $\rho > 0$  and one in  $c$ . We have

**Proposition 1** *Consider the fundamental ODE, (19).*

- *Given  $\rho$ , and  $q(x_0) = c$  for some  $x_0$ , there is a unique solution to (19),  $z_{\rho,c}(x)$ , defined on the whole of  $\mathbb{R}$ .*
- *Given  $\rho_1, \rho_2, c_1$ , and  $c_2$ , such that  $\rho_1 \neq \rho_2$  or  $c_1 \neq c_2$ , then the solutions to (19) with parameters  $\rho_1, c_1$ , and  $\rho_2, c_2$ , respectively, are distinct,  $z_{\rho_1,c_1} \neq z_{\rho_2,c_2}$ .*

Proposition 1 makes precise the concept that there is in general sufficient information to reduce the indeterminacy of the recovery problem down to two degrees of freedom. The second part suggests that without further knowledge of  $\rho$  and  $q(x_0)$ , for some  $x_0$ , recovery is not possible. We have still not used the fact that  $m$  must be positive though. The second—and fundamental—insight of Ross (2013a) in the discrete setting is that positivity allows for recovery, because the Perron-Frobenius Theorem guarantees that only one solution to the eigenvalue problem is strictly positive.

In our diffusion setting, it is a priori unclear how far positivity will take us. Given that there is an infinite number of unknowns ( $m(x)$  for all  $x$ ), as well as conditions (relating  $q'(x)$  to  $q(x)$  for all  $x$  in (18)), we cannot simply count the number of equations and unknowns to see

whether there is sufficient information for recovery. The following proposition shows that the behavior of the diffusion process as  $x$  tends to  $\pm\infty$  determines whether recovery is possible:

**Proposition 2** *A necessary and sufficient condition for recovery of  $m(x)$  is that*

$$\int_{-\infty}^0 e^{-\int_0^x \frac{\mu(s)}{D(s)} ds} dx = \infty, \quad \text{and} \quad \int_0^{\infty} e^{-\int_0^x \frac{\mu(s)}{D(s)} ds} dx = \infty. \quad (25)$$

Here, we use the identity  $\int_0^x \frac{\mu(s)}{D(s)} ds = -\int_x^0 \frac{\mu(s)}{D(s)} ds$  for  $x < 0$ . Thus, the behavior of  $\frac{\mu(x)}{D(x)}$  for large (negative or positive)  $x$  determines whether recovery is possible. A sufficient but not necessary condition for (25) is that  $X$  is mean reverting. An example in which  $X$  is not mean reverting but recovery is still possible is when  $X$  is a standardized Brownian motion, i.e., when  $\mu(x) = 0$ ,  $\sigma(x) = \sigma > 0$ . It is easy to verify that (25) is satisfied in this case. The drift term,  $\mu(x)$ , can also be positive for  $x > 0$ , and/or negative for  $x < 0$ , as long as it approaches zero quickly enough, or  $D$  grows fast enough.

An intuition for why the positivity requirement reduces the number of degrees of freedom in the recovery problem from two to zero is provided as follows: It is straightforward to verify that the solutions to  $W[s|\lambda] = 0$  can be written as  $s = u(x)v(x)$ , where  $u(x)$  solves the ODE

$$u'' = \left( \frac{1}{4} \left( \frac{\mu}{D} \right)^2 + \frac{1}{2} \frac{d}{dx} \left( \frac{\mu}{D} \right) - \frac{\lambda - \rho}{D} \right) u, \quad (26)$$

and  $v(x) = e^{-\frac{1}{2} \int_0^x \frac{\kappa(y)}{D(y)} dy}$ . This ODE, which also arises in the the model of Carr and Yu (2012) with bounded domains, provides a convenient separation into a part,  $v$ , that depends on the representative investor's marginal utility, and a part,  $u(x)$ , that solely depends on the diffusion process, and specifically on  $\frac{\mu}{D}$  as seen in (26). Moreover,  $v(x)$  is always positive, so negativity of the solution must come from  $u$ . This explains the result that the condition for recovery does not depend on  $m$ , but only on the diffusion process through  $\frac{\mu}{D}$ .

Now, it is easy to check that a solution in the case when  $\lambda = \rho$  is given by  $u_{\rho,1} = e^{\frac{1}{2} \int_0^x \frac{\mu(y)}{D(y)} dy}$ ,



and in the proof it is moreover shown that if (25) is satisfied, then for any  $c \neq 1$ , the range of the other solution to this second order ODE is the whole real line. Therefore, the range of any combination of the two solutions must also be the whole real line, violating the positivity constraint. This reduces the number of degrees of freedom from two to one, by forcing  $c = 1$ .

The final part of the argument, allowing us to nail down  $\rho$ , is that a higher  $\lambda$  in (26) will have a negative effect on  $u$ , at any point where  $u(x)$  is positive, by decreasing  $u''$ . As shown in the proof, as long as  $u_{\rho,1}$  does not grow too fast, this negative effect on  $u''$  of having  $\lambda > \rho$  eventually makes  $u(x)$  become negative. The conditions (25) are such that  $\frac{\mu}{D}$ , and thereby  $u$ , does not grow too fast. Altogether, this implies that  $\rho$  can be identified as the largest  $\lambda$  for which there is a positive solution to  $\mathcal{W}[z|\lambda] = 0$ , which in turn will be unique. Recovery is therefore possible.

We next study two examples in more detail, one for which recovery is possible and one for which it is not, to provide additional insight and intuition.

### 2.3 Brownian motion example

Consider the classical Black-Scholes economy with  $dX = \mu dt + \sigma d\omega$ , where  $\mu > 0$  and  $\sigma > 0$  are constants. It follows that  $\int_{-\infty}^0 e^{-\int_0^x \frac{\mu(s)}{D(s)} ds} dx = \frac{D}{\mu} < \infty$ , so (25) is not satisfied, and recovery is therefore not possible. This is in line with what has been reported earlier in Ross (2013a) and Dubynskiy and Goldstein (2013).

We verify non-recovery for two utility functions. We first study the standard Lucas economy with power utility, where  $g(x) = e^x$ , and  $u'(g) = g^{-\gamma}$ ,  $\gamma > 0$ . It follows from previous definitions that  $m(x) = e^{-\gamma x}$ ,  $q(x) = -\gamma$ , and  $z(x) = e^{\gamma x}$ . From (13,16), we then have

$$\begin{aligned} r &= \rho + \gamma\mu - \gamma^2 \frac{\sigma^2}{2}, \\ \kappa &= \mu - \gamma\sigma^2, \end{aligned}$$

in line with standard results. The solutions to (24) are

$$\begin{aligned} z_1^\lambda(x) &= e^{-x \frac{\kappa + \sqrt{\kappa^2 + 2\sigma^2(r-\lambda)}}{\sigma^2}}, \\ z_2^\lambda(x) &= e^{-x \frac{\kappa - \sqrt{\kappa^2 + 2\sigma^2(r-\lambda)}}{\sigma^2}}. \end{aligned}$$

For  $\lambda \leq r + \frac{\kappa^2}{2\sigma^2} = \rho + \frac{\mu^2}{2\sigma^2}$ , there are two distinct positive solutions to the equation. Thus, the possible marginal utilities are

$$m_{\lambda,c} = \frac{1}{z_{\lambda,c}} = \frac{1}{cz_1^\lambda(x) + (1-c)z_2^\lambda(x)}, \quad \lambda \leq r + \frac{\kappa^2}{2\sigma^2}, \quad 0 \leq c \leq 1.$$

As previously discussed, there are two remaining degrees of freedom but the size of the set of possible solutions is drastically reduced compared with what would be possible if one only used the “pointwise” estimate  $\kappa(x) = \mu(x) + q(x)\sigma^2(x)$  for every  $x$ .

Even if  $\rho$  is known,  $m$  is not uniquely recovered. In fact, it is easy to check that in this case the possible solutions are

$$m_{\rho,c} = \frac{1}{z_{\rho,c}} = \frac{1}{ce^{x\rho} + (1-c)e^{x(\rho - \frac{2\mu}{\sigma^2})}}, \quad 0 \leq c \leq 1.$$

So, in addition to the correct solution,  $m = m_{\rho,1}$ , there is a whole set of other possible positive solutions. In Figure 1, some possible functional forms of  $m$  are shown, given that  $\rho$  is known. In Figure 2, the corresponding possible relative risk aversion coefficients as a function of  $x$  are shown.

We next consider the case where  $u(x) = x + \frac{x^3}{3}$ ,  $g(x) = x$ , so that  $m(x) = 1 + x^2$ . Note that this utility function is quite nonstandard in that it is not concave. It shows the strength of the methodology that no additional restrictions on  $m$  are needed, beyond positivity. In this case,

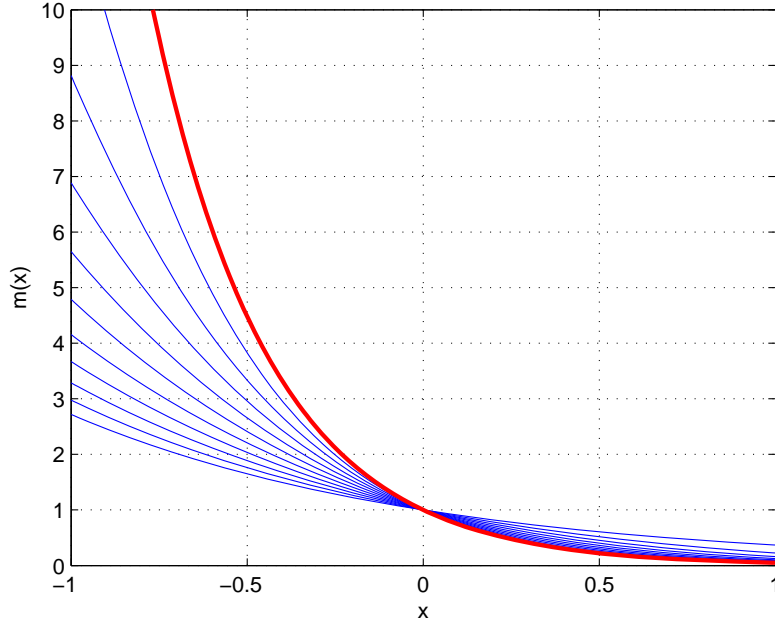


Figure 1: **Some candidate  $m$  functions, given that  $\rho$  is known,  $c = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.8, 1$  shown. The solid (red) line represents the true  $m = e^{-\gamma x}$ , corresponding to  $c = 1$ . Recovery is not possible in this case. Parameter values:  $\gamma = 3$ ,  $\mu = 0.01$ ,  $\rho = 0.01$ ,  $\sigma = 0.1$ .**

we get

$$r = \rho - \frac{2x\mu + \sigma^2}{1 + x^2}, \quad (27)$$

$$\kappa = \mu + \frac{2x\sigma^2}{1 + x^2}. \quad (28)$$

These expressions are also nonstandard. For example, the short interest rate is highly negative for large  $x$ . Again, a strength of the methodology is that it allows us to analyze recovery under very general conditions.

The stochastic process is still a Brownian motion, so Proposition 2 again implies that recovery is not possible with this utility specification either. The general solution to  $\mathcal{W} \left[ \frac{1}{m} | \rho \right] = 0$  in this case is

$$m_{\rho,c} = \frac{1}{z_{\lambda,c}} = \frac{1 + x^2}{c + (1 - c)e^{-\frac{2\mu}{\sigma^2}x}}, \quad 0 \leq c \leq 1,$$

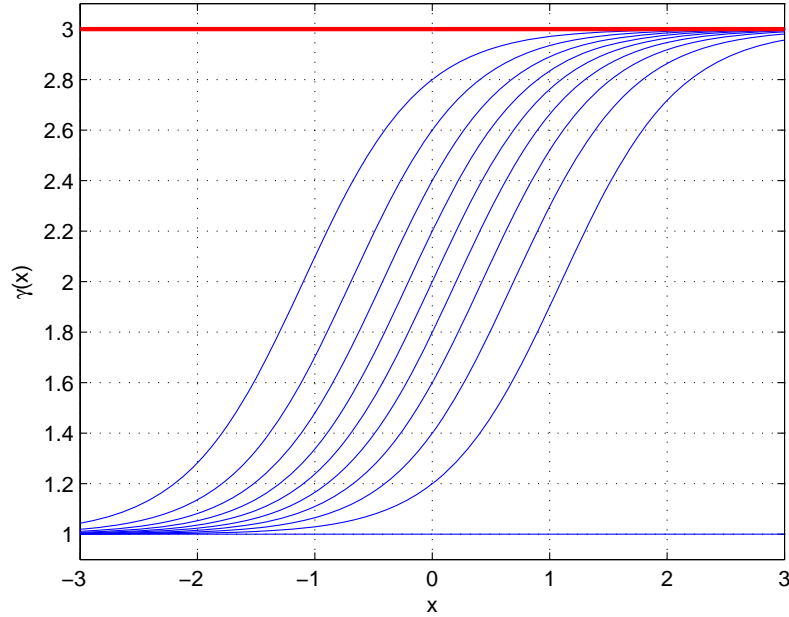


Figure 2: **Some possible risk-aversion functions,  $\gamma(x)$ , given that the correct  $\rho$  is known,  $c = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.8, 1$ . The solid red line at the top represents the true  $\gamma(x) = 3$ ,  $c = 1$ . Parameter values:  $\gamma = 3$ ,  $\mu = 0.01$ ,  $\rho = 0.01$ ,  $\sigma = 0.1$ .**

so there are multiple possible  $m(x)$  functions, even if  $\rho$  is known.

