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# Recovery with Unbounded Diffusion Processes*

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## Abstract

We analyze the problem of recovering the pricing kernel and real probability distribution from observed option prices, when the state variable is an unbounded diffusion process. We derive necessary and sufficient conditions for recovery. In the general case, these conditions depend on the properties of the diffusion process, but not on the pricing kernel. We also show that the same conditions determine whether recovery works in practice, when the continuous problem is approximated on a bounded or discrete domain without further specification of boundary conditions. Altogether, our results suggest that recovery is possible for many interesting diffusion processes on unbounded domains.

**JEL classification:** : G12, G13

**Keywords:** Recovery theorem, Ross recovery, Asset pricing

## 1. Introduction

In a remarkable paper, Ross (2015) shows that it is possible to recover the pricing kernel and real probabilities from prices of contingent claims alone, contrary to what has long been the common belief. The result relies on two insights: first that under so-called transition independence, observed prices link the pricing kernel across states; second that the positivity of the pricing kernel provides important additional restrictions. The two effects together allow for unique recovery in Ross's model.

Such information about preferences and risk in the market obtained by recovery would of course be highly valuable to investors, policy makers, and society in general, and it is therefore of fundamental importance to understand under which conditions recovery works. The state space in Ross (2015) is finite in contrast to many work-horse models in finance, for example, models in continuous time with diffusion processes. It is an open

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question if, and if so when, recovery works in a setting with a larger, unbounded state space. This question is important, since it is a priori unclear which approach, bounded or unbounded, best models financial markets and, from a robustness perspective, results that hold in one setting but not in the other may be viewed with some concern.

Of course, a rationale for not worrying too much about whether the state space is bounded or not is that even if it is unbounded, it may be possible to simply "truncate" the state space far enough out—for very rare events—without affecting the results in the interior more than marginally. Such a rationale is often too simplistic, however. Dubynskiy and Goldstein (2013) provide an example in which the assumptions made at the boundaries have a first-order effect on the solution in the interior, even for states that are very far away from these boundaries. Such dependence on boundary conditions is well-known in the study of the dynamic problems that arise in finance, for example, parabolic partial differential equations (PDEs) in case of diffusion processes; see John (1991) and DiBenedetto (1995).

In the example in Dubynskiy and Goldstein (2013), the boundary conditions provide important information about the preferences of the representative investor—exactly the information that the method was designed to recover. Similarly, Carr and Yu (2012) show that for bounded diffusion processes, under appropriate exogenously specified boundary conditions, recovery is possible. Again, boundary conditions are needed in their setting.[1] Even if the true state space is bounded, truncation may still be present because of a limited number of observable asset prices. The bounds may even be unknown. For example, one may argue that in our world with finite resources, there must be an upper bound on the value of the stock market, GDP, etc. However, it seems virtually impossible to determine whether the correct bound to use for the Dow Jones Industrial Average is at 48,000, a million, a billion, or even higher.

The potential importance of rare events for the recovery problem is related to several fragility results for equilibrium asset pricing models in finance that have been put forward in recent years. We mention a few examples. Barro (2005), building on Rietz (1989), shows that the risk for catastrophic events far out in the tail of the return distribution may have large asset pricing effects, potentially explaining the equity premium puzzle. Parlour, Stanton, and Walden (2011) show that adding a very small risk-free consumption stream to an otherwise standard Lucas economy can have drastic effects on stock prices and discount rates, because of the insurance such an asset provides in rare bad states. Kogan *et al.* (2006) show how a small number of irrational investors in the market can have a disproportionate impact on asset prices by entering into extreme bets on rare events. Such models, several of which assume diffusion processes, are therefore fragile with respect to combined assumptions about rare events and agent preferences. For the recovery problem there is a similar potential fragility, namely whether the assumption about boundedness of the state space fundamentally impacts the feasibility of the method.

We analyze the recovery problem in a representative agent economy where the state evolves in continuous time according to a time homogeneous univariate diffusion process on an unbounded domain. Our first contribution is to derive necessary and sufficient conditions for unique recovery in this setting. We derive properties of the diffusion process that alone determine whether recovery is possible; the form of the pricing kernel, that is, the

1 Carr and Yu (2012) mention the extension of the recovery methodology to unbounded domains as an interesting extension.

marginal utility of the representative agent, is not important. In general, for recovery to be possible, the process cannot be allowed to drift off toward infinity too quickly in that it needs to be recurrent. A sufficient but not necessary condition is that the diffusion process has a stationary distribution. This complete independence between the feasibility of recovery and the functional form of the pricing kernel is a priori quite surprising, and adds to the strength of the recovery method.

We also study the recovery problem when additional restrictions on the pricing kernel are imposed. Specifically, when we require marginal utility to be bounded from above and below, the drift of the diffusion process only needs to be restricted in one direction. Finally, we demonstrate that state prices alone cannot be used to determine whether recovery is possible, that is, that some knowledge about the underlying process is needed. Altogether, our results show that recovery is possible for a wide class of interesting diffusion processes, but that there are also interesting cases for which it fails, for example, models with growth and unbounded utility.

Our second contribution is to show that the recovery conditions for the unbounded case are also important in determining whether the method works with bounded and with discrete state spaces. If option prices are only known on a bounded domain, as long as this domain is large enough, approximate recovery is possible on this bounded domain if and only if recovery is possible on the unbounded domain. Specifically, if recovery is possible on the unbounded domain, an approximate pricing kernel can be constructed from truncated observations of option prices on a bounded interval, and as the length of this interval grows, the approximation converges pointwise to the true kernel. Importantly, no boundary conditions are needed for this approximation method. The result is promising for the use of recovery methods in practice. We show in several examples that the numerical method works well, and also provide Matlab code for the approximation method in the Appendix.

We also show that the solution to a discrete approximation of the continuous problem is sensitive to small perturbations when recovery fails in the continuous case. Thus, even though a unique solution always exists in the discrete case, the solution may be "wrong" whenever the conditions for continuous recovery fail. Our approach may potentially be used to further our understanding of the robustness of the discrete recovery problem. The examples with truncated and discrete state spaces also shed further light on the continuous and discrete approximations in Ross (2015).

As a third contribution, our reformulation of the problem in a setting with diffusion processes allows for additional insight about how recovery works in a fairly standard framework. Throughout the paper, we provide examples that underline the theoretical results, and discuss the results extensively to provide further intuition and insight about how they arise.

The paper closest to ours is by Qin and Linetsky (2016), who derive sufficient conditions for recovery. Specifically, using the theory of general right Borel processes, they show that recurrence is a sufficient condition for recovery within this class of processes. In contrast, we use the theory of differential equations, and specifically Sturm–Liouville theory, to derive necessary and sufficient conditions for recovery within the narrower class of diffusion processes, and show that recurrence is in general not necessary for such processes. We also analyze recovery when further restrictions are imposed on the pricing kernel, providing what we believe is a fruitful framework for analyzing unique recovery under joint restrictions on the class of processes and on the pricing kernel. Finally, we analyze the approximate recovery problem when the available state prices are truncated and/or discrete, and

show how it relates to the continuous problem. Altogether, our approach and results therefore complement those in Qin and Linetsky (2016).

Borovicka, Hansen, and Scheinkman (2016) discuss the question of identification challenges that arise when the pricing kernel may also contain a nontrivial martingale component, in which case transition independence cannot be taken for granted. If the martingale component is nontrivial, the recovered probabilities and pricing kernel are distorted. The question we analyze is separate in that we assume transition independence of the pricing kernel, and focus on the question of what conditions are needed for recovery to be possible with unbounded state spaces, given such transition independence. We further expand upon the differences in the body of the paper.