## 2.4 Ornstein-Uhlenbeck example

Now consider the Ornstein-Uhlenbeck process  $dX = -\theta(a - X)dt + \sigma d\omega$ ,  $\theta > 0$ ,  $\sigma > 0$ . Without loss of generality, we assume that  $a = 0$ , since we can always define  $\hat{x} = x - a$  for non-zero  $a$ , and solve in  $\hat{x}$  coordinates. We then have  $\mu = -\theta x$ , and since  $\frac{\mu}{D} = -\frac{\theta}{\sigma}x$ , the conditions for recovery in Proposition 2 are satisfied. Again, we assume that  $m(x) = 1 + x^2$ . We calculate

$$\begin{aligned}
 r &= \rho + \frac{2x^2\theta - \sigma^2}{1 + x^2}, \\
 \kappa &= -x\theta + \frac{2x\sigma^2}{1 + x^2},
 \end{aligned}$$

and (24) then takes the form

$$z'' + \frac{x}{D} \left( \frac{2\sigma^2}{1+x^2} - \theta \right) z' + 2 \frac{\lambda + x^2(\lambda - \rho - 2\theta) - \rho + \sigma^2}{\sigma^2(1+x^2)} z = 0. \quad (29)$$

The solutions to (29) are

$$\begin{aligned} z_1^\lambda(x) &= \frac{1}{1+x^2} H_{\frac{\lambda-\rho}{\theta}} \left( \frac{x\sqrt{\theta}}{\sigma} \right), \\ z_2^\lambda(x) &= \frac{1}{1+x^2} {}_1F_1 \left( \frac{\rho-\lambda}{2\theta}, \frac{1}{2}, \frac{x^2\theta}{\sigma^2} \right). \end{aligned}$$

Here,  $H_a$  is the Hermite function, and  ${}_1F_1$  is the confluent hypergeometric function (see Gradshteyn and Ryzhik (2000), p. 986 and 1013). In the case when  $\lambda = \rho$ , this reduces to

$$\begin{aligned} z_1^\rho(x) &= \frac{1}{1+x^2}, \\ z_2^\rho(x) &= \frac{1}{1+x^2} \operatorname{Erfi} \left( \frac{x\theta}{\sigma} \right), \end{aligned}$$

where  $\operatorname{Erfi}$  is the imaginary error function,  $\operatorname{Erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{\frac{s^2}{2}} ds$ . Now, since  $\operatorname{Erfi}(-\infty) = -\infty$  and  $\operatorname{Erfi}(\infty) = \infty$ , the only way to make  $z_{\rho,c} = cz_1^\rho(x) + (1-c)z_2^\rho(x)$  strictly positive for all  $x$  is to choose  $c = 1$ . Moreover, for any  $\lambda > \rho$ , all candidate  $z_{\lambda,c} = cz_1^\lambda + (1-c)z_2^\lambda$  are negative for some  $x$ , and therefore “disqualified” as candidate  $z$  functions. This follows from the proof of Proposition 2. An example is shown in Figure 3, where candidate  $z^\lambda$  for a specific  $\lambda > \rho$  are shown. Since all candidates are negative for some  $x$ , they cannot represent the correct  $z$  function.

## 2.5 Relationship between recovery and stationary distribution

The condition for recovery (25) is related to the existence of a stationary distribution of the diffusion process. Necessary and sufficient conditions for a function  $\varphi(y)$  to be a stationary

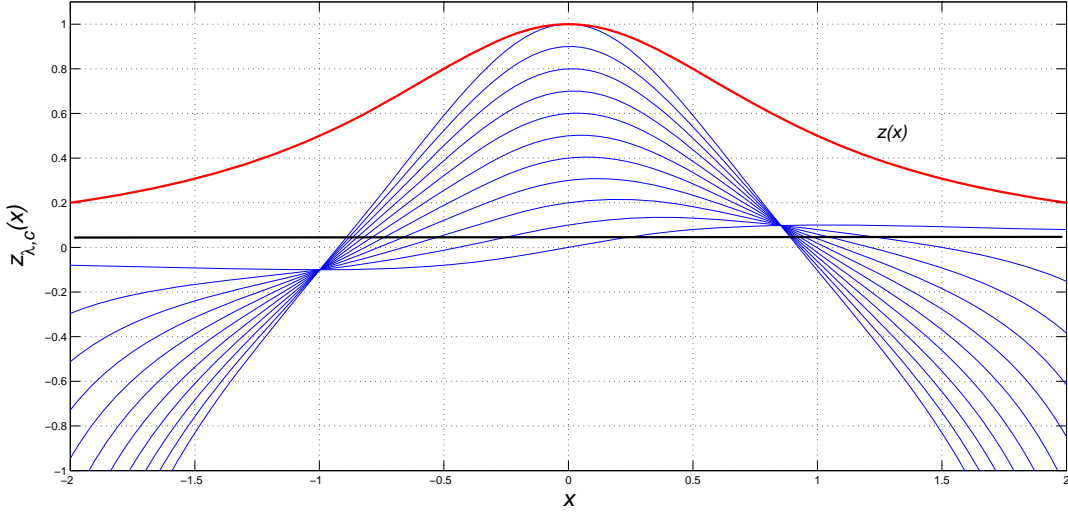


Figure 3: **Candidate functions,  $z_{\lambda,c}(x)$ ,  $c = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.8, 1$ . The solid (red) line above the other lines represents the true  $z = \frac{1}{m}$ . All candidate functions with  $\lambda > \rho$  eventually become negative, which means that they cannot represent  $\frac{1}{m}$ . Parameter values:  $\lambda = 0.02$ ,  $\theta = 0.01$ ,  $\rho = 0.01$ ,  $\sigma = 0.1$ .**

distribution is that  $\varphi(y) \geq 0$ ,  $\int \varphi(y)dy = 1$ , and that  $\mathcal{L}^*\varphi = 0$ .

Let us define  $Q(x) \stackrel{\text{def}}{=} e^{-\int_0^x \frac{\mu(s)}{D(s)} ds}$ , and (25) can then be written as  $\int_0^\infty Q(x)dx = \int_{-\infty}^0 Q(x)dx = \infty$ . It is easy to verify that the general solution to  $\mathcal{L}^*\varphi = 0$  is

$$\varphi(y) = \frac{1}{Q(y)D(y)} \left( c_1 + c_2 \int_0^y Q(s)ds \right). \quad (30)$$

and therefore that a necessary and sufficient condition for the existence of a stationary distribution is that

$$\int_{-\infty}^0 \frac{1}{Q(x)D(x)} dx < \infty, \quad \text{and} \quad \int_0^\infty \frac{1}{Q(x)D(x)} dx < \infty. \quad (31)$$

Now, the link between (25) and (31) is clear: the faster  $Q$  increases for large  $y$ , the larger the right integral in (25), and the smaller the right integral in (31). An identical argument holds for the left integrals. However, the two conditions are not equivalent. The existence of a stationary

distribution implies that recovery is possible, but the reverse causality is not true. We have

**Proposition 3** *If the diffusion process has a stationary distribution, then recovery is possible.*

An example for which recovery is possible, but there is no stationary distribution is the standardized Brownian motion,  $\mu = 0$ ,  $\sigma = 1$ , leading to  $Q(x) \equiv 1$ . Clearly condition (31) fails in this case, but the condition for recovery, (25) is satisfied. Indeed, the solutions to the fundamental ODE in this case are

$$\begin{aligned} z_{\rho,c} &= \frac{c + (1-c)x}{m(x)}, \quad \text{and} \\ z_{\lambda,c} &= \frac{c \times \cos\left(\sqrt{2(\lambda-\rho)}x\right) + (1-c) \sin\left(\sqrt{2(\lambda-\rho)}x\right)}{m(x)}, \quad \lambda > \rho. \end{aligned}$$

Thus, the only positive solution is  $z_{\rho,1}$ , and recovery is therefore possible.

Interestingly, although recovery is possible in the economy above, it is not possible to recover the personal discount rate directly from the yield of long-term bonds. This is in contrast to the result in Martin and Ross (2013), who show that such direct recovery is possible in the finite state space model. For example, in the case above with  $\mu = 0$ ,  $\sigma = 1$ , and CRRA preferences with risk aversion coefficient  $\gamma$ ,  $m(x) = e^{-\gamma x}$ , it follows from Section 2.3 that the short rate is  $r = \rho - \gamma^2/2$  and, since the yield curve is flat in this standard Lucas economy, this is also the long rate. Thus,  $r$  does not provide sufficient information to directly back out  $\rho$ . The reason is that although the drift is  $\mu = 0$ , the risk averse agent behaves as if the drift is  $-\gamma/2$  (see, e.g., Parlour et al. 2011 for a discussion), which brings down the risk-free rate by introducing a precautionary savings motive. Such a precautionary savings motive is of course also present in the finite state model, but since there are bounds on marginal utility, i.e., on  $\frac{m(X_T)}{m(X_0)}$ , in that setting, the dominant term of the pricing kernel in the long run is the personal discount rate,  $e^{-\rho T}$ , allowing direct recovery of  $\rho$  from long yields in that case.

In the finite dimensional case, the existence of a unique stationary distribution is both

necessary and sufficient for recovery. The situation is thus different for the case with unbounded diffusions. In the diffusion case, a stationary distribution, if it exists, is unique, so that part of the causality is the same in both models. The reverse causality, however, is different, since recovery does not imply the existence of a stationary distribution in the diffusion model. The reason for the difference is clear: The eigenfunction relationship for recovery is defined by the operator relationship (21). The corresponding finite relationship is (1). The existence of a stationary distribution is governed by the adjoint equation,  $F^*\varphi = \varphi$  in the finite case, and  $\mathcal{L}^*\varphi = 0$ , in the diffusion case. But, whereas it is always possible to rescale  $\varphi$  such that  $\sum_i \varphi_i = 1$  in the finite case, there is no guarantee that  $\int \varphi(y)dy < \infty$  in the diffusion case. In the terminology of functional analysis: there is no guarantee that the positive solution to the adjoint equation  $\mathcal{L}^*\varphi = 0$  belongs to the space  $L^1(\mathbb{R})$  of integrable functions. Therefore, recovery may be possible even without a stationary distribution.

## 2.6 Recovery from a restricted class of utility functions

Since condition (25) in Proposition 1 is necessary and sufficient for recovery, there is nothing more to say about the general recovery problem. However, if we are willing to rule out some candidate pricing kernels by imposing stricter requirements than mere positivity of  $m$ , we may weaken the requirements on the diffusion process for recovery.

So far, we have considered any  $m \in \mathcal{C}_+^2$  as a candidate function for the pricing kernel  $\Lambda_t = e^{-\rho t} \frac{m(X_t)}{m(X_0)}$ , where  $\mathcal{C}_+^2$  is the class of strictly positive, twice continuously differentiable functions on the real line. By requiring  $m$  to belong to a smaller set, recovery becomes easier. Specifically, assume that for a specific class of diffusion processes (characterized by  $\mu$  and  $D$ ), and a set  $\mathcal{B} \subset \mathcal{C}_+^2$ , if  $m$  belongs to  $\mathcal{B}$ , then no other function in  $\mathcal{B}$  satisfies  $\mathcal{W}[1/m|\lambda] = 0$ , for  $\lambda \geq \rho$ . In this case we say that *unique recovery within  $\mathcal{B}$*  is possible for this class of diffusion processes.

One fruitful restriction is to focus on bounded marginal utilities,  $\mathcal{B} = \{m \in \mathcal{C}_+^2 : 0 < c_1 \leq$



$m(x) \leq c_2 < \infty$ . Here, we require that the bound below is strictly positive ( $c_1 > 0$ ).<sup>6,7</sup> We have:

**Proposition 4** *Unique recovery within  $\mathcal{B}$  is possible if and only if at least one of the conditions in (25) is satisfied.*

As a consequence, the classical Black-Scholes process studied in Section 2.3, which satisfies (25) on the left interval but not on the right, allows for unique recovery within  $\mathcal{B}$ . An example is given in Figure 4, where the bounded function  $m(x) = \left(1 + \frac{\tan^{-1}(x)}{\pi}\right)^{-1}$  is recovered within  $\mathcal{B}$ .<sup>8</sup>

We note that there is a trade-off here, in that the more we restrict the class of candidate functions, the more we are effectively taking a stand on what the pricing kernel looks like, going against the philosophy that the kernel and objective probability distribution should be inferred from data alone.

### 3 Backing out $r$ , $\kappa$ and $D$ from option prices

At a specific point in time,  $t$ , we only observe  $p^s(x_t, y)$  for general  $s > t$  and  $y \in \mathbb{R}$ . However, in our previous derivation of  $r$ ,  $D$ , and  $\kappa$  we needed  $p^t(x, y)$  for general  $x \in \mathbb{R}$ . Using an approach similar to Dupire's method (Dupire (1994)) for backing out local volatility, we can show that it is sufficient to know  $p^t(x_0, y)$ , as shown by the following proposition.