Other related work includes the rapidly growing literature on empirical recovery, which may shed light on whether the conditions needed for recovery are satisfied in practice, see Audrino, Huitema, and Ludwig (2014); Tran and Xia (2014); Bakshi, Chabo-Yo, and Gao (2015); Massacci, Williams, and Zhang (2016), as well as Jensen, Lando, and Pedersen (2016) who generalize the recovery framework to multiple time-period models with non-Markovian finite state spaces.

The recovery approach in Ross (2015) and its extensions, including the approach taken in this paper, are based on specific assumptions about the underlying physical process, for example, it being Markovian, the growth conditions analyzed in this paper in case of unbounded state spaces, and also on assumptions about the pricing kernel (transition independence). The strength of the recovery method is that no further parametric restrictions on the state space are needed, in that positivity of the pricing kernel alone ensures unique recovery when the assumptions are satisfied. An alternative, almost model-free, recovery approach proposed in Schneider and Trojani (2016), makes only weak—empirically verifiable—economic moment constraints on physical returns, and then identifies a minimum-variance Hansen–Jagannathan pricing kernel projection consistent with these constraints, as well as an associated unique physical probability distribution with a minimal state space. As discussed in Schneider and Trojani (2016), neither recovery approach is guaranteed to recover the actual pricing kernel and physical probabilities in general. Specifically, although very few restrictions on the underlying physical process are made in Schneider and Trojani (2016), without the minimum-variance condition, the actual kernel is no longer unique. Both recovery approaches therefore provide valuable information about the kernel and physical probabilities, based on the prices observed in the market, but neither approach can be used to guarantee recovery under completely general conditions.

Finally, the recovery problem is related to the literature that uses Perron–Frobenius theory to study the general link between long-term growth and asset prices under very general conditions, see Alvarez and Jerman (2005), Hansen and Scheinkman (2009), Hansen (2012), and Hansen and Scheinkman (2013). Although methodologically very similar to the recovery framework, the main focus of this literature has been on the long-term properties of the economy and pricing kernel, for example, in terms of risk pricing.

The rest of the paper is organized as follows. In the next section, we give a brief summary of recovery with a finite number of states, as introduced in Ross (2015). In Section 3, we analyze the recovery problem for a diffusion process on an unbounded domain. In Section 4, we show that when recovery is possible in the unbounded state space, approximate recovery is possible when the state space is truncated, and we also relate the robustness of the discrete

recovery problem to our continuous results. Finally, Section 5 concludes. Proofs, the Matlab code for approximate recovery, and some details, are delegated to the Appendix.

## 2. Recovery with Finite State Space

We summarize the approach in Ross (2015), which is based on a model in discrete time with a finite number of states. We use the same terminology as Ross, except for the vector of marginal utilities, for which we use $m$ for instead of $d$, because $d$ is easy to mistake for the differential operator in the continuous model.

There are $N$ states, and a stochastic, irreducible, aperiodic, matrix, $F$, such that $F_{ij}$ denotes the probability of moving from state $i$ to $j$. Since $F$ is stochastic,

$$F1 = 1, \tag{1}$$

where 1 is an $N$-vector of ones. There is a representative agent, with time separable expected utility, discount rate $\delta < 1$, and marginal utility $m_i > 0$ in state $i$. We define the vector $\mathbf{m} = (m_1, \ldots, m_N)^T$, and its reciprocal $\mathbf{z} = (1/m_1, \ldots, 1/m_N)^T$. Let $P_{ij}$ denote the time-0 price in state $i$ of an AD security that pays a dollar at time 1 if the state is $j$. In a Walrasian complete market equilibrium, the price can then be expressed as

$$P_{ij} = \delta \frac{m_j}{m_i} F_{ij}, \tag{2}$$

or in matrix form

$$P = \delta M^{-1} F M, \tag{3}$$

where $M$ is the diagonal matrix, $M = \text{diag}(\mathbf{m})$. From Equation (3), it follows that

$$F = \delta^{-1} M P M^{-1}, \tag{4}$$

which when plugged into Equation (1) yields

$$\delta^{-1} P M^{-1} 1 = M^{-1} 1, \text{ that is,} \tag{5}$$

$$P\mathbf{z} = \delta \mathbf{z}. \tag{6}$$

From the Perron–Frobenius theorem, it follows that there is a unique strictly positive pair $\delta$ and $\mathbf{z}$ that solves the eigenvector problem (6),[2] via Equation (4) allowing $F$ to be recovered. On pricing kernel form, Equation (2) can written as

$$P_{ij} = E\left[\frac{\Lambda_{t+1}}{\Lambda_t} \delta_j\right], \qquad \text{where} \tag{7}$$

$$\Lambda_t = \delta^t m(X_t), \tag{8}$$

$X_t \in \{1, 2, \ldots, N\}$ is a Markov process representing the state at time $t$, $m(X_i) = m_i$, $i = 1, \ldots, N$, $\delta_j = 1$ if $X_{t+1} = j$ and 0 otherwise, and $P_{ij}$ is the price at time $t$ of the AD security that pays a dollar if $X_{t+1} = j$, given that $X_t = i$. If the representation is possible, then the pricing kernel is said to be on transition independent form.

As discussed in Borovicka, Hansen, and Scheinkman (2016), multiplicative pricing relations on the form $P_{ij} = \delta s_{ij} F_{ij}$, $s_{ij} > 0$, are more general than the transition independent form

2 Uniqueness ensured, because $F$ is irreducible and aperiodic.

(2), which may lead to misspecification. Specifically, the recovery methodology cannot distinguish between pricing relationships on the form (2) and on the form $P_{ij} = \delta \frac{m_i}{m_i} H_{ij} F_{ij}$, where $\sum_j H_{ij} F_{ij} = 1$ for all states $j$, that is, the process $h_{t+1} = H_{X_t, X_{t+1}} h_t$ is a martingale. Only when $h$ is trivially identically equal to one is the pricing kernel transition independent. The authors provide an example with stochastic consumption growth, where

$$P_{ij} = \delta s_{ij} F_{ij} = E\left[\delta \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} \delta_j\right], \qquad \text{where} \tag{9}$$

$$s_{ij} = \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} = \Phi(X_{t+1} = j, X_t = i), \tag{10}$$

for some function $\Phi$, and $C_t$ is the aggregate consumption at time $t$. Clearly, with the state represented by $X_t$, the pricing kernel formulation (9, 10) is in general not on transition independent form.

Our focus is not on whether transition independence holds, but on the conditions needed for recovery to work with unbounded state spaces, given transition independence. We note, however, that whether transition independence in Equations (9, 10) is satisfied depends on how the state space is defined. Specifically, defining $\hat{X}_t = (X_t, C_t)$, it follows that $\hat{X}_t$ follows an (unbounded) Markov process, and that the pricing kernel under $\hat{X}_t$ has the transition independent form $m(\hat{X}_t) = C_t^{-\gamma}$. We will subsequently return to this point in an example with stochastic growth and unbounded state space.

## 3. Recovery with Unbounded Diffusion Process

The state evolves according to a univariate time homogeneous diffusion process:

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}\omega, \qquad t \geq 0. \tag{11}$$

It will be convenient to define the function

$$D(x) = \frac{\sigma^2(x)}{2}.$$

We make the technical assumptions that $\mu$ and $\sigma$ are continuously differentiable, and that there are constants, $C_1$, $C_2$, and $C_3$, such that $|\mu(x) - \mu(y)| \leq C_1|x - y|$, $0 < C_2 \leq \sigma(x)$, $|\sigma(x) - \sigma(y)| \leq C_3|x - y|$, for all $x$ and $y$, to ensure that a strong solution exists and that any interval on the real line, $\mathbb{R}$, is covered with positive probability.[3]

Associated with the diffusion process is a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, $t \geq 0$, satisfying the usual assumptions. We define the transition density function $f^t(x, y) = \frac{\partial F^t}{\partial y}$, where $F^t(x, y) = \mathbb{P}(X_t \leq y | X_0 = x)$, and it then follows that $f^t$ satisfies the Fokker–Planck equation

$$\begin{aligned} \frac{\partial f^t}{\partial t} &= \mathcal{L}^* f^t, \\ f^0(x, y) &= \delta_x(y), \end{aligned} \tag{12}$$

3 These technical conditions are standard for guaranteeing the existence of strong solutions to Equation (11), see Oksendahl (1998), but exclude some interesting examples for which strong solutions are known to exists, for example, the univariate square root process (CIR process). Our approach based on Sturm–Liouville theory can be extended to include such examples.

where the operator $\mathcal{L}^*$ is defined as

$$\mathcal{L}^* f = -\frac{\mathrm{d}}{\mathrm{d}y}(\mu(y)f(y)) + \frac{\mathrm{d}^2}{\mathrm{d}y^2}\left(\frac{\sigma^2(y)}{2}f(y)\right). \tag{13}$$

Here, $\delta_x(y)$ is the Dirac distribution centered at $x$, defined by the conditions $\delta_x(y) = 0$, $x \neq y$, and $\int \delta_x(y)\mathrm{d}y = 1$. The fact that $\sigma \geq C_2 > 0$ implies that $f^t(x,y) > 0$ for all $t > 0$, $x \in \mathbb{R}$, and $y \in \mathbb{R}$, that is, that for any $x_0$, $y$, and $t$, the probability density of $X_t$ at $y$ is strictly positive. The function $f$ corresponds to the matrix $F$ in the discrete case.

The instantaneous flow of a single consumption good at time $t$ is $g(X_t)\mathrm{d}t$, where $g$ is a strictly increasing, twice continuously differentiable, function. A price taking representative agent seeks to maximize expected utility of a consumption flow, $c_t$:

$$U = E\left[\int_0^T \mathrm{e}^{-\rho t} u(c_t)\mathrm{d}t\right], \tag{14}$$

for some $0 < T \leq \infty$. Here, the constant $\rho > 0$ is the agent's personal discount rate, and $u$ is a strictly increasing, three times continuously differentiable function, such that $|U| < \infty$, when $c_t = g(X_t)$.

A complete financial market of AD securities exists (e.g., implemented through dynamic trading of a finite number of assets). The time 0 price of an AD security that pays $\delta_y(X_t)$ at time $t$, given that $X_0 = x$, is defined as $p^t(x,y)$. Absence of arbitrage then implies that the time 0 price of a simple contingent claim with time $t$ payoff $\Phi(X_t)$ is $P = \int_{-\infty}^{\infty} p^t(x,y)\Phi(y)\mathrm{d}y$.

A standard argument implies that the Walrasian equilibrium prices of the AD securities are

$$p^t(x,y) = \mathrm{e}^{-\rho t}\frac{m(y)}{m(x)}f^t(x,y), \tag{15}$$

where $m(x) = u'(g(x))$ is strictly positive and twice continuously differentiable. This corresponds to Equation (3) in the finite case. It will be convenient to define the functions $q(x) = \frac{m'(x)}{m(x)}$ and $z(x) = \frac{1}{m(x)}$. Clearly, $q$ is closely related to the representative agent's relative risk-aversion coefficient at $x$, $\gamma(x)$, since $-q = -\frac{m'(x)}{m(x)} = -g'(x)\frac{u''(g(x))}{u'(g(x))} = \frac{g'(x)}{g(x)}\gamma(x)$. The extra factor $\frac{g'(x)}{g(x)}$ arises because $g(x)$, rather than $x$ represents, units of the consumption good (which in turn allows us to cover both arithmetic and geometric consumption processes within a unified framework). We note that $m(x)$ is only unique up to multiplication with an arbitrary positive constant, given the equivalence of two utility functions that are positive affine transformations of each other. However, $q$ is unique, since any constant will occur both in the dominator and numerator of $q$ and therefore cancel out.

The function

$$\Lambda_t = \mathrm{e}^{-\rho t}\frac{m(X_t)}{m(X_0)} \tag{16}$$

is the pricing kernel in the economy, leading to the standard pricing formula

$$P = E\left[\frac{\Lambda_t}{\Lambda_0}\Phi(X_t)\right], \tag{17}$$

for the time 0 price of a simple contingent claim with time $t$ payoff $\Phi(X_t)$. In the terminology of Ross (2015), the specific kernel is transition independent, being the product of a constant discount rate depreciation factor, and the fraction of a function evaluated at $X_t$ and $X_0$, respectively. We thus take the existence of a pricing kernel on the form (16) as given.

Assume that the prices of all AD securities are known, that is, that the function $p^t(x,y)$ is known for all $t>0$, $x\in\mathbb{R}$ and $y\in\mathbb{R}$. It is well-known that we can draw inferences about the underlying parameters, $\rho$, $\mu(x)$, $\sigma(x)$, and $m(x)$ from $p$, using standard equilibrium conditions and risk neutral pricing. In the Appendix, we expand further on how these parameters can be inferred from state prices. Standard equilibrium arguments, see, for example, Cochrane (2005), imply that the short risk-free rate is

$$r(x)=\rho-q(x)\mu(x)-(q'(x)+q(x)^2)D(x). \tag{18}$$

Moreover, it is well-known that volatility, and thereby $D(x)$, can be uniquely identified in a complete market diffusion setting. The drift term, $\mu(x)$, is not directly identifiable, but its risk neutral counterpart can be inferred from state prices. Specifically, define

$$\kappa(x)\overset{\text{def}}{=}\mu(x)+2q(x)D(x), \tag{19}$$

and the infinitesimal generator $\mathcal{A}[m]=\mu\frac{\mathrm{d}m}{\mathrm{d}x}+D\frac{\mathrm{d}^2m}{\mathrm{d}x^2}$. The pricing formula (16) for an asset that pays $X_t$ at time $t$ is on risk neutral form written as

$$P=E^{\mathbb{Q}}\left[\mathrm{e}^{-\int_0^t r(s)\mathrm{d}s}X_t\right]=E\left[\mathrm{e}^{-\rho t}\frac{m(X_t)}{m(X_0)}X_t\right],$$

see Duffie (2001), or on differential form

$$E^{\mathbb{Q}}[\mathrm{d}P]=-rX\mathrm{d}t+E^{\mathbb{Q}}[\mathrm{d}X]=-\rho X\mathrm{d}t+E\left[\frac{\mathrm{d}(mX)}{m}\right]=-\rho X\mathrm{d}t+\frac{\mathcal{A}[m]}{m}X\mathrm{d}t+\mu\mathrm{d}t+2qD\mathrm{d}t,$$

which via Equation (18) leads to $E^{\mathbb{Q}}[\mathrm{d}X]=\kappa\mathrm{d}t$. To summarize, in the complete market equilibrium, $r$, $D$, and $\kappa$ are directly observable, whereas $\mu$ is not.