**Proposition 5** *Assume that at time 0, the prices  $p^t(x_0, y)$  are observed for all  $y$ , for all  $t \in (0, T)$ , for some  $T > 0$ . Define  $V(t, y) = p^t(x_0, y)$ . Then, for each  $y$  and  $t > 0$ ,  $V$  satisfies the*

---

<sup>6</sup>The study of this class was inspired by a discussion with Steve Ross, who in a working paper, Ross (2013b), assume boundedness when analyzing recovery in a model in discrete time with a continuous state space, using the Krein-Rutman theorem.

<sup>7</sup>Note that we do not require that the limits of  $m(x)$  exist as  $x$  tends to plus or minus infinity. Indeed,  $m$  may oscillate for large  $x$  without convergence.

<sup>8</sup>Equivalently, in units of the consumption good,  $g = e^x > 0$ , the marginal utility is  $\left(1 + \frac{\tan^{-1}(\ln(g))}{\pi}\right)^{-1}$ .

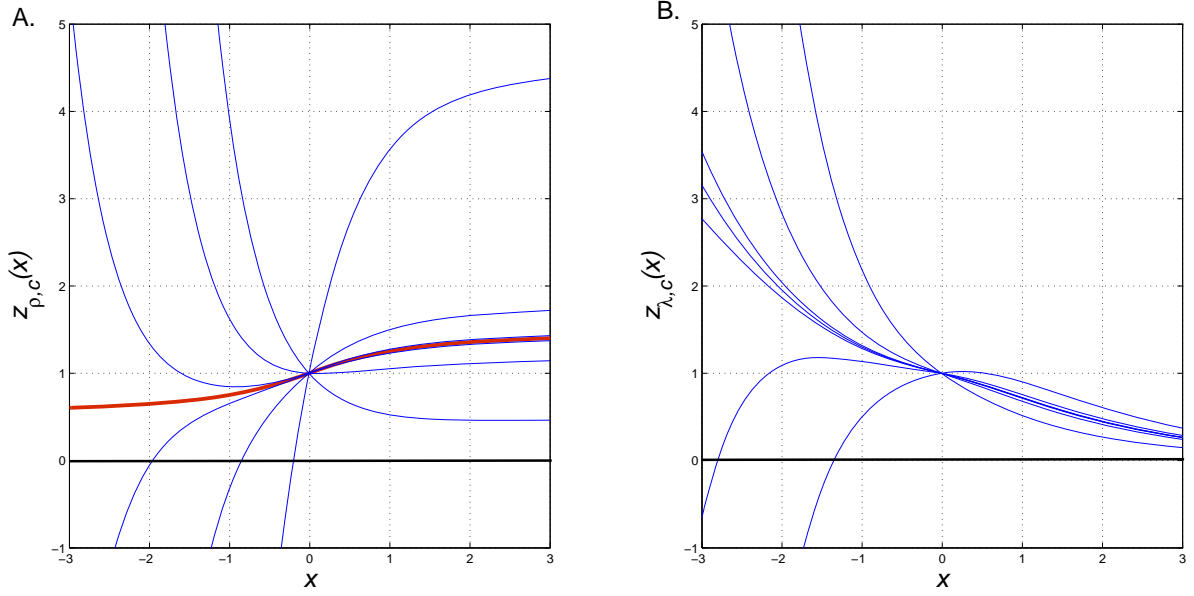


Figure 4: **Black-Scholes economy,  $dX = \mu dt + \sigma d\omega$ , with bounded marginal utility,  $m = \left(1 + \frac{\tan^{-1}(x)}{\pi}\right)^{-1}$ .** In the left panel (A), candidate functions,  $z_{\rho,c}(x)$  are shown. The only candidate function that is both bounded and positive is the solid (red) line  $z = \frac{1}{m} = 1 + \frac{\tan^{-1}(x)}{\pi}$ . In the right panel (B), candidate functions are shown for  $\lambda > \rho$ . No candidate function satisfies the condition of being positive, bounded from above, and bounded from below. Thus, unique recovery within  $\mathcal{B}$  is possible. Parameter values:  $\mu = 0.01$ ,  $\sigma = 0.1$ ,  $\rho = 0.01$ ,  $\lambda = 0.0104$ .

*PDE:*

$$V_t = D(y)V_{yy} + \alpha_1(y)V_y + \alpha_0(y)V, \quad (32)$$

where

$$\alpha_1(y) = 2D'(y) - \kappa(y), \quad (33)$$

$$\begin{aligned} \alpha_2(y) &= \alpha_1'(y) - r(y) \\ &= D''(y) - \kappa'(y) - r(y). \end{aligned} \quad (34)$$

Thus, by observing  $V(t, y)$ , we can calculate  $V_t$ ,  $V_y$ , and  $V_{yy}$ , and use (32) to solve for  $D(y)$ ,  $\alpha_1(y)$ , and  $\alpha_0(y)$ . Since there are three unknowns, for each  $y$ ,  $V$ ,  $V_t$ ,  $V_y$ , and  $V_{yy}$  need to

be known for three different  $t$ , to calculate  $D(y)$ ,  $\alpha_1(y)$ , and  $\alpha_0(y)$ . Once  $D(y)$  is known in a neighborhood of  $y$ ,  $\kappa(y)$  can be calculated, using (33), and given that  $\kappa(y)$  is known in a neighborhood of  $y$ ,  $r(y)$  can be calculated, using (34).

We note that the prices of AD securities,  $V(t, K)$ , can be inferred from the prices,  $C^t(K)$ , of call options with strike price  $K$  and maturity  $t$ ,  $0 < t < T$ ,  $K \in \mathbb{R}$ ,

$$C^t(K) \stackrel{\text{def}}{=} \int_K^\infty (y - K) \frac{m(y)}{m(x_0)} f^t(x_0, y) dy. \quad (35)$$

The price  $V(t, K) = C_{KK}^t(K)$ , is the second derivative of the price of the call option, with respect to the strike price, so  $D$ ,  $r$ , and  $\kappa$  can thus equivalently be calculated from call option prices. The proof of Proposition 5 follows similar lines as Dupire's method for backing out volatility in the local volatility model. We also note that the method is local in the sense that to back out  $D$ ,  $\kappa$ , and  $r$  at  $x$ , only option prices with strike prices around  $x$  are needed.

As an example, consider Brownian motion with  $m(x) = 1 + x^2$ , and assume that  $x_0 = 0$ .

We then get

$$V(t, y) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\rho t} (1 + y^2) e^{-\frac{y^2}{2\sigma^2 t}}.$$

Calculating  $V_t$ ,  $V_y$ , and  $V_{yy}$ , and backing out  $D$ ,  $\alpha_1$ , and  $\alpha_0$  from (32) leads to

$$\begin{aligned} D &= \frac{\sigma^2}{2}, \\ \alpha_1 &= -\left(\mu + \frac{2y\sigma^2}{1+y^2}\right), \\ \alpha_0 &= -\rho + \mu \frac{2y}{1+y^2} - \sigma^2 \frac{1-3y^2}{(1+y^2)^2}. \end{aligned}$$

Since  $D' \equiv 0$ , we get  $\kappa = -\alpha_1$ , in line with (28), and  $r = \alpha_1' - \alpha_2 = -2\sigma^2 \frac{1-y^2}{(1+y^2)^2} - \left(-\rho + \mu \frac{2y}{1+y^2} - \sigma^2 \frac{1-3y^2}{(1+y^2)^2}\right) = \rho - \frac{2y\mu + \sigma^2}{1+y^2}$ , in line with (27).

## 4 Approximate recovery on finite domains

In practice, option prices are not available for arbitrarily large prices, so we would not be able to observe  $p^t(x, y)$  for all  $x$  and  $y$ . An open question is then whether “approximate” recovery of  $m$  is possible given that  $p^t(x, y)$  is only known on some domain,  $-N \leq x, y \leq N$ , for some  $N > 0$ . Of course, in the case when recovery is not possible even if  $p^t(x, y)$  is known for all  $x, y$ , i.e., when  $N = \infty$ , recovery can never be possible when  $N < \infty$ . We therefore focus on the case when recovery is possible when  $N = \infty$ .

The question of approximate recovery is important: if few inferences about  $m$  can be drawn even for arbitrarily large but finite  $N$ , then for all practical purposes, recovery in the case with unbounded diffusion processes will not work. An example of such a situation is given in Dubynskiy and Goldstein (2013), where additional information about the representative agent’s preference parameters is needed for recovery to work. But their example is exactly one for which recovery does not work even if  $N = \infty$ , and is therefore of limited use for us.

The following result shows that as long as recovery is possible when  $N = \infty$ , strong inferences can be drawn about  $m$  in the case when  $N < \infty$ , without any additional information, e.g., in the form of boundary conditions.

**Proposition 6** *Assume that  $m$  can be uniquely recovered when  $N = \infty$ . Then, given a finite  $N > 0$ ,  $m(x)$  and  $\rho$  can be approximated by functions  $\hat{m}_N(x)$ , defined on  $(-N, N)$ , and  $\hat{\rho}_N$ , such that*

- $\hat{\rho}_N$  is nonincreasing in  $N$ , and  $\lim_{N \rightarrow \infty} \hat{\rho}_N = \rho$ ,
- for each  $x$ ,  $\lim_{N \rightarrow \infty} \hat{m}_N(x) = m(x)$ .

Thus, as long as recovery is possible on the unbounded domain, approximate recovery is possible on a bounded subdomain.

The intuition behind the result is simple. When  $N < \infty$ , we can solve for all candidate functions  $z_{\rho,c}$ , which satisfy (19) on  $[-N, N]$ , and are positive. Any candidate  $z_{\lambda,c}$  for  $\lambda > \rho$  will eventually become negative, and can therefore be ruled out if we have a large enough domain of observation. It follows from standard theory of ODEs that the larger  $\lambda > \rho$  is, the faster  $z$  will become negative, so for large domains, only  $z_{\lambda,c}$  for  $\lambda$  very close to  $\rho$  stay positive on the whole observed domain. However, these candidate  $z_{\lambda,c}$ 's are then also close to the true  $z$ , because of continuity. Therefore, as  $N$  increases, tighter and tighter bounds on both  $m = \frac{1}{z}$ , and  $\rho$  can be inferred.

We show how such approximate recovery works for the Ornstein-Uhlenbeck example with  $m(x) = 1 + x^2$ . In Figure 5, we assume that  $r$ ,  $\kappa$  and  $D$  are observed on  $x \in [-3, 3]$ , and calculate the approximate  $m$  function, as well as the approximated  $\hat{\rho}$ . We see that for  $|x| \leq 2$ , the approximation is very close to the correct solution, whereas the error is larger when we approach the boundary. This is typical: At  $x = N$ , the upper bound on  $m$  is infinity at one of the boundaries, since the only condition we have is that  $z > 0$  (i.e.,  $m < \infty$ ) on the whole domain. The approximated  $\hat{\rho} = 0.010002$  is very close to the true  $\rho = 0.01$ . We stress that no additional information was needed in this approximation, i.e., we imposed no “artificial” boundary conditions.

In the appendix, we provide Matlab code for approximating the pricing kernel on a finite domain, given  $D$ ,  $r$  and  $\kappa$  evaluated at  $N$  equidistant points,  $x_0, x_0 + \Delta x, \dots, x_0 + (N - 1)\Delta x$ , where  $\Delta x > 0$ . The code consists of two parts: the first part calculates the general solutions to the ODE, given a conjectured  $\hat{\rho}$ , using a standard finite difference method. The second part tests for whether a positive kernel can be constructed as a linear combination of the general solutions, and updates the conjectured  $\hat{\rho}$  accordingly. If multiple positive solutions exist, this means that the conjectured  $\hat{\rho}$  was too low, and if no positive solutions exist, this means that the conjectured  $\hat{\rho}$  was too high.

The code has performed well for the examples in this paper, and several other examples.

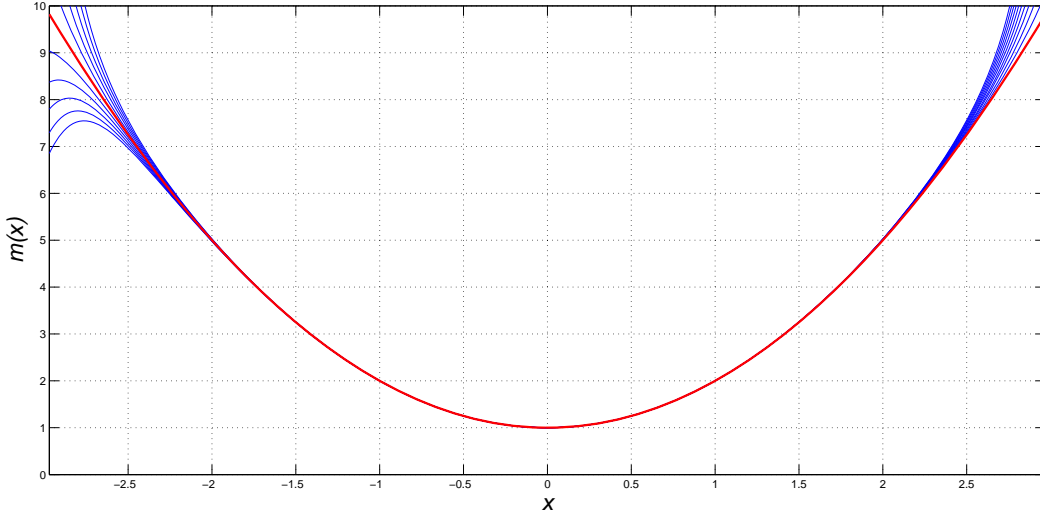


Figure 5: **Candidate functions,  $m(x)$ ,  $c = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.8, 1$  in the Ornstein-Uhlenbeck economy. The red line is the true  $m(x) = 1 + x^2$ . Given that  $r$ ,  $\kappa$ , and  $D$  are observed on  $x \in [-3, 3]$ , the candidate functions are those consistent with positivity of  $z$  on  $[-3, 3]$ , for the estimated discount rate,  $\hat{\rho}$ . For  $x \in [-2, 2]$ ,  $m$  is well approximated. The parameter values are:  $\theta = 0.01$ ,  $\rho = 0.01$ ,  $\sigma = 0.1$ . The estimated discount rate is  $\hat{\rho} = 0.010002$ .**

Convergence to  $\hat{\rho}$  is typically obtained in 15–30 iterations. In Figure 6, we show the approximation error for several different examples, as the interval,  $[-N, N]$  increases. We use the Ornstein-Uhlenbeck process (OU), the classical Black-Scholes process (BS), and the Brownian motion process without drift (BM) where  $dX = 0.1d\omega$ , as previously analyzed. We also introduce two new examples, that are close to the threshold of where recovery is possible, but on different sides of this threshold. The first is a slow growth process (SG),  $dX = 0.05dt + \sqrt{0.1(1 + X^2)}d\omega$ , and the second is a fast growth process (FG),  $dX = 0.1dt + \sqrt{0.1(1 + X^2)}d\omega$ . It is easily checked that the first economy satisfies the conditions for recovery, whereas the second does not. In all five examples, we use the nonstandard pricing kernel  $m = 1 + x^2$ . For robustness, we have also used standard power utility,  $m = e^{-\gamma x}$ , with similar results (not reported).

The left panel of Figure 6 shows the relative error of the approximated  $\hat{m}$ , evaluated at  $X = 1$ ,  $\frac{|\hat{m}(1) - m(1)|}{m(1)}$ , when  $N$  varies from 2 – 50. We see that the error decreases quickly as  $N$

increases for the Ornstein-Uhlenbeck and Brownian motion processes.<sup>9</sup> For the Black-Scholes process, the error is large and basically constant, in line with recovery failing for this process. For the intermediate slow growth and fast growth processes, the error decreases, but it is hard to draw inferences about ultimate convergence. A similar picture emerges for the error in the approximate discount rate,  $\hat{\rho} - \rho$ , in the right panel of the figure.

From the above results, it is clear that (25) influences how large an interval is needed to get accurate approximate recovery. If  $\int_{-N}^N Q(y)dy$  grows quickly as  $N$  grows, as in OU and BM, then a small interval is sufficient for good recovery. If  $\int_{-N}^N Q(y)dy$  converges quickly to a finite value, as in BS, then no convergence occurs. In the intermediate cases, SG and FG, the error decreases very slowly, and very large intervals are needed to draw inferences. The slow-growth example, SG, is chosen such that  $Q(y) = \frac{1}{\sqrt{1+x^2}}$ , leading to  $\int_{-N}^N Q(y)dy = 2 \sinh^{-1}(N) \approx 2 \ln(1 + N)$ . It is difficult to draw inferences about whether this function is ultimately bounded or unbounded from its behavior on a finite domain. Thus, even though approximate recovery works for this function in theory, stronger constraints on  $Q$  may be needed in practice for the method to work. We leave a detailed analysis of the convergence properties of the numerical method for future work.

## 5 Extensions and further discussion

The condition for recovery (25) depends on the behavior of  $\mu$  and  $\sigma$  for large  $|x|$ . It is natural to ask whether how the process is defined may influence whether recovery is possible. Specifically, in an economy for which recovery is not possible given an  $X$  process, could it be that by defining  $Y_t = G(X_t)$  for some smooth, strictly increasing transformation,  $G$ , with the whole real line as range (to keep the process unbounded), recovery is possible under the  $Y$  process?

It is easy to see that the answer to this question is no, because the requirement for recovery

---

<sup>9</sup>For the OU process, the error for  $N > 2$  is very small, but constant as  $N$  increases. This is because of the error introduced by using a finite difference method to solve the ODE, which is independent of the observation range,  $N$ . By decreasing  $\Delta x$ , this error can be decreased.

is invariant to such transformations, and therefore cannot “restore” the recovery property. The diffusion for  $Y$ , by Itô’s lemma, is

$$dY = (G'(X)\mu(X) + G''(x)D(X)) dt + G'(X)\sigma(X)d\omega \stackrel{\text{def}}{=} \hat{\mu}dt + \hat{\sigma}d\omega.$$

We then have  $\hat{D} = \frac{\sigma(X)^2 G'(X)^2}{2}$ , and  $\frac{\hat{\mu}}{\hat{D}} = \frac{\mu(X)}{G'(X)D(X)} + \frac{G''(X)}{G'(X)^2}$ . Here,  $X = G^{-1}(Y)$ , and without loss of generality we assume that  $G(0) = 0$ . For (25), we then have

$$\begin{aligned} \int_0^\infty e^{-\int_0^y \frac{\hat{\mu}}{\hat{D}} ds} dy &= \int_0^\infty e^{-\int_0^y \left( \frac{\mu(G^{-1}(s))}{G'(G^{-1}(s))D(G^{-1}(s))} + \frac{G''(G^{-1}(s))}{G'(G^{-1}(s))^2} \right) ds} dy \\ &= \int_0^\infty e^{-\int_0^{G^{-1}(y)} \left( \frac{\mu(v)}{D(v)} + \frac{G''(v)}{G'(v)^2} \right) dv} dy \\ &= \int_0^\infty e^{-\int_0^{G^{-1}(y)} \frac{\mu(v)}{D(v)} dv - [G'(v)]_0^{G^{-1}(y)}} dy \\ &= \int_0^\infty \frac{G'(0)}{G'(G^{-1}(y))} e^{-\int_0^{G^{-1}(y)} \frac{\mu(v)}{D(v)} dv} dy \\ &= G'(0) \int_0^\infty \frac{1}{G'(x)} e^{-\int_0^x \frac{\mu(v)}{D(v)} dv} G'(x) dx \\ &= G'(0) \int_0^\infty e^{-\int_0^x \frac{\mu(v)}{D(v)} dv} dx, \end{aligned}$$

where we used the changes of variables,  $v = G^{-1}(s)$ , and  $x = G^{-1}(y)$ . The finiteness of the positive integral is thus invariant to the transformation, as is by an identical argument the finiteness of the negative integral. Such transformations will therefore not change the recovery properties.

Another transformation is to introduce long-term growth into the specification. A recent literature has studied the link between long-term growth and asset prices under very general conditions, using Perron-Frobenius theory, see Alvarez and Jerman (2005), Hansen and Scheinkman (2009), Hansen (2012), and Hansen and Scheinkman (2013). Our narrow objective here is to consider whether long-term growth may be incorporated into our specific diffusion setting with the given pricing kernel specification.



Consider the time dependent production function  $g(t, X)$ . The function  $g$  can obviously not be chosen arbitrarily, since such arbitrary choices will be inconsistent with transition independent pricing kernels. However, for the specific choice of  $g(t, X)$ , such that  $u'(g(t, X)) = e^{\alpha t}v(X)$  for some constant  $\alpha$ , and function  $v(x)$ , the kernel will be on the form  $\Lambda_t = e^{(\alpha-\rho)t} \frac{v(X_t)}{v(X_0)}$ , i.e., on transition independent form. In this case, the recovery problem for  $\rho - \alpha$ ,  $v(x)$  and  $f^t$  is identical to our previous recovery problem. However, since the AD securities are defined on  $X_t$ , the function  $g(t, X)$  must be known in this case, to back out  $X$  from the observed  $g$ .

As an example, let us consider the growing economy with a representative agent with power utility,  $u'(c) = c^{-\gamma}$ ,  $g(t, X) = e^{at+X}$ ,  $a > 0$ , where  $dX = -\theta dt + \sigma d\omega$  is an Ornstein-Uhlenbeck process, leading to the pricing kernel

$$\frac{\Lambda_t}{\Lambda_0} = e^{-(\rho-a)t} \frac{v(X_t)}{v(X_0)}, \quad \alpha = -a\gamma, \quad v(x) = e^{-\gamma x}.$$

Here,  $e^{at}$  is a long-term growth component,  $a$  is assumed to be known,  $X_t$  is the deviation from this long-term growth, and its dynamics is not a priori known.

Recovery in this case is possible, i.e.,  $\gamma$  and  $f^t(x, y)$  can be recovered from observed option prices. However, it is clear that much stronger restrictions on the kernel are made in this case. In our previous analysis, no assumptions about  $m$  were needed. In this case, we are assuming that  $v(x)$  takes the form  $e^{-\gamma x}$  for some  $\gamma$ , and can then recover  $\gamma$  and  $\rho$ , as well as the diffusion process for  $X$ .

Extending the approximate recovery method to multi-dimensional state spaces should also be straightforward. For example, in the two-dimensional case, the pricing kernel would be on the form  $e^{-\rho t} \frac{m(X_{1t}, X_{2t})}{m(X_{10}, X_{20})}$ , where  $X_{1t}$  and  $X_{2t}$  are time-homogeneous diffusion processes on the form

$$\begin{aligned} dX_1 &= \mu_1(X_1, X_2)dt + \sigma(X_1, X_2)d\omega_1, \\ dX_2 &= \mu_2(X_1, X_2)dt + \sigma(X_1, X_2)d\omega_2, \end{aligned}$$

and  $\omega_1$  and  $\omega_2$  are standardized Brownian motions with  $Cov(d\omega_1, d\omega_2) = \rho(X_{1t}, X_{2t})dt$ . Of course, the theoretical analysis would need to be extended to ensure that recovery is possible. However, given that recovery is possible, we expect approximate recovery to work on bounded domains in multiple dimensions, along the lines of Section 4. A practical issue is that in the multidimensional case, AD securities on the joint realization of multiple variables would typically be needed, e.g., the prices  $p^t(x_1, x_2, y_1, y_2)$  in the case with two processes,  $X_{1t}$ , and  $X_{2t}$ .

So, altogether, where does this leave us with respect to recovery? It is clear from our results that for recovery to work, the setting cannot be too close to that in the classical Black-Scholes economy. If recovery among a perfectly general class of pricing kernels is aimed for, then the asymptotic growth of the distribution needs to be a lot slower than in the Black-Scholes model, as shown by Proposition 2. Indeed, recovery will fail in any model with stochastic growth, where the long-term growth rate is positive and unknown, by Proposition 2. We can of course restrict the class of feasible pricing kernels, but to cover the growth process of the Black-Scholes model, we expect quite strong restrictions to be needed, e.g., in the form of marginal utilities bounded strictly away from zero, as shown in Proposition 4.

One may argue that these restrictions on the growth of the state variable, or on boundedness of marginal utility, concern events very far out in the state space, since (25) is effectively an asymptotic condition, and that they are therefore unimportant in practice. However, from our analysis in Section 4, it is clear that approximate recovery only has “bite,” if the observed domain is sufficiently large to separate processes with “slow” growth from those with “fast” growth. Similarly, approximate recovery within the class of bounded marginal utility functions will only work if the observed domain is large enough to separate bounded from unbounded marginal utilities. Thus, the bounds, although technically asymptotic, effectively need to be pronounced within the range of observed option prices for recovery to work in practice in this setting.

Altogether, our results therefore suggest three areas for future research. First, additional restrictions may potentially be imposed on the set of feasible pricing kernels, allowing for recovery among a broader class of diffusion processes, including those in traditional asset pricing models. Second, the assumption of positive long-term growth, although a common feature of traditional asset pricing models, is not set in stone; other processes may also be explored. Third, along the lines of the discussion earlier in this section, it may potentially be possible to renormalize the economy, such that the state variable is interpreted as the deviation from a known long-term growth rate, thereby allowing for recovery in growing economies.

## 6 Concluding remarks

We have provided a general characterization of when recovery of the pricing kernel and objective probability distribution is possible in a model with a time homogeneous diffusion process on an unbounded domain. Mean reversion, for example, is a sufficient but not necessary condition for recovery. With further restrictions on marginal utility, long-term growth can be incorporated. When recovery works on an unbounded domain, then even if prices are only observed on a bounded subdomain, the kernel and probability distribution on this subdomain can be approximated well without imposing additional boundary conditions. Altogether, our results suggest that recovery is possible for many interesting cases, but that it will not work in economies that are “too close” to the classical Black-Scholes setting with positive long-term growth and unbounded marginal utility.

Our focus has mainly been theoretical. In future work we plan to address in more detail the numerical properties and practical applications of the recovery method in the setting with a diffusion process on an unbounded domain.

# Proofs

**Proof of Proposition 1:** The result follows from the standard properties of solutions to second order linear ODEs, see, e.g., Simmons (1988), pp. 72-78.

**Proof of Proposition 2: Necessity:** Assume that  $z_1$  is a strictly positive solution to (19). From (1), we know that the general solution (up to a multiplication by a constant) is on the form  $z_c = z_1 + cz_2$ , where  $z_2$  is also a solution. It is sufficient to show that any other solution,  $z_c$ ,  $c \neq 0$  must be negative at some point.