For any given $x$, Equations (18) and (19) provide two equations for the three unknown $\rho$, $\mu(x)$, and $m(x)$, and it may therefore seem as if there is one degree of freedom at each point $x$. For example, such pointwise indeterminacy arises in a one-factor time homogeneous term structure model, where an unknown risk-premium process $\lambda(X_t)$ is introduced, and the function $\lambda$ may be quite arbitrary. An insight in Ross (2015) is that a pricing kernel on the form (16) leads to strong constraints on how the marginal utility can change with $x$. We can see this in our context by rewriting Equation (18) as

$$(\rho-r)\frac{1}{m}=\frac{\mathcal{A}[m]}{m^2}=\kappa\frac{m'}{m^2}+D\left(\frac{m''}{m^2}-2\frac{(m\prime)^2}{m^3}\right),$$

and defining $z=\frac{1}{m}$, such that $z'=-\frac{m'}{m^2}$, $z''=-\frac{m''}{m^2}+2\frac{(m\prime)^2}{m^3}$, altogether obtaining the second-order ordinary differential equation (ODE) in $z$:

$$z''+\frac{\kappa}{D}z'+\frac{\rho-r}{D}z=0. \tag{20}$$

We note that all functions and variables in Equations (20) are observable, except for $\rho$. Thus, if we define the fundamental ODE for the recovery problem of the diffusion process:

$$z''+\frac{\kappa}{D}z'+\frac{\lambda-r}{D}z=0,\qquad \lambda\in\mathbb{R}, \tag{21}$$

then the true $z$ is a solution to the ODE with the parameter value $\lambda = \rho$.[4]

In Appendix A, we show that the eigenvector formulation in Ross (2015) and the ODE formulation in our setting are actually very similar, in that the fundamental ODE is the differential form of the integral equation eigenfunction problem for the process.

Following the insight in Ross (2015), we next study how positivity can be used to decrease the number of degrees of freedom even further.

### 3.1 Recovering Pricing Kernel from Fundamental ODE

We address the question of under which conditions there is sufficient information to uniquely recover $m(x)$ and $\rho$ from the fundamental ODE. Here, uniqueness of $m$ is defined up to scaling with an arbitrary positive constant, in line with our previous discussion of invariance under positive affine transformations of the utility function. In this case, we say that recovery is possible.

We define the operator $\mathcal{W}[s|\lambda] = \frac{\mathrm{d}^2 s}{\mathrm{d}x^2} + \frac{\kappa}{D}\frac{\mathrm{d}s}{\mathrm{d}x} + \frac{\lambda - r}{D}s$, and can for general $\lambda$ solve

$$\mathcal{W}[s|\lambda] = 0. \tag{22}$$

Of course, from Equation (21), $\mathcal{W}\left[\frac{1}{m}|\rho\right] = 0$.

Under general conditions, given $\rho$, the solution to Equation (21) is on the form $c_1 z_1(x) + c_2 z_2(x)$, for arbitrary constants, $c_1$, and $c_2$. But, since $z$ is only unique up to multiplication by a finite constant, there is effectively only one degree of freedom: $z = cz_1 + (1 - c)z_2$. Thus, in general, Equation (21) has only two degrees of freedom, one degree in $\rho > 0$ and one in $c$. We have

**Proposition 1.** Consider the fundamental ODE, Equation (21):

- *Given $\rho$, and $q(x_0) = c$ for some $x_0$, there is a unique solution to* Equation (21), *$z_{\rho,c}(x)$, defined on the whole of $\mathbb{R}$.*
- *Given $\rho_1$, $\rho_2$, $c_1$, and $c_2$, such that $\rho_1 \neq \rho_2$ or $c_1 \neq c_2$, then the solutions to Equation (21) with parameters $\rho_1, c_1$, and $\rho_2, c_2$, respectively, are distinct, $z_{\rho_1,c_1} \neq z_{\rho_2,c_2}$.*

Proposition 1 makes precise the concept that there is in general sufficient information to reduce the indeterminacy of the recovery problem down to two degrees of freedom. The second part suggests that without further knowledge of $\rho$ and $q(x_0)$ for some $x_0$, recovery is not possible. We have still not used the fact that $m$ must be positive though. The second—and fundamental—insight of Ross (2015) in the discrete setting is that positivity allows for recovery, because the Perron–Frobenius Theorem guarantees that only one solution to the eigenvalue problem is strictly positive.

In our diffusion setting, it is a priori unclear how far positivity will take us. Given that there is an infinite number of unknowns ($m(x)$ for all $x$), as well as conditions (relating $z''(x)$, $z'(x)$, and $z(x)$ for all $x$ in Equation (21)), we cannot simply count the number of equations and unknowns to see whether there is sufficient information for recovery.

4 The fundamental ODE also appears in Carr and Yu (2012), Tran (2013), and Dubynskiy and Goldstein (2013). We note that there is nothing in this ODE formulation or our subsequent analysis that restricts $\rho$ to be positive. However, for the pricing kernel formulation to have economic meaning, the expected utility of the representative agent needs to be well defined, which may not be the case if $\rho < 0$.

We define the function $Q(x) \stackrel{\text{def}}{=} e^{-\int_0^x \frac{\mu(s)}{D(s)} ds}$. The following proposition shows that the behavior of the diffusion process as $x$ tends to $\pm\infty$, via its influence on the so-called scale function, $\int Q(x)dx$ (see Karlin and Taylor, 1981), determines whether recovery is possible:

**Proposition 2.** A necessary and sufficient condition for recovery of m(x) is that

$$\int_{-\infty}^{0} Q(x)dx = \infty \qquad \text{and} \qquad \int_{0}^{\infty} Q(x)dx = \infty. \tag{23}$$

Here, we use the identity $\int_0^x \frac{\mu(s)}{D(s)} ds = -\int_x^0 \frac{\mu(s)}{D(s)} ds$ for $x < 0$. Thus, the behavior of $\frac{\mu(x)}{D(X)}$ for large (negative or positive) $x$ determines whether recovery is possible. A sufficient but not necessary condition for (23) to hold is that $X$ is mean reverting. An example in which $X$ is not mean reverting but recovery is still possible is when $X$ is a standardized Brownian motion (BM). The drift term, $\mu(x)$, can also be positive for $x > 0$, and/or negative for $x < 0$, as long as it approaches zero quickly enough, or $D$ grows fast enough. We note that Condition (23) means that the process is recurrent in that it returns to any state as time increases, see Pinsky (1995, p. 208). Processes that are not recurrent are transient. Thus, for transient processes recovery is not possible.

The positivity requirement reduces the number of degrees of freedom in the recovery problem from two to zero, as follows: It is straightforward to verify that the solutions to $\mathcal{W}[s|\lambda] = 0$ can be written as $s = u(x)v(x)$, where $u(x)$ solves the ODE

$$u'' = \left(\frac{1}{4}\left(\frac{\mu}{D}\right)^2 + \frac{1}{2}\frac{d}{dx}\left(\frac{\mu}{D}\right) - \frac{\lambda - \rho}{D}\right)u, \tag{24}$$

and $v(x) = e^{-\frac{1}{2}\int_0^x \frac{\kappa(y)}{D(y)} dy}$. This ODE, which also arises in the model of Carr and Yu (2012) with bounded domains, provides a convenient separation into a part, $v$, that depends on the representative investor's marginal utility, and a part, $u(x)$, that solely depends on the diffusion process, and specifically on $\frac{\mu}{D}$ as seen in Equation (24). Moreover, $v(x)$ is always positive, so negativity of the solution must come from $u$. This explains why the condition for recovery does not depend on $m$, but only on the diffusion process through $\frac{\mu}{D}$.

Now, it is easy to check that a solution in the case when $\lambda = \rho$ is given by $u_{\rho,1} = e^{\frac{1}{2}\int_0^x \frac{\mu(y)}{D(y)} dy}$, and in the proof it is moreover shown that if Condition (23) is satisfied, then for any $c \neq 1$, the range of the other solution to this second order ODE is the whole real line. Therefore, the range of any combination of the two solutions must also be the whole real line, violating the positivity constraint. This reduces the number of degrees of freedom from two to one, by forcing $c = 1$.