As discussed in Simmons (1988), pp. 81-83,  $z_2$  can be solved for, once  $z_1$  is known. The general solution,  $z_c$  can then be written as

$$\begin{aligned}
 z_c(x) &= z_1(x) \left( 1 + c \int_0^x \frac{1}{z_1(y)^2} e^{-\int_0^y (\frac{\mu(s)}{D(s)} + 2q(s)) ds} dy \right) \\
 &= z_1(x) \left( 1 + c \int_0^x \frac{z_1(y)^2}{z_1(y)^2} e^{-\int_0^y \frac{\mu(s)}{D(s)} ds} dy \right) \\
 &= z_1(x) \left( 1 + c \int_0^x e^{-\int_0^y \frac{\mu(s)}{D(s)} ds} dy \right) \\
 &= z_1(x)(1 + cR(x)), \tag{36}
 \end{aligned}$$

where  $R(x) \stackrel{\text{def}}{=} \int_0^x e^{-\int_0^y \frac{\mu(s)}{D(s)} ds} dy$ . Of course, the sign of  $z_c(x)$  is the same as the sign of  $1 + cR(x)$ , so strict positivity of  $z_c$  is equivalent to strict positivity of  $1 + cR(x)$ . Now,  $R(x)$  is a strictly increasing function such that  $R(0) = 0$ . If  $R(\infty) < \infty$ , then for small  $c < 0$ ,  $z_c$  is strictly positive, as is the case for small  $c > 0$ , if  $R(-\infty) > -\infty$ . In this case, recovery is not possible, even if  $\rho$  is known, since there are multiple candidate solutions that are all strictly positive, so necessity follows.

*Sufficiency:* The argument above implies that if  $R(-\infty) = -\infty$ , and  $R(\infty) = \infty$ , then recovery is possible, given that  $\rho$  is known. If we show that there are no strictly positive solutions to  $W[z|\lambda] = 0$  for  $\lambda > \rho$  in this case, then recovery follows automatically, since  $\rho$  must be the largest  $\lambda$  for which the solution to  $W[z|\lambda] = 0$  has exactly one strictly positive solution.

We transform the ODE

$$\frac{d^2 s}{dx^2} + \frac{\kappa}{D} \frac{ds}{dx} + \frac{\lambda - r}{D} s = 0 \tag{37}$$

to normal form (see Simmons (1988), p. 119-120), to get  $s = uv$ , where  $v(x) = e^{-\frac{1}{2} \int_0^x \frac{\kappa}{D} dy} = ze^{-\frac{1}{2} \int_0^x \frac{\mu}{D} dy}$ , and  $u$  is the general solution to the ODE

$$u'' + \left( \tau(x) + \frac{\lambda - \rho}{D(x)} \right) u = 0, \quad \tau \stackrel{\text{def}}{=} -\frac{1}{4} \left( \frac{\mu}{D} \right)^2 - \frac{1}{2} \frac{d}{dx} \left( \frac{\mu}{D} \right). \tag{38}$$

For  $\lambda = \rho$ , it is easy to see that the strictly positive function  $u_\rho(x) = e^{\frac{1}{2} \int_0^x \frac{\mu}{D} dy}$  solves (38), which in turn has  $u_\rho(0) = 1$ , and  $u'_\rho(0) = \frac{1}{2} \frac{\mu(0)}{D(0)}$ .

Define the function  $u_\lambda(x)$ , as the solution to (38), with parameter  $\lambda > 0$ , and initial conditions  $u_\lambda(0) = u_\rho(0)$ ,  $u'_\lambda(0) = u'_\rho(0)$ . Then, if we can show that  $u_\lambda$  has at least two roots, i.e., that there are two points,  $x_1$ , and  $x_2$ , for which  $u_\lambda(x_1) = u_\lambda(x_2) = 0$ , it follows from the Sturm comparison theorem (see Simon (2005)) that any solution to (38) has at least one root. Moreover, since  $s = uv$ , and  $v > 0$  for all  $\lambda$  and  $x$ , this in turn implies that any solution to (37) with  $\lambda > \rho$  has at least one root, and is therefore disqualified as a candidate solution for  $1/m$ . Therefore,  $z$  and  $\rho$  can be uniquely recovered. Specifically, in this case,  $z$  is the unique positive solution to  $W[z|\rho] = 0$ , and for no  $\lambda > \rho$ , is there a positive solution to  $W[s|\lambda] = 0$ .

To show that  $u_\lambda$  has at least two roots for all  $\lambda > \rho$ , we proceed as follows. We define  $w_\lambda(x) = \frac{u'_\lambda(x)}{u_\lambda(x)}$ . Since  $u$  is continuous and defined on the whole of  $\mathbb{R}$ , it must be that if  $|w_\lambda|$  tends to infinity at some finite  $x$ , then  $u_\lambda(x) = 0$ . Of course,  $w_\rho(x) = \frac{u'_\rho(x)}{u_\rho(x)} = \frac{1}{2} \frac{\mu(x)}{D(x)}$ .

From (37), it follows that

$$\frac{u''_\lambda}{u_\lambda} = \frac{u''_\rho}{u_\rho} - \frac{\lambda}{D}. \quad (39)$$

Moreover, since  $w'_\lambda = \frac{u''_\lambda}{u_\lambda} - w_\lambda^2$ , we can rewrite (39) as

$$w'_\lambda + w_\lambda^2 = w'_\rho + w_\rho^2 - \frac{\lambda}{D},$$

or

$$\begin{aligned} w'_\lambda - w'_\rho &= -(w_\lambda^2 - w_\rho^2) - \frac{\lambda}{D} \\ &= -(w_\lambda^2 + w_\rho^2 - 2w_\rho w_\lambda - 2w_\rho^2 + 2w_\rho w_\lambda) - \frac{\lambda}{D} \\ &= -(w_\lambda^2 - w_\rho^2)^2 - 2w_\rho(w_\lambda - w_\rho) - \frac{\lambda}{D}. \end{aligned}$$

Since  $w_\rho(0) = w_\lambda(0)$ , this means that if we define  $\Gamma(x) = w_\lambda - w_\rho$ ,  $\Gamma$  satisfies the following ODE:

$$\begin{aligned} \Gamma' &= -\Gamma^2 - \frac{\mu}{D}\Gamma - \frac{\lambda}{D}, \\ \Gamma(0) &= 0. \end{aligned}$$

Of course, regardless of  $\mu$  and  $D$ , the solution must satisfy  $\Gamma(x) < 0$  for all  $x > 0$ , since if  $\Gamma$  ever gets close to 0, the term  $-\frac{\lambda}{D}$  dominates the right hand side of the equation. Thus, we can assume that  $\Gamma(x_0) = -\epsilon$  for some  $x_0 > 0$ ,  $\epsilon > 0$ . Now, consider the ODE

$$\begin{aligned} \hat{\Gamma}' &= -\hat{\Gamma}^2 - \frac{\mu}{D}\hat{\Gamma}, \\ \hat{\Gamma}(x_0) &= -\epsilon. \end{aligned} \quad (40)$$

Clearly, it must be that the differential inequality  $\Gamma \leq \hat{\Gamma}$  is satisfied for all  $x \geq x_0$ , since whenever  $\hat{\Gamma} = \Gamma$ ,  $\Gamma' < \hat{\Gamma}'$ . Therefore, if  $\Gamma$  is defined for all  $x_0 \geq x$ , then so is  $\hat{\Gamma}$ . Let us assume that this is the case.

We define  $Z = -\hat{\Gamma} \geq 0$ , and we can then rewrite (40) as

$$\frac{\mu}{D} = Z - \frac{Z'}{Z},$$

which upon integration yields

$$-\int_{x_0}^y \frac{\mu(x)}{D(x)} dx = -\int_{x_0}^y Z(x) dx + [\ln(Z(x))]_{x_0}^y,$$

in turn leading to

$$e^{-\int_{x_0}^y \frac{\mu(x)}{D(x)} dx} = \frac{1}{\epsilon} Z(y) e^{-\int_{x_0}^y Z(x) dx}.$$

Let us define  $Q(x) = e^{-\int_0^x \frac{\mu(x)}{D(x)} dx}$ , so that we can write

$$\begin{aligned} \int_0^\infty Q(x) dx &= \int_0^{x_0} Q(y) dy + Q(x_0) \int_{x_0}^\infty e^{-\int_{x_0}^y \frac{\mu(x)}{D(x)} dx} \\ &= \int_0^{x_0} Q(y) dy + \frac{Q(x_0)}{\epsilon} \int_{x_0}^\infty Z(y) e^{-\int_{x_0}^y Z(x) dx} dy \\ &\leq \int_0^{x_0} Q(y) dy + \frac{Q(x_0)}{\epsilon} \\ &< \infty. \end{aligned}$$

Here, we used the inequality  $\int_a^b Z(y) e^{-\int_a^y Z(x) dx} dy = \int_a^b -\frac{d}{dy} \left( e^{-\int_a^y Z(x) dx} \right) dy = \left[ e^{-\int_a^y Z(x) dx} \right]_b^a = 1 - e^{-\int_a^b Z(x) dx} \leq 1$ , since  $Z \geq 0$ . Thus, to summarize, if  $\Gamma$  is defined on the whole of  $\mathbb{R}_+$ , then it must be that the right integral in (25) is finite. So, if (25) is infinite, it must be that  $\Gamma \rightarrow -\infty$  for some finite  $x > 0$ , in turn implying that  $u_\lambda(x) = 0$ .

An identical argument for  $x < 0$ , shows that  $u_\lambda(x) = 0$  for some finite  $x < 0$ . Thus, in line with the previous argument,  $u_\lambda$  has at least two roots, and any other solution to (37) has at least one root, when  $\lambda > \rho$ . We are done. ■

**Proof of Proposition 3:** The existence of a stationary distribution is equivalent to (31). Define  $K_N = \int_0^N \frac{1}{QD} dx < \infty$ , and consider the variational problem  $\min_{\{Q(x)\}} \int_0^N Q(x) dx$  subject to  $\int_0^N \frac{1}{QD} dx = K_N$ . The solution to this problem is

$$Q^*(y) = \frac{\sqrt{D(y)}}{K_N} \times \int_0^N \frac{1}{\sqrt{D}} dx.$$

So

$$\int_0^N Q dx \geq \int_0^N Q^* dx = \frac{1}{K_N} \int_0^N \sqrt{D} dx \times \int_0^N \frac{1}{\sqrt{D}} dx \geq \frac{1}{K_\infty} N,$$

and thus as  $N$  tends to infinity,  $\int_0^N Q dx$  must too. An identical argument holds for the negative integral,  $\int_{-N}^0 Q dx$ . ■

**Proof of Proposition 4:** From Proposition 2, we know that general recovery, and therefore recovery within  $\mathcal{B}$ , is possible if both conditions in (25) are satisfied. We therefore study the case in which exactly one condition is satisfied. We note that  $m \in \mathcal{B} \Leftrightarrow z \in \mathcal{B}$ .

*Necessity:* From the representation (36) of the general solution, it is clear that if both integrals in (25) are finite, given that  $z_1 \in \mathcal{B}$ , for small enough  $|c|$ ,  $z_c \in \mathcal{B}$ , so recovery is not possible within  $\mathcal{B}$  in this case.

*Sufficiency:* Without loss of generality, assume that the left integral in (25) is infinite, the right integral is finite, and that  $z_1 \in \mathcal{B}$  in (36). Then, because  $R(-\infty) = -\infty$  for any  $c > 0$ ,  $z_c \notin \mathcal{C}_+^2$  as the function eventually turns negative for negative  $x$ . Moreover, for  $c < 0$ ,  $z_c$  is everywhere positive but unbounded,  $\limsup_{x \rightarrow -\infty} z_c(x) = \infty$ , so  $z_c \notin \mathcal{B}$ . Thus, given  $\lambda = \rho$ , the only function in  $\mathcal{B}$  that is a candidate for the inverse of  $m$  is  $z_1$ .

For  $\lambda > \rho$ , we proceed as follows. Recall that  $Q(x) = e^{-\int_0^x \frac{\mu(y)}{D(y)} dy}$  is positive,  $R(x) = \int_0^x Q(y) dy$  is increasing, and define the limit  $K = R(\infty) < \infty$ . Moreover, define the function  $z_{\lambda, \alpha}$ , as the solution to  $W[z|\lambda] = 0$ , given initial conditions  $z_{\lambda, \alpha}(0) = 1$ ,  $z'_{\lambda, \alpha}(0) = \alpha$ . Given that  $z = 1/m$  is the correct reciprocal of  $m$ , normalized such that  $z(0) = 1$ , and that  $z'(0) = \beta$ , it is easy to verify the relationship with  $z_c$  in (36),  $z_{\rho, c+\beta} = z_c$ . Now,  $0 < C_1 \leq z \leq C_2 < \infty$ , since  $z \in \mathcal{B}$ . Defining  $\beta^* = \beta - \frac{1}{K}$ , it then follows immediately that for  $\alpha > \beta$ , for  $x \geq 0$ ,  $C_1 \leq z_{\rho, \alpha} \leq C_2(1 + (\alpha - \beta)K)$ , that for  $\beta^* < \alpha \leq \beta$ ,  $0 < C_1(1 - (\beta - \alpha)K) \leq z_{\rho, \alpha} \leq C_2$ ,  $x \geq 0$ , and that for  $\alpha \leq \beta^*$ ,  $\limsup_{x \rightarrow \infty} z_{\rho, \alpha} \leq 0$ .

Now, similar to the approach in the proof of Proposition 2, we can write

$$z_{\lambda,\alpha}(x) = z(x)\sqrt{Q(x)}u_{\lambda,\alpha}(x), \quad (41)$$

where

$$u''_{\lambda,\alpha} + \left( \tau(x) + \frac{\lambda - \rho}{D(x)} \right) u_{\lambda,\alpha} = 0, \quad u_{\lambda,\alpha}(0) = 1, \quad u'_{\lambda,\alpha}(0) = \alpha + \frac{1}{2} \frac{\mu(0)}{D(0)} - \beta. \quad (42)$$

It is easy to verify that for  $\lambda = \rho$ , the solution is  $u_{\rho,\alpha} = e^{\frac{1}{2} \int_0^x \frac{\mu}{D} dy} (1 + (\alpha - \beta)R(x))$ .

Following the proof of Proposition 2, we define  $w_{\lambda,\alpha}(x) = \frac{u'_{\lambda,\alpha}(x)}{u_{\lambda,\alpha}(x)}$ , which is well defined as long as  $u_{\lambda,\alpha}(x) > 0$ . We then have  $w_{\rho,\alpha}(x) = \frac{u'_{\rho,\alpha}(x)}{u_{\rho,\alpha}(x)} = \frac{1}{2} \frac{\mu(x)}{D(x)} + \frac{d}{dx} (\ln(1 + (\alpha - \beta)R(x))) = \frac{1}{2} \frac{\mu(x)}{D(x)} + \frac{(\alpha - \beta)Q(x)}{1 + (\alpha - \beta)R(x)}$ . Similar steps as in the proof of Proposition 2 lead to

$$\begin{aligned} \Gamma' &= -\Gamma^2 - A(x)\Gamma - \frac{\lambda}{D}, \\ \Gamma(0) &= 0, \end{aligned}$$

where  $\Gamma(x) \stackrel{\text{def}}{=} w_{\lambda,\alpha}(x) - w_{\rho,\alpha}(x)$ , and  $A(x) = \left( \frac{\mu(x)}{D(x)} + 2 \frac{d}{dx} (\ln(1 + (\alpha - \beta)R(x))) \right)$ .

As before, the solution must satisfy  $\Gamma(x) < 0$  for all  $x > 0$ , since if  $\Gamma$  ever gets close to 0, the term  $-\frac{\lambda}{D}$  dominates the right hand side of the equation. This, means that we can immediately rule out any  $z_{\lambda,\alpha}$  for  $\alpha \leq \beta^*$  as candidate solutions, since as long as  $u_{\rho,\alpha} > 0$  and  $u_{\lambda,\alpha} > 0$ ,

$$0 > \int_0^x \Gamma(y) dy = \int_0^x \left( \frac{u'_{\lambda,\alpha}(y)}{u_{\lambda,\alpha}(y)} - \frac{u'_{\rho,\alpha}(y)}{u_{\rho,\alpha}(y)} \right) dy = \ln(u_{\lambda,\alpha}(x)) - \ln(u_{\rho,\alpha}(x)),$$

in turn implying that  $u_{\lambda,\alpha}(x) < u_{\rho,\alpha}(x)$ . As long as both  $z_{\lambda,\alpha}$ , and  $z_{\rho,\alpha}$  are positive, via (41) we have  $\frac{z_{\lambda,\alpha}}{z_{\rho,\alpha}} = \frac{u_{\lambda,\alpha}}{u_{\rho,\alpha}}$ , so this means that  $z_{\lambda,\alpha} < z_{\rho,\alpha}$ . Since  $\limsup_{x \rightarrow \infty} z_{\rho,\alpha} \leq 0$  when  $\alpha \leq \beta^*$ , it must either be that  $z_{\lambda,\alpha}$  reaches zero for a finite  $x$ , or approaches zero as  $x$  tends to infinity, in both cases disqualifying  $z_{\lambda,\alpha}$  as a candidate function in  $\mathcal{B}$ .

It remains to be shown that  $z_{\lambda,\alpha} \notin \mathcal{B}$  when  $\alpha > \beta^*$  and  $\lambda > \rho$ . In this case,  $A(x)$  is well defined for all  $x > 0$ . Of course, if  $u_{\lambda,\alpha} = 0$  for some  $x > 0$ , then  $z_{\lambda,\alpha} \notin \mathcal{B}$ , so we assume that  $u_{\lambda,\alpha} > 0$ . As in the proof of Proposition 2, we can assume that  $\Gamma(x_0) = -\epsilon$  for some  $x_0 > 0$ ,  $\epsilon > 0$ . Now, assume that for  $x \geq x_0$ ,  $A(x)$  satisfies the bound  $A(x) \leq Cx$ , for some  $C < \infty$ . Define  $\xi = \frac{\lambda}{\inf_x D(x)} > 0$ , and consider the ODE

$$\begin{aligned} \hat{\Gamma}' &= -Cx\hat{\Gamma} - \xi, \\ \hat{\Gamma}(x_0) &= -\epsilon. \end{aligned} \quad (43)$$

It must be that  $\Gamma \leq \hat{\Gamma}$  for all  $x \geq x_0$ , since whenever  $\hat{\Gamma} = \Gamma$ ,  $\Gamma' < \hat{\Gamma}'$  (similar to the argument in the proof of Proposition 2). Now, the solution to (43) is

$$\hat{\Gamma}(x) = -e^{-\frac{Cx^2}{2}} \left( \epsilon + \xi \sqrt{\frac{2\pi}{4C}} \text{Erfi} \left( \sqrt{\frac{C}{2}} x \right) \right),$$

and it is easy to verify that  $\hat{\Gamma}(x) = -\frac{\xi}{Cx} + O(x^{-2})$  for large  $x$ , and thus that  $\int_{x_0}^y \hat{\Gamma}(x) dx$  tends to  $-\infty$  as  $y$  grows. Since  $\Gamma \leq \hat{\Gamma}$ , it must be that  $\int_{x_0}^y \Gamma(x) dx$  tends to  $-\infty$  too. But,  $\int_{x_0}^y \Gamma(x) dx = \ln \left( \frac{u_{\lambda,\alpha}(y)}{u_{\rho,\alpha}(y)} \right) - \ln \left( \frac{u_{\lambda,\alpha}(x_0)}{u_{\rho,\alpha}(x_0)} \right)$ , so this implies that  $\frac{u_{\lambda,\alpha}(y)}{u_{\rho,\alpha}(y)} \rightarrow 0$ , as  $y$  grows. Now, since  $z_{\lambda,\alpha} > 0$  and  $z_{\rho,\alpha} > 0$ ,  $\frac{z_{\lambda,\alpha}}{z_{\rho,\alpha}} = \frac{u_{\lambda,\alpha}}{u_{\rho,\alpha}}$ . Moreover,  $z_{\rho,\alpha}(x) \leq C_2(1 + (\alpha - \beta)K) < \infty$ .

It must therefore be that  $z_{\lambda,\alpha}(x) \rightarrow 0$  for large  $x$ , so  $z_{\lambda,\alpha} \notin \mathcal{B}$ .

The only part remaining is to show that  $A(x) \leq Cx$  for  $x \geq x_0$ , for some constant  $C < \infty$ . We have

$$A(x) = \frac{\mu(x)}{D(x)} + 2 \frac{(\alpha - \beta)Q(x)}{1 + (\alpha - \beta)R(x)}.$$

Since, per assumption,  $D(x) \geq C_2^2/2 > 0$ , and  $\mu(x) \leq C_1(1+x) \leq C_1(x_0^{-1}+1)x = C'_1x$ , it follows that such a bound exists for the first term,  $\frac{\mu(x)}{D(x)} \leq Cx$ . For the second term, the denominator is bounded below by a strictly positive constant, since  $\alpha > \beta^*$ . Therefore, as long as  $\limsup_{x \rightarrow \infty} \frac{Q(x)}{x} < \infty$ , the second term can also be bounded by  $Cx$  for  $x \geq x_0$ .

Intuitively, since  $\int_0^\infty Q(x)dx < \infty$ , it should not be possible for  $\frac{Q(x)}{x}$  to be large infinitely often. This intuition can be formalized as follows. Since the integral of  $Q$  is finite,  $Q(x) \leq \frac{C'}{x}$  infinitely often for any constant  $C' > 0$ . Now, assume that also  $Q(x) = C'x$  infinitely often, for some  $C' > 0$ . Then, consider a large  $x_1$ , such that  $Q(x_1) = C'x_1$ , and an even larger  $x_2 = x_1 + \delta$ , such that  $Q(x_2) = \frac{C'}{x_2}$ . Since  $\frac{C'}{x_1} = Q(x_1 + \delta) = Q(x_1)e^{-\int_{x_1}^{x_1+\delta} \frac{\mu}{D} ds} = C'x_1 e^{-\int_{x_1}^{x_1+\delta} \frac{\mu}{D} ds}$ , it follows that  $\int_{x_1}^{x_1+\delta} \frac{\mu}{D} ds = 2 \ln(x_1)$ , and since  $\frac{\mu(s)}{D(s)} \leq cs$ , that  $\delta \geq \frac{2 \ln(x_1)}{cx_1} \left(1 - \frac{k}{x_1^2}\right)$ , for some constant,  $k > 0$ .

We now have

$$\begin{aligned} \int_{x_1}^{x_2} Q(x)dx &= \int_{x_1}^{x_1+\delta} Q(x)dx \\ &= \int_{x_1}^{x_1+\delta} e^{-\int_0^y \frac{\mu}{D} ds} dy \\ &= \int_{x_1}^{x_1+\delta} Q(x_1) e^{-\int_{x_1}^y \frac{\mu}{D} ds} dy \\ &= C'x_1 \int_{x_1}^{x_1+\delta} e^{-\int_{x_1}^y \frac{\mu}{D} ds} dy \\ &\geq C'x_1 \int_{x_1}^{x_1+\delta} e^{-\int_{x_1}^y cs ds} dy \\ &= C'x_1 \int_{x_1}^{x_1+\delta} e^{-\frac{c}{2}(y^2 - x_1^2)} dy \\ &= C'x_1 \int_0^\delta e^{-cx_1p - cp^2/2} dp \\ &\geq C''x_1 \int_0^{\frac{2 \ln(x_1)}{cx_1} \left(1 - \frac{k}{x_1^2}\right)} e^{-cx_1p - cp^2/2} dp \\ &\geq C'''x_1 \int_0^{\frac{2 \ln(x_1)}{cx_1} \left(1 - \frac{k}{x_1^2}\right)} e^{-cx_1p} dp \\ &= C'''x_1 \left[ \frac{e^{-cx_1p}}{cx_1} \right]_0^{\frac{2 \ln(x_1)}{cx_1} \left(1 - \frac{k}{x_1^2}\right)} \\ &\geq \frac{C'''}{c} \left( 1 - x_1^{-2 \left(1 - \frac{k}{x_1^2}\right)} \right) \\ &\geq C'''' > 0. \end{aligned}$$

Thus, every time  $Q(x)$  reaches  $C'x_1$ , the contribution to  $R(x)$  on the subsequent interval,  $[x_1, x_1 + \delta]$ , over which  $Q(x)$  decreases to  $\frac{C'}{x_1}$  is bounded below by a strictly positive constant,  $C''''$ , and if there are infinitely many such intervals it must then be that  $R(\infty) = \infty$ , contradicting the assumption that  $R(\infty)$  is finite. Therefore,  $\frac{Q(x)}{x} \rightarrow 0$ , for large  $x$ , in



turn implying that  $A(x) \leq Cx$ , and that, in extension,  $\limsup_{x \rightarrow \infty} z_{\lambda, \alpha}(x) \geq 0$  for  $\alpha > \beta^*$ . This completes the proof.  $\blacksquare$