The final part of the argument, allowing us to nail down $\rho$, is that a higher $\lambda$ in Equation (24) will have a negative effect on $u$, at any point where $u(x)$ is positive, by decreasing $u''$. As shown in the proof, as long as $u_{\rho,1}$ does not grow too fast, this negative effect on $u''$ of having $\lambda > \rho$ eventually makes $u(x)$ become negative. Condition (23) is such that $\frac{\mu}{D}$, and thereby $u$, does not grow too fast. Altogether, this implies that $\rho$ can be identified as the largest $\lambda$ for which there is a positive solution to $\mathcal{W}[z|\lambda] = 0$, which in turn will be unique. Recovery is therefore possible.

The proof of Proposition 2 uses the theory of differential equations, and specifically Sturm–Liouville theory, to explicitly construct multiple positive solutions when Condition (23) fails, thereby showing not only the sufficiency but also the necessity of the condition. This is in contrast to Qin and Linetsky (2016), who shows that recurrence is a sufficient condition for recovery for the larger class of right Borel processes. In the univariate case,

Condition (23) is both necessary and sufficient for recurrence, so our result implies that recurrence is both necessary and sufficient.

In higher dimensions, however, recurrence is no longer necessary, that is, there are transient (non-recurrent) diffusion processes that allow for unique recovery. Consider the economy in which a three-dimensional standardized BM governs the state space, $(\mathrm{d}X_1, \mathrm{d}X_2, \mathrm{d}X_3) = (\mathrm{d}\omega_1, \mathrm{d}\omega_2, \mathrm{d}\omega_3)$. It is well-known that this process is transient, so that the possibility of recovery cannot be inferred from recurrence in this case. Total instantaneous consumption is $X_1 + X_2 + X_3$, and the pricing kernel corresponds to a risk-neutral representative agent, $m \equiv 1$. It then follows that the risk-free rate is $r = \rho$, and that the three-dimensional version of the fundamental ODE (21) is the so-called Helmholtz PDE:

$$\Delta z + (\lambda - \rho) z = 0, \qquad \lambda \in \mathbb{R}, \tag{25}$$

where $\Delta$ is the Laplace operator, $\Delta z = \frac{\partial^2 z}{\partial X_1^2} + \frac{\partial^2 z}{\partial X_2^2} + \frac{\partial^2 z}{\partial X_3^2}$.

The solutions to Equation (25) are thus the eigenfunctions of the Laplace operator. For $\lambda < \rho$, there are multiple positive eigenfunctions, for example, $z = e^{X_i \sqrt{\rho - \lambda}}$, $i = 1, 2, 3$. For $\lambda = \rho$, Liouville's theorem for harmonic functions implies that the only positive eigenfunction is the constant function, $z \equiv 1$. For $\lambda > \rho$, there are only oscillating eigenfunctions (i.e., eigenfunctions that take on both negative and positive values).[5] Thus, $z \equiv 1$, with corresponding parameter value $\lambda = \rho$, is the unique positive solution to Equation (25) among all $\lambda \geq \rho$. The same approach as in the univariate case therefore leads to unique recovery in this example, suggesting that the PDE analysis is useful for cases when recovery is possible but does not follow directly from recurrence of the process. Altogether, our results and the results in Qin and Linetsky (2016) therefore complement each other.

We next study three examples in more detail, one for which recovery is possible and one for which it is not, and also a bivariate example, to provide additional insight.

### 3.2 BM Example

Consider the classical Black–Scholes (BS) economy with $\mathrm{d}X = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}\omega$, where $\mu > 0$ and $\sigma > 0$ are constants. It follows that $\int_{-\infty}^{0} Q(x)\mathrm{d}x = \frac{D}{\mu} < \infty$, so Condition (23) is not satisfied, and recovery is therefore not possible. This is in line with what has been reported earlier in Ross (2015) and Dubynskiy and Goldstein (2013).

We verify non-recovery for two utility functions. We first study the standard Lucas economy with power utility, where $g(x) = \mathrm{e}^x$, and $u'(g) = g^{-\gamma}$, $\gamma > 0$. It follows from previous definitions that $m(x) = \mathrm{e}^{-\gamma x}$, $q(x) = -\gamma$, and $z(x) = \mathrm{e}^{\gamma x}$. From Equations (18) and (19), we then have

$$r = \rho + \gamma\mu - \gamma^2 \frac{\sigma^2}{2},$$

$$\kappa = \mu - \gamma\sigma^2,$$

in line with standard results. The solutions to Equation (22) are

$$z_1^\lambda(x) = \mathrm{e}^{-x\frac{\kappa + \sqrt{\kappa^2 + 2\sigma^2(r-\lambda)}}{\sigma^2}},$$

$$z_2^\lambda(x) = \mathrm{e}^{-x\frac{\kappa - \sqrt{\kappa^2 + 2\sigma^2(r-\lambda)}}{\sigma^2}}.$$

5 This follows, for example, from the extension of the mean value theorem for Helmholtz equation, see Courant and Hilbert (1962, p. 288).

For $\lambda \leq r + \frac{\kappa^2}{2\sigma^2} = \rho + \frac{\mu^2}{2\sigma^2}$, there are two distinct positive solutions to the equation. Thus, the possible marginal utilities are

$$m_{\lambda,c} = \frac{1}{z_{\lambda,c}} = \frac{1}{cz_1^{\lambda}(x) + (1-c)z_2^{\lambda}(x)}, \qquad \lambda \leq r + \frac{\kappa^2}{2\sigma^2},\ 0 \leq c \leq 1.$$

Even if $\rho$ is known, $m$ is not uniquely recovered. In fact, it is easy to check that in this case the possible solutions are

$$m_{\rho,c} = \frac{1}{z_{\rho,c}} = \frac{1}{ce^{x\gamma} + (1-c)e^{x\left(\gamma - \frac{2\mu}{\sigma^2}\right)}}, \qquad 0 \leq c \leq 1.$$

So, in addition to the correct solution, $m = m_{\rho,1}$, there is a whole range of other possible positive solutions. In Figure 1, some possible functional forms of $m$ are shown, given that $\rho$ is known. In Figure 2, the corresponding possible relative risk aversion coefficients as a function of $x$ are shown.

We next consider the case where $u(x) = x + \frac{x^3}{3}$, $g(x) = x$, so that $m(x) = 1 + x^2$. Note that this utility function is quite nonstandard in that it is not concave. It shows the strength of the methodology that no additional restrictions on $m$ are needed, beyond positivity. In this case, we get

$$r = \rho - \frac{2x\mu + \sigma^2}{1 + x^2}, \tag{26}$$

$$\kappa = \mu + \frac{2x\sigma^2}{1 + x^2}. \tag{27}$$

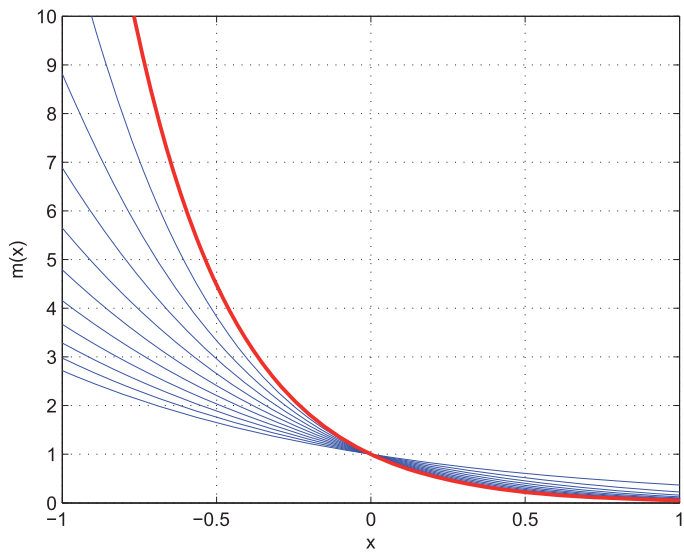


**Figure 1.** Some candidate m functions, given that $\rho$ is known, $c = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.8, 1$ shown. The solid (red) line represents the true $m = e^{-\gamma x}$, corresponding to $c = 1$. Recovery is not possible in this case. Parameter values: $\gamma = 3$, $\mu = 0.01$, $\rho = 0.01$, $\sigma = 0.1$.

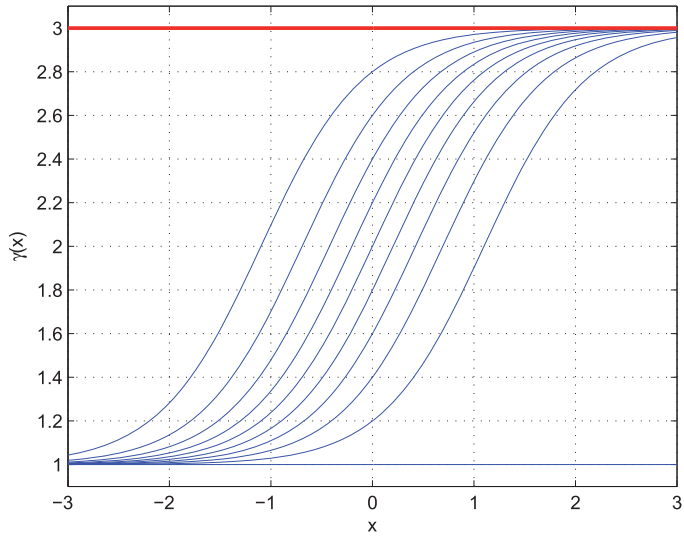


**Figure 2.** Some possible risk-aversion functions, $\gamma(x)$, given that the correct $\rho$ is known, $c = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.8, 1$. The solid (red) line at the top represents the true $\gamma(x) = 3$, $c = 1$. Parameter values: $\gamma = 3$, $\mu = 0.01$, $\rho = 0.01$, $\sigma = 0.1$.

These expressions are also nonstandard. For example, the short interest rate is highly negative for large $x$. Again, a strength of the methodology is that it allows us to analyze recovery under very general conditions.

The stochastic process is still a BM with positive drift, so Proposition 2 implies that recovery is not possible with this utility specification either. The general solution to $\mathcal{W}\left[\frac{1}{m}|\rho\right] = 0$ in this case is

$$m_{\rho,c} = \frac{1}{z_{\lambda,c}} = \frac{1+x^2}{c+(1-c)e^{-\frac{2\mu}{\sigma^2}x}}, \qquad 0 \leq c \leq 1,$$

so there are multiple possible $m(x)$ functions, even if $\rho$ is known.

### 3.3 Ornstein–Uhlenbeck Example

Consider the Ornstein–Uhlenbeck (OU) process $dX = \theta(a - X)dt + \sigma d\omega$, $\theta > 0$, $\sigma > 0$. Without loss of generality, we assume that $a = 0$, since we can always define $\hat{x} = x - a$ for nonzero $a$, and solve in $\hat{x}$ coordinates. We then have $\mu = -\theta x$, and since $\frac{\mu}{D} = -\frac{\theta}{\sigma}x$, the conditions for recovery in Proposition 2 are satisfied. Again, we assume that $m(x) = 1 + x^2$. We calculate

$$r = \rho + \frac{2x^2\theta - \sigma^2}{1+x^2},$$
$$\kappa = -x\theta + \frac{2x\sigma^2}{1+x^2},$$

and Equation (22) then takes the form

$$z'' + \frac{x}{D}\left(\frac{2\sigma^2}{1+x^2} - \theta\right)z' + \frac{1}{D}\left(\lambda - \rho - \frac{2x^2\theta - \sigma^2}{1+x^2}\right)z = 0. \tag{28}$$

The solutions to Equation (28) are

$$z_1^\lambda(x) = \frac{1}{1+x^2} H_{\frac{\lambda-\rho}{\theta}}\left(\frac{x\sqrt{\theta}}{\sigma}\right),$$

$$z_2^\lambda(x) = \frac{1}{1+x^2} {}_1F_1\left(\frac{\rho-\lambda}{2\theta}, \frac{1}{2}, \frac{x^2\theta}{\sigma^2}\right).$$

Here, $H_a$ is the Hermite function and ${}_1F_1$ is the confluent hypergeometric function [see Gradshteyn and Ryzhik (2000, p. 986 and 1013)]. In the case when $\lambda = \rho$, this reduces to

$$z_1^\rho(x) = \frac{1}{1+x^2},$$

$$z_2^\rho(x) = \frac{1}{1+x^2} \mathrm{Erfi}\left(\frac{x\theta}{\sigma}\right),$$

where Erfi is the imaginary error function, $\mathrm{Erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{\frac{s^2}{2}} ds$. Now, since $\mathrm{Erfi}(-\infty) = -\infty$ and $\mathrm{Erfi}(\infty) = \infty$, the only way to make $z_{\rho,c} = cz_1^\rho(x) + (1-c)z_2^\rho(x)$ strictly positive for all $x$ is to choose $c = 1$. Moreover, for any $\lambda > \rho$, all candidate $z_{\lambda,c} = cz_1^\lambda + (1-c)z_2^\lambda$ are negative for some $x$, and therefore disqualified as candidate $z$ functions. This follows from the proof of Proposition 2. An example is shown in Figure 3, where candidate $z^\lambda$ for a specific $\lambda > \rho$ are shown. Since all candidates are negative for some $x$, they cannot represent the correct $z$ function, and thus recovery is possible in this case.

### 3.4 An Example with Stochastic Growth

We also study a two-dimensional example with stochastic growth, related to the previously discussed discrete example in Borovicka, Hansen, and Scheinkman (2016). The two-dimensional state space follows the bivariate diffusion process

$$dX = -\alpha Y dt + \sigma_X dw_X, \tag{29}$$

$$dY = -Y dt + \sigma_Y dw_Y, \tag{30}$$

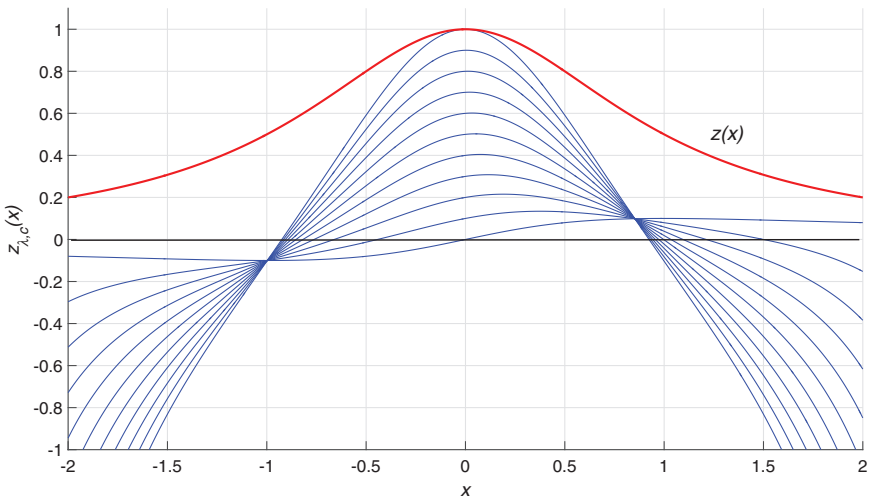


**Figure 3.** Candidate functions, $z_{\lambda,c}(x)$, $c = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.8, 1$. The solid (red) line above the other lines represents the true $z = \frac{1}{m}$. All candidate functions with $\lambda > \rho$ eventually become negative, which means that they cannot represent $\frac{1}{m}$. Parameter values: $\lambda = 0.02$, $\theta = 0.01$, $\rho = 0.01$, $\sigma = 0.1$.