**Proof of Proposition 5:** Following Dupire, we study the time  $t$  price of a call option with strike price  $K$ , Thus,  $C^t(K)$  is the time zero price of the option that pays  $(X_t - K)_+$  at time  $t$ . Differentiating leads to

$$C_K^t(K) = - \int_K^\infty e^{-\rho t} \frac{m(y)}{m(x)} f^t(x, y) dy, \quad (44)$$

$$C_{KK}^t(K) = e^{-\rho t} \frac{m(K)}{m(x)} f^t(x, K). \quad (45)$$

We now use (35,7,45) to write

$$\begin{aligned} C_t^t(K) = \frac{\partial C^t(K)}{\partial t} &= -\rho C^t(K) + \int_K^\infty (y - K) e^{-\rho t} \frac{m(y)}{m(x)} \frac{\partial f^t(x, y)}{\partial t} dy \\ &= -\rho C^t(K) + \int_K^\infty (y - K) e^{-\rho t} \frac{m(y)}{m(x)} \left[ -\frac{\partial}{\partial y} (\mu(y) f^t(x, y)) + \frac{\partial^2}{\partial y^2} \left( \frac{\sigma^2(y)}{2} f^t(x, y) \right) \right] dy \\ &\stackrel{I.B.P.}{=} -\rho C^t(K) - \left[ e^{-\rho t} (y - K) \frac{m(y)}{m(x)} \mu(y) f^t(x, y) \right]_K^\infty + \int_K^\infty e^{-\rho t} \frac{m(y)}{m(x)} \mu(y) f^t(x, y) dy \\ &\quad + \int_K^\infty (y - K) e^{-\rho t} \frac{m'(y)}{m(x)} \mu(y) f^t(x, y) dy + \left[ (y - K) e^{-\rho t} \frac{m(y)}{m(x)} \frac{\partial}{\partial y} \left( \frac{\sigma^2(y)}{2} f^t(x, y) \right) \right]_K^\infty \\ &\quad - \int_K^\infty e^{-\rho t} \frac{m(y)}{m(x)} \frac{\partial}{\partial y} \left( \frac{\sigma^2(y)}{2} f^t(x, y) \right) dy - \int_K^\infty (y - K) e^{-\rho t} \frac{m'(y)}{m(x)} \frac{\partial}{\partial y} \left( \frac{\sigma^2(y)}{2} f^t(x, y) \right) dy \\ &= -\rho C^t(K) + \int_K^\infty e^{-\rho t} \frac{m(y)}{m(x)} \mu(y) f^t(x, y) dy + \int_K^\infty (y - K) e^{-\rho t} \frac{m'(y)}{m(x)} \mu(y) f^t(x, y) dy \\ &\quad - \int_K^\infty e^{-\rho t} \frac{m(y)}{m(x)} \frac{\partial}{\partial y} \left( \frac{\sigma^2(y)}{2} f^t(x, y) \right) dy - \int_K^\infty (y - K) e^{-\rho t} \frac{m'(y)}{m(x)} \frac{\partial}{\partial y} \left( \frac{\sigma^2(y)}{2} f^t(x, y) \right) dy. \end{aligned}$$

The derivation above depends the decay of  $f^t(x, y)m(y)$  as  $y$  approaches  $\pm\infty$ , allowing for integration by parts. Specifically, for a fixed  $t > 0$ ,  $f^t(x, y)m(y)$  approaches zero as  $y$  tends to infinity, for all  $x$ . This condition must be satisfied for expected utility to be well defined, as we already assumed earlier.

Taking the derivative with respect to  $K$ , we get

$$\begin{aligned} C_{tK}^t(K) &= -\rho C_K^t(K) - e^{-\rho t} \frac{m(K)}{m(x)} \mu(K) f^t(x, K) - \int_K^\infty e^{-\rho t} \frac{m'(y)}{m(x)} \mu(y) f^t(x, y) dy \\ &\quad + e^{-\rho t} \frac{m(K)}{m(x)} \frac{\partial}{\partial K} \left( \frac{\sigma^2(K)}{2} f^t(x, K) \right) + \int_K^\infty e^{-\rho t} \frac{m'(y)}{m(x)} \frac{\partial}{\partial y} \left( \frac{\sigma^2(y)}{2} f^t(x, y) \right) dy \\ &= -\rho C_K^t(K) - \mu(K) C_{KK}^t(K) - \int_K^\infty e^{-\rho t} \frac{m'(y)}{m(x)} \mu(y) f^t(x, y) dy \\ &\quad + m(K) \frac{\partial}{\partial K} \left( \frac{\sigma^2(K)}{2m(K)} C_{tK}^t(K) \right) + \int_K^\infty e^{-\rho t} \frac{m'(y)}{m(x)} \frac{\partial}{\partial y} \left( \frac{\sigma^2(y)}{2} f^t(x, y) \right) dy. \end{aligned}$$

Taking the derivative with respect to  $K$  again, we get

$$\begin{aligned}
C_{tKK}^t(K) &= -\rho C_{KK}(K) - \frac{\partial}{\partial K} [\mu(K)C_{KK}^t(K)] + \frac{m'(K)}{m(K)}\mu(K)C_{KK}^t(K) \\
&+ \frac{\partial}{\partial K} \left[ m(K) \frac{\partial}{\partial K} \left( \frac{\sigma^2(K)}{2m(K)} C_{KK}^t(K) \right) \right] - m'(K) \frac{\partial}{\partial K} \left( \frac{\sigma^2(K)}{2m(K)} C_{KK}^t(K) \right) \\
&= -\rho C_{KK}(K) - \frac{\partial}{\partial K} [\mu(K)C_{KK}^t(K)] + \frac{m'(K)}{m(K)}\mu(K)C_{KK}^t(K) + m(K) \frac{\partial^2}{\partial K^2} \left( \frac{\sigma^2(K)}{2m(K)} C_{KK}^t(K) \right).
\end{aligned} \tag{46}$$

Expanding the the differential operators, we get

$$\begin{aligned}
C_{tKK}^t &= -\rho C_{KK}^t - \mu' C_{KK}^t - \mu C_{KKK}^t + z\mu C_{KK}^t \\
&+ m \left( \frac{\sigma^2}{2m} C_{KKKK}^t + 2 \left( \frac{\sigma\sigma'}{m} - \frac{\sigma^2 m'}{2m^2} \right) C_{KKK}^t + \left( \frac{\sigma\sigma'' + (\sigma')^2}{m} - 2\frac{\sigma\sigma' m'}{m^2} - \frac{\sigma^2 m''}{2m^2} + \frac{\sigma^2 (m')^2}{m^3} \right) C_{KK}^t \right) \\
&= -\rho C_{KK}^t - \mu' C_{KK}^t - \mu C_{KKK}^t + q\mu C_{KK}^t \\
&+ \frac{\sigma^2}{2} C_{KKKK}^t + 2 \left( \sigma\sigma' - \frac{\sigma^2}{2} q \right) C_{KKK}^t + \left( \sigma\sigma'' + (\sigma')^2 - 2\sigma\sigma'q - \frac{\sigma^2}{2} q' + \frac{\sigma^2}{2} q^2 \right) C_{KK}^t \\
&= \frac{\sigma^2}{2} C_{KKKK}^t + (2\sigma\sigma' - \sigma^2 q - \mu) C_{KKK}^t + \left( \sigma\sigma'' + (\sigma')^2 - 2\sigma\sigma'q - \frac{\sigma^2}{2} q' + \frac{\sigma^2}{2} q^2 + q\mu - \mu' - \rho \right) C_{KK}^t \\
&= DC_{KKKK}^t + \alpha_1 C_{KKK}^t + \alpha_0 C_{KK}^t.
\end{aligned}$$

We define  $V(t, K) = C_{KK}^t = e^{-\rho t} \frac{m(K)}{m(x_0)} f^t(x_0, K) = p^t(x_0, K)$ , and can then write

$$V_t = DV_{KK} + \alpha_1 V_K + \alpha_0 V.$$

This equation can be used to solve for  $D$ ,  $\alpha_1$ , and  $\alpha_0$  for a specific  $K$ , by observing  $V_t$ ,  $V$ ,  $V_K$ , and  $V_{KK}$  for 3 different values of  $t$ , and solving a set of 3 linear equations with 3 unknowns.

We identify coefficients, to get

$$\alpha_1 = 2D' - \kappa, \tag{47}$$

$$\begin{aligned}
\alpha_0 &= \frac{\sigma^2}{2} (q^2 - q') + q\mu - \mu' - 2\sigma\sigma'q - \rho \\
&= -r - (\sigma^2 q' + \mu' + 2\sigma\sigma'q) \\
&= -r - \kappa'.
\end{aligned} \tag{48}$$

Thus, from knowledge about  $C^t(K)$ , or, equivalently  $p^t(x_0, y)$ , we can calculate  $D(y)$ ,  $r(y)$ , and  $\kappa(y)$  for arbitrary  $y$ . We are done.  $\blacksquare$

**Proof of Proposition 6:** We will use a specific parametrization of the general independent solutions,  $z_\lambda^i$ ,  $i = 1, 2$ ,

$\lambda \geq \rho$ , to the ODE  $\mathcal{W}[z|\lambda] = 0$ . We define  $z_\lambda^i$  to be the solution to

$$\begin{aligned} W[z_\lambda^i] &= 0, \\ z_\lambda^i(0) &= z_{\rho,i}(0), \\ z_\lambda^{i'}(0) &= z'_{\rho,i}(0), \end{aligned}$$

where  $z_{\rho,1}$ , as before, is the strictly positive solution to  $\mathcal{W}[z|\rho] = 0$ , and  $z_{\rho,2}$  is another solution, which given that recovery is possible is chosen to be zero and increasing at  $x = 0$ . Finally, define the general solution  $z_{\lambda,c} = cz_\lambda^1 + (1-c)z_\lambda^2$ ,  $c \in [0, 1]$ ,  $\lambda \geq \rho$ . It follows from standard properties of linear second order ODEs that for any  $x$ ,  $z_{\lambda,c}(x)$  depends continuously on  $\lambda$  and  $c$  (see, e.g., Simmons (1988)).

The correct  $z = \frac{1}{m}$ , is then the only positive function,  $z_{\rho,1}$ , whereas  $z_{\rho,c}(x) = 0$  for some  $x$ , if either  $c \neq 1$ , or  $\lambda > \rho$ . We are interested in how strong inferences we can draw about  $z$  from observing  $D$ ,  $\kappa$ , and  $r$  on the domain  $[-N, N]$ . Candidate  $z$ 's are then solutions  $z_{\lambda,c}$ , that are strictly positive on  $[-N, N]$ .

We define  $N_{\lambda,c} \stackrel{\text{def}}{=} \inf\{|x| : z_{\lambda,c} = 0\}$ . It follows that if  $N_{\lambda,c} \leq N$ ,  $z_{\lambda,c}$  cannot be a candidate  $z$ , since it is not strictly positive on the observable domain. The following properties of  $N_{\lambda,c}$  follow:

1. For  $\lambda = \rho$ ,  $N_{\rho,c}$  is continuous and strictly increasing in  $c$ , for  $0 \leq c < 1$ . Moreover,  $N_{\rho,1} = \infty$ . This follows from Proposition 2, and the definition of  $z_{\rho,c}$  as a linear combination of the strictly positive  $z_{\rho,1}^1$ , and  $z_{\rho,2}$  which has exactly one root.
2. For  $(\lambda, c) \neq (\rho, 1)$ ,  $N_{\lambda,c}$  is continuously differentiable in  $\lambda$ , and  $\frac{dN_{\lambda,c}}{d\lambda} < 0$ . This follows from the Sturm comparison theorem, see e.g., Simon (2005).
3. For  $\lambda > \rho$ ,  $N_{\lambda,c}$  is a continuous function of  $c \in [0, 1]$ , and therefor also bounded,  $R_\lambda \stackrel{\text{def}}{=} \sup_c N_{\lambda,c} < \infty$ .
4.  $R_\lambda$  is nonincreasing in  $\lambda$ . This follows directly from point 2. above.

Point 3. follows from the following argument: From the proof of Proposition 2 it follows that  $z_{\lambda,1}$  has at least two roots for any  $\lambda > \rho$ , one for  $x$  less than zero, and one for  $x$  greater than zero. Let us call these two roots  $v_1 < 0$  and  $v_2 > 0$ . From the Sturm separation theorem (see, Simmons (1988), p. 118), it follows that  $z_{\lambda,0}$  has exactly one root in  $(v_1, v_2)$ , which from the construction of  $z_{\lambda,0}$  in Proposition 2, lies at  $x = 0$ . Moreover,  $z_{\lambda,c}$  has exactly one root in  $(v_1, 0)$ , for  $0 < c < 1$ . We denote this root by  $v_1(c)$ . Clearly, if we define  $c_1(x) = \frac{z_{\lambda,0}(x)}{z_{\lambda,0}(x) - z_{\lambda,1}(x)}$ , for  $x \in [v_1, 0]$ , we have  $c_1(x)z_{\lambda,1}(x) + (1 - c_1(x))z_{\lambda,0}(x) = 0$ , i.e.,  $v_1(c_1(x)) = x$ . Now,  $c_1$  is continuous,  $c_1(v_1) = 1$ ,  $c_1(0) = 0$ , and  $\frac{dc_1(x)}{dx} = \frac{1}{(z_{\lambda,0}(x) - z_{\lambda,1}(x))^2} (z'_{\lambda,0}(x)z_{\lambda,1}(x) - z'_{\lambda,1}(x)z_{\lambda,0}(x))$ . Since the Wronskian,  $z'_{\lambda,0}(x)z_{\lambda,1}(x) - z'_{\lambda,1}(x)z_{\lambda,0}(x) \neq 0$  (see, Simmons (1988)), it follows that  $c_1(x)$  is strictly decreasing on  $[v_1, 0]$ , and therefore its inverse,  $v_1(c)$  is a continuous function on  $c \in [0, 1]$ . If  $|v_1| \leq v_2$ , then clearly  $N_{\lambda,c} = |v_1(c)|$  but if  $|v_1| > v_2$ , we must also consider a potential root to the right of  $v_2$  as a candidate for being closest to zero. If  $z_{\lambda,0}$  has a root at  $x = v_3 > v_2$ , then an identical argument as that above can be made to infer that there is a unique root of  $z_{\lambda,c}$ ,  $v_3(c) \in (v_2, v_3]$ , for all  $c \in [0, 1]$ , which decreases continuously in  $c$ . If  $z_{\lambda,0}$  has no such root to the right of  $v_2$ , then neither does  $z_{\lambda,1}$  (again by the Sturm separation theorem). In this case it follows that  $c_2(x) \stackrel{\text{def}}{=} \frac{z_{\lambda,0}(x)}{z_{\lambda,0}(x) - z_{\lambda,1}(x)}$ , for  $x \geq v_2$  is a continuous, strictly decreasing (because of the nonzero Wronskian) function, and that its inverse  $v_2(c)$  can be defined on  $c \in [c_2(|v_1|), 1]$ . The function  $v_2(c)$  can then be continuously extended to the domain  $c \in [0, 1]$ , so that for  $0 \leq c < c_2(|v_1|)$ ,  $v_2(c) = v_2(c_2(|v_1|))$ . It now follows that  $N_{\lambda,c} = \min(v_1(c), v_2(c))$  is also continuous in  $c \in [0, 1]$ . Since the domain of  $c$ ,  $[0, 1]$ , is compact, boundedness of  $R_\lambda$  follows immediately. We also define  $A_\lambda = \{c : N_{\lambda,c} = R_\lambda\}$ , and note that  $A_\lambda$  must be nonempty, again since  $N_{\lambda,c}$  is continuous in  $c$ .