where $w_X$ and $w_Y$ are independent Wiener processes, and the pricing kernel is on the form $\Lambda_t = \mathrm{e}^{-\rho t}\frac{m(X_t,Y_t)}{m(X_0,Y_0)} = \mathrm{e}^{-\rho t}\frac{\mathrm{e}^{-\gamma X_t}}{\mathrm{e}^{-\gamma X_0}}$. The example corresponds to an economy with a representative investor with power utility and risk aversion coefficient $\gamma$, and aggregate consumption $C_t = \mathrm{e}^{X_t}$ that experiences stochastic, mean reverting, growth rates. The fundamental PDE in this case is on the form:

$$\frac{\sigma_X^2}{2}z_{XX} + \frac{\sigma_Y^2}{2}z_{YY} - Yz_Y - (\alpha Y + \gamma\sigma_X^2)z_X + \left(\lambda - \rho + \gamma\alpha Y + \gamma^2\frac{\sigma_X^2}{2}\right)z = 0. \tag{31}$$

Decomposing $z(X, Y)$ into $z(X, Y) = \mathrm{e}^{\gamma X}w(X, Y)$, it follows that $w$ satisfies the PDE

$$\frac{\sigma_X^2}{2}w_{XX} + \frac{\sigma_Y^2}{2}w_{YY} - Yw_Y - \alpha Yz_X + (\lambda - \rho)w = 0. \tag{32}$$

It follows from Corollary 3.5 and Theorem 3.8 in chapter 4 of Pinsky (1995) that the unique positive solution to Equation (32) when $\lambda \geq \rho$ is $w \equiv 1$ and $\lambda = \rho$, leading to $z(X, Y) = \mathrm{e}^{\gamma X}$, and the correct pricing kernel $m = \frac{1}{z}$. Recovery is thus possible in this case with unbounded state space and stochastic growth.

The above example relates to the discrete example 1.1 with stochastic growth in Borovicka, Hansen, and Scheinkman (2016), discussed in their Section 1.4. Consider the special case of their example where there are two states, $X_t \in \{1, 2\}$, the transition probabilities are $F_{1,1} = F_{2,2} = 3/4$, $F_{1,2} = F_{2,1} = 1/4$, there representative agent has logarithmic utility, $u(C) = \log_2(C)$, and discount factor $\delta = 1$, and the pricing kernel is of the form $s_{1,1} = \frac{1}{2}$, $s_{1,2} = s_{2,1} = 1$, $s_{2,1} = 2$, corresponding to $C_{t+1} = 2C_t$ if $X_{t+1} = X_t = 1$, $C_{t+1} = C_t$ in case $X_{t+1} = 2, X_t = 1$ or $X_{t+1} = 1, X_t = 2$, and $C_{t+1} = \frac{1}{2}C_t$ in case $X_{t+1} = X_t = 2$. The time-$t$ price in state $i$ of a AD security that pays a dollar at time $t+1$ in state $j$ is then $P_{ij}$, where

$$P = \begin{bmatrix} \frac{3}{8} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{2} \end{bmatrix}.$$

It is easy to verify that the solution to the eigenvector problem $Pz = \lambda z$ with maximal eigenvalue will neither identify the correct pricing kernel, transition probabilities, nor the discount rate, since the pricing kernel does not have transition independent form under the state space $X \in \{1, 2\}$.

However, when the state space is defined as $\hat{X}_t = (X_t, \log(C_t)) = (i, j) \in \{1, 2\} \times \mathbb{Z}$, the pricing kernel is on transition independent form, $M_{t,t+1} = \frac{C_t}{C_{t+1}}$. The function $z_{\hat{X}} = 2^j$ now satisfies the eigenfunction problem:

$$\lambda z_{1,j} = P_{1,1}z_{1,j+1} + P_{1,2}z_{2,j},$$
$$\lambda z_{2,j} = P_{2,1}z_{1,j} + P_{2,2}z_{2,j-1},$$

$j \in \mathbb{Z}$ and, as shown in the Appendix, the same type of egienfunction analysis as carried out for diffusion processes in this paper allows unique recovery of the discount rate $\lambda = \delta = 1$, and the function $z_{\hat{X}} = 2^j$.

### 3.5 Relationship between Recovery and Stationary Distribution

The condition for recovery (23) is related to the existence of a stationary distribution of the diffusion process. Necessary and sufficient conditions for a function $\phi(y)$ to be a stationary distribution is that $\phi(y) \geq 0$, $\int\phi(y)\mathrm{d}y = 1$, and that $\mathcal{L}^*\phi = 0$.

It is easy to verify that the general solution to $\mathcal{L}^*\phi = 0$ is

$$\phi(y) = \frac{1}{Q(y)D(y)}\left(c_1 + c_2\int_0^y Q(s)\mathrm{d}s\right). \tag{33}$$

and therefore that

$$\int_{-\infty}^{0} \frac{1}{Q(x)D(x)}\mathrm{d}x < \infty, \qquad \text{and} \qquad \int_{0}^{\infty} \frac{1}{Q(x)D(x)}\mathrm{d}x < \infty, \tag{34}$$

is a necessary and sufficient condition for the existence of a stationary distribution.[6] Now, the link between Equations (23) and (34) is clear: the faster $Q(x)$ increases for large $x$, the larger the right integral in Equation (23), and the smaller the right integral in Equation (34). An identical argument holds for the left integrals. However, the two conditions are not equivalent. The existence of a stationary distribution implies that recovery is possible, but the reverse causality is not true. We have

**Proposition 3.** *If the diffusion process has a stationary distribution, then recovery is possible.*

As mentioned, an example for which recovery is possible but there is no stationary distribution is the standardized BM, $\mu = 0$, $\sigma = 1$, leading to $Q(x) \equiv 1$. Clearly Condition (34) fails in this case, but the condition for recovery (23) is satisfied. Indeed, the solutions to the fundamental ODE in this case are

$$z_{\rho,c} = \frac{c + (1-c)x}{m(x)}, \quad \text{and}$$

$$z_{\lambda,c} = \frac{c \times \cos\left(\sqrt{2(\lambda-\rho)}x\right) + (1-c)\sin\left(\sqrt{2(\lambda-\rho)}x\right)}{m(x)}, \qquad \lambda > \rho.$$

Thus, the only positive solution for $\lambda \geq \rho$ is $z_{\rho,1}$, and recovery is therefore possible.

The nonstationarity follows from the fact that the process is null recurrent rather than positive recurrent (see Pinsky, 1995, p. 185). Null recurrent and positive recurrent processes are both recurrent, but the expected time it takes to revisit a state for a null recurrent process is infinite, whereas it is finite for a positive recurrent process.