The results above are sufficient to imply that as  $N$  grows, the set of candidate functions both over  $c$  and  $\lambda$  shrinks so that ultimately only  $z = \frac{1}{m}$  remains. Specifically, define  $\mathcal{G}_N = \{(\lambda, c) : z_{\lambda,c}(x) > 0, |x| \leq N\}$ . This set contains

the candidate  $z$ -functions, given that  $D$ ,  $r$ , and  $\kappa$  are observed on  $[-N, N]$ . Clearly,  $\mathcal{G}_{N'} \subset \mathcal{G}_N$ , for  $N' > N$ , and from Proposition 1,  $\mathcal{G}_\infty = \{(\rho, 1)\}$ . We wish to show that  $\mathcal{G}_N$  converges to  $\mathcal{G}_\infty$  as  $N \rightarrow \infty$ .

Define

$$\begin{aligned} c_N &= \inf\{c : N_{\rho,c} \geq N\}, \\ \lambda_N &= \inf\{\lambda : R_\lambda \geq N\}. \end{aligned}$$

It then follows immediately that  $\mathcal{G}_N \subset [c_N, 1] \times [\rho, \lambda_N]$ . Moreover, Proposition 2 implies that  $\lim_{N \rightarrow \infty} c_N = 1$ , and  $\lim_{N \rightarrow \infty} \lambda_N = \rho$ , since otherwise there would be other strictly positive solutions to the fundamental ODE with  $\lambda \geq \rho$ . Thus,  $\lim_{N \rightarrow \infty} \bigcap_{n=1}^N \mathcal{G}_n = \mathcal{G}_\infty = \{(\rho, 1)\}$ , as claimed.

The results in the proposition follow immediately. By choosing  $\hat{\rho} = \lambda_N$  (which of course is observable, given that  $D$ ,  $\kappa$ , and  $r$  are on  $[-N, N]$ ), we get the first result. Next, we choose a  $\hat{z}_N = z_{\hat{\rho}_N, w_N}$ , and  $\hat{m}_N = \frac{1}{\hat{z}_N}$ , where  $w_N \in A_{\hat{\rho}_N}$ . Since  $z_{\lambda,c}(x)$  depends continuously on  $\lambda$  and  $c$ , which converge to  $\hat{\rho}$  and 1, respectively, as  $N$  tends to infinity, it follows that  $\lim_{N \rightarrow \infty} z_{\hat{\rho}_N, w_N}(x) = z(x)$  for any  $x$ , and since  $z$  is strictly positive, also that  $\hat{m}_N(x) \rightarrow m(x)$ . We are done. ■

## Matlab code for recovery algorithm in Section 4

The results in Section 4 were based on the following Matlab code, which approximates the pricing kernel and personal discount rate from  $D$ ,  $r$ , and  $\kappa$ .

```
% Filename: Recovery.m
% By Johan Walden, November 7, 2013
% Recovery method for diffusion process
% Described in: "Recovery with diffusions on unbounded domains"
% Original method with finite state space described in Ross (2013)
%
% Input:
% dx: stepsize (e.g. 1E-4)
% rhomax: Assumed maximum possible personal discount rate
% NoSteps: Number of iterations (e.g., 30)
% D: Vector of D values [D(0), D(dx), ..., D(N*dx)];
% r: Vector of r values [r(0), r(dx), ..., r(N*dx)];
% k: Vector of kappa values [kappa(0), kappa(dx), ..., kappa(N*dx)];
%
% Output:
% rho: Approximate personal discount rate
% m: Vector of approximate marginal utility [m(0), m(dx), ..., m(N*dx)];
function [rho,m]=Recovery(dx,rhomax,NoSteps,D,r,k)
rhomin=0; %Lower bound on personal discount rate
N=length(D);
zapp=zeros(N,1);
FoundPositive=0;
for n=1:NoSteps % Iterate over conjectured discount rate
    rho=(rhomax+rhomin)/2; %Conjectured rho
    %Solve ODEs
    z=zeros(N,2); %Two solutions
    Mid=floor(N/2);
    z(Mid-1:Mid+1,1)=[1,1,1]; %Solution with initial condition z'=0;
    z(Mid-1:Mid+1,2)=[1-dx,1,1+dx]; %Solution with initial condition z'=1;
    for j=Mid+1:N-1
        vj=k(j)*dx/(2*D(j));
        z(j+1,:) = 1/(1+vj)*((2-dx^2*(rho-r(j)))/D(j))*z(j,:)-(1-vj)*z(j-1,:);
    end
    for j=Mid-1:-1:2
        vj=k(j)*dx/(2*D(j));
        z(j-1,:) = 1/(1-vj)*((2-dx^2*(rho-r(j)))/D(j))*z(j,:)-(1+vj)*z(j+1,:);
    end
    % Check number of roots of solutions to infer new rho
    Roots=sum(z(2:N,:).*z(1:N-1,:)<= 0);
    if (Roots(1)>1 || Roots(2)>1) % Too high rho, since multiple roots
        rhomax=rho;
    elseif (Roots(1)==0) %Too low rho (weakly), since positive solution
        rhomin=rho;
        zapp=z(:,1); %Update approximate kernel
        FoundPositive=1;
    elseif (Roots(2)==0)
        rhomin=rho;
        zapp=z(:,2); %Update approximate kernel
    else %No solution with two roots, at least one with one, check for linear combination
        A1=angle(z(:,1)+i*z(:,2));
        A2=angle(-(z(:,1)+i*z(:,2))); %Rotate angle by pi
        if ((max(A1)-min(A1) < pi) %Positive possible
            rhomin=rho;
            A=1/2*(max(A1)+min(A1));
            zapp=cos(A)*z(:,1)+sin(A)*z(:,2);
            FoundPositive=1;
        elseif (max(A2)-min(A2) < pi) %Positive possible
            rhomin=rho;
            A=1/2*(max(A2)+min(A2))+pi;
            zapp=cos(A)*z(:,1)+sin(A)*z(:,2);
            FoundPositive=1;
        else
            rhomax=rho;
        end
    end
end;
end
if(FoundPositive==0)
    disp('Did not find a positive kernel')
end;
m=1./zapp;
```

## References

- Alvarez, F., and J. Jerman, 2005, Using asset prices to measure the persistence of the marginal utility of wealth, *Econometrica* 73, 1977–2016.
- Barro, R., 2005, Rare events and the equity premium, Working Paper, Harvard University.
- Carr, P., and J. Yu, 2012, Risk, return, and Ross recovery, *Journal of Derivatives* 20, 38–59.
- Dubynskiy, A., and R. S. Goldstein, 2013, Recovering drifts and preference parameters from financial derivatives, Working Paper, University of Minnesota.
- Dupire, B., 1994, Pricing with a smile, *Risk* 7, 18–20.
- Gradsteyn, L. S., and I. M. Ryzhik, 2000, *Tables of Integrals, Series and Products* (Academic Press).
- Hansen, L. P., 2012, Dynamic valuation decomposition within stochastic economies, *Econometrica* 80, 911–987.
- Hansen, L. P., and J. A. Scheinkman, 2009, Long-term risk: An operator approach, *Econometrica* 77, 177–234.
- Hansen, L. P., and J. A. Scheinkman, 2013, Stochastic compounding and uncertain valuation, Working Paper, Princeton University.
- Ibragimov, R., 2009, Heavy-tailed densities, in S. N. Durlauf, and L. E. Blume, eds., *The New Palgrave Dictionary of Economics* (MacMillan).
- John, F., 1991, *Partial differential equations* (Springer-Verlag).
- Kogan, L., S. Ross, J. Wang, and M. Westerfield, 2006, The price impact and survival of irrational traders, *Journal of Finance* 61, 195–229.
- Martin, I., and S. Ross, 2013, The long bond, Working Paper, Stanford GSB.
- Parlour, C., R. Stanton, and J. Walden, 2011, Revisiting asset pricing puzzles in an exchange economy, *Review of Financial Studies* 24, 629–674.
- Rietz, T. A., 1989, The equity premium: A solution, *Journal of Monetary Economics* 22, 117–131.
- Ross, S., 2013a, The recovery theorem, *Journal of Finance*, forthcoming.
- Ross, S., 2013b, The recovery theorem in a continuous space model, Working Paper, MIT.
- Simmons, G., 1988, *Differential Equations* (McGraw Hill).
- Simon, B., 2005, Sturm oscillation and comparison theorems, in W. O. Amrein, A. M. Hinz, and D. B. Pearson, eds., *Sturm-Liouville Theory: Past and Present* (Birkhäuser Verlag, Basel, Switzerland).

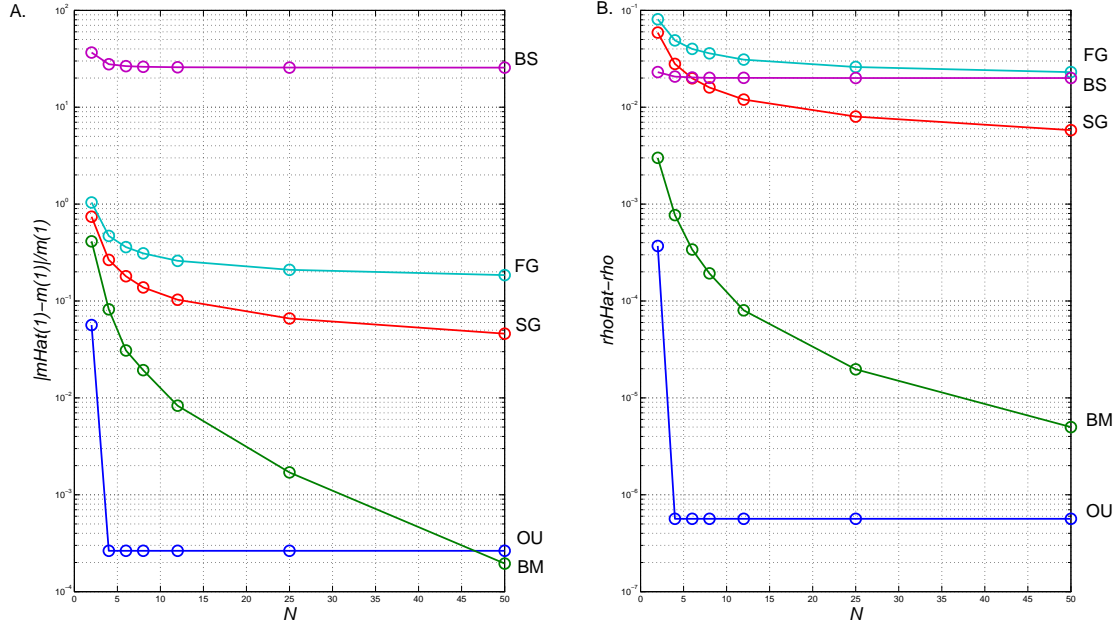


Figure 6: **Approximate recovery for different diffusion processes.** The processes are **1. Ornstein-Uhlenbeck (OU)**,  $dX = -0.01dt + 0.1d\omega$ , **2. Brownian motion (BM)**,  $dX = 0.01d\omega$ , **3. slow growth (SG)**,  $dX = 0.05dt + \sqrt{0.1(1 + X^2)}d\omega$ , **4. fast growth (FG)**,  $dX = 0.1dt + \sqrt{0.1(1 + X^2)}d\omega$ , and **5. Black Scholes (BS)**,  $dX = 0.01dt + 0.1d\omega$ . Processes 1-3 satisfy the conditions for recovery, whereas processes 4 and 5 do not. The left panel (A.) shows the relative error of the approximated  $\hat{m}$  at  $X = 1$ ,  $\frac{|\hat{m}(1) - m(1)|}{m(1)}$ , as a function of the interval  $-[N, N]$  observed, for the five processes. The right panel (B.) shows the error in the approximate personal discount rate,  $\hat{\rho} - \rho$ . The pricing kernel  $m(x) = 1 + x^2$  is used, and the personal discount rate is  $\rho = 0.01$ . A small step-length of  $\Delta x = 10^{-4}$  is used, to focus on the error introduced by bounded observations. In both panels, the convergence to the correct solution is fast for 1. and 2., no convergence occurs for 5., and it is unclear from the figure whether convergence occurs for 3. and 4.