Interestingly, although recovery is possible in the economy above, it is not possible to recover the personal discount rate directly from the yield of long-term bonds. This is in contrast to the result in Martin and Ross (2013), who show that such direct recovery is possible in the finite state space model. For example, in the case above with $\mu = 0$, $\sigma = 1$, and CRRA preferences with risk aversion coefficient $\gamma$, $m(x) = \mathrm{e}^{-\gamma x}$, it follows from Section 3.2 that the short rate is $r = \rho - \gamma^2/2$ and, since the yield curve is flat in this standard Lucas economy, this is also the long rate. Thus, $r$ does not provide sufficient information to directly back out $\rho$. The reason is that although the drift is $\mu = 0$, the risk averse agent behaves as if the drift is $-\gamma/2$ [see, e.g., Parlour, Stanton, and Walden (2011) for a discussion], which brings down the risk-free rate by introducing a precautionary savings motive. Such a precautionary savings motive is of course also present in the finite state model, but since there are bounds on marginal utility, that is, on $\frac{m(X_T)}{m(X_0)}$, in that setting, the dominant term of the pricing kernel in the long run is the personal discount rate, $\mathrm{e}^{-\rho T}$, allowing direct recovery of $\rho$ from long yields in that setting.

6 In other words, the speed measure needs to be finite, see Karlin and Taylor (1981).

In the finite dimensional case, the existence of a unique stationary distribution is both necessary and sufficient for recovery. The situation is thus different for the case with unbounded diffusions. In the diffusion case, a stationary distribution, if it exists, is unique, so that part of the causality is the same in both models. The reverse causality, however, is different, since recovery does not imply the existence of a stationary distribution in the diffusion model. The reason for the difference is clear: The eigenfunction relationship for recovery is defined by the operator relationship (56). The corresponding finite relationship is (1). The existence of a stationary distribution is governed by the adjoint equation, $F^*\phi = \phi$ in the finite case, and $\mathcal{L}^*\phi = 0$, in the diffusion case. But, whereas it is always possible to rescale $\phi$ such that $\sum_i \phi_i = 1$ in the finite case, there is no guarantee that $\int \phi(y)\mathrm{d}y < \infty$ in the diffusion case. In the terminology of functional analysis: there is no guarantee that the positive solution to the adjoint equation $\mathcal{L}^*\phi = 0$ belongs to the space $L^1(\mathbb{R})$ of integrable functions. Therefore, recovery may be possible even without the existence of a stationary distribution.

### 3.6 Recovery from a Restricted Class of Utility Functions

Since Condition (23) in Proposition 1 is necessary and sufficient for recovery, there is nothing more to say about the general recovery problem. However, if we are willing to rule out some candidate pricing kernels by imposing stricter requirements than mere positivity of $m$, we may weaken the requirements on the diffusion process for recovery.

So far, we have considered any $m \in \mathcal{C}^2_+$ as a candidate function for the pricing kernel $\Lambda_t = \mathrm{e}^{-\rho t}\frac{m(X_t)}{m(X_0)}$, where $\mathcal{C}^2_+$ is the class of strictly positive, twice continuously differentiable functions on the real line. By requiring $m$ to belong to a smaller set, recovery becomes easier. Specifically, assume that for a specific class of diffusion processes (characterized by $\mu$ and $D$), and a set $\mathcal{B} \subset \mathcal{C}^2_+$, if $m$ belongs to $\mathcal{B}$, then no other function in $\mathcal{B}$ satisfies $\mathcal{W}[1/m|\lambda] = 0$, for $\lambda \geq \rho$. In other words, given that there exists a possible pricing kernel, $m \in \mathcal{B}$, no other possible pricing kernel can lie in $\mathcal{B}$. In this case, we say that unique recovery within $\mathcal{B}$ is possible for this class of diffusion processes.

One fruitful restriction is to focus on bounded marginal utilities, $\mathcal{B} = \{m \in \mathcal{C}^2_+ : 0 < c_1 \leq m(x) \leq c_2 < \infty\}$. Here, we require that the bound below is strictly positive ($c_1 > 0$).[7,8] We have

**Proposition 4.** *Unique recovery within $\mathcal{B}$ is possible if and only if at least one of the conditions in Equation (23) is satisfied.*

As a consequence, the classical BS process studied in Section 3.2, which satisfies Condition (23) on the left interval but not on the right, allows for unique recovery within $\mathcal{B}$. An example is given in Figure 4, where the bounded function $m(x) = \left(1 + \frac{\tan^{-1}(x)}{\pi}\right)^{-1}$ is recovered within $\mathcal{B}$.[9] We stress that Proposition 4 does not guarantee that there exists solution in $\mathcal{B}$ to the recovery problem, just that if such a solution exists, it is unique. We also

7 The study of this class was inspired by a discussion with Steve Ross, who in a working paper, Ross (2013), assumes boundedness when analyzing recovery in a model in discrete time with a continuous state space, using the Krein–Rutman theorem.

8 Note that we do not require that the limits of $m(x)$ exist as $x$ tends to plus or minus infinity. Indeed, $m$ may oscillate for large $x$ without convergence.

9 Equivalently, in units of the consumption good, $g = \mathrm{e}^x > 0$, the marginal utility is $\left(1 + \frac{\tan^{-1}(\ln(g))}{\pi}\right)^{-1}$.

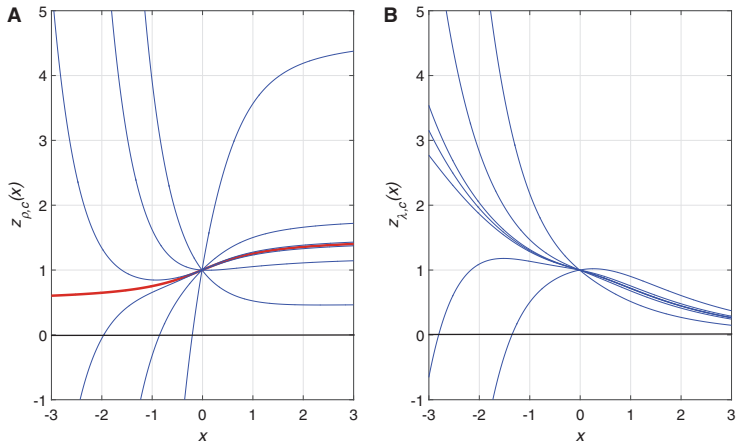


**Figure 4.** BS economy, $dX = \mu dt + \sigma d\omega$, with bounded marginal utility, $m = \left(1 + \frac{\tan^{-1}(x)}{\pi}\right)^{-1}$. In the left panel (A), candidate functions, $z_{\rho,c}(x)$ are shown. The only candidate function that is both bounded and positive is the solid (red) line $z = \frac{1}{m} = 1 + \frac{\tan^{-1}(x)}{\pi}$. In the right panel (B), candidate functions are shown for $\lambda > \rho$. No candidate function satisfies the condition of being positive, bounded from above, and bounded from below. Thus, unique recovery within $\mathcal{B}$ is possible. Parameter values: $\mu = 0.01$, $\sigma = 0.1$, $\rho = 0.01$, $\lambda = 0.0104$.

note that there is a trade-off here, in that the more we restrict the class of candidate functions, the more we are effectively taking a stand on what the pricing kernel looks like, going against the philosophy that the kernel and real probability distribution should be inferred from data alone.

### 3.7 On the Need for Conditions on Real Probabilities

Proposition 2 provides conditions on the real probabilities, probabilities which in turn are to be recovered, making the recovery argument somewhat circular. It would be very valuable if there were conditions on the observable variables alone, that is, on $\kappa$, $r$, and $D$, which guaranteed recovery.

Proposition 5 implies that such conditions do not exist, by showing that whenever recovery is possible (because the conditions on $\mu$ and $D$ in Proposition 2 are satisfied), there is also an infinite number of processes, none of which satisfy the conditions for recovery, with associated positive pricing kernels, that are also consistent with the fundamental ODE.

**Proposition 5.** *Consider an economy in which Condition (23) is satisfied, thus allowing for recovery. Then, for each $\lambda < \rho$, the fundamental ODE, $\mathcal{W}[s|\lambda] = 0$, has a strictly positive solution.*

**Corollary 1.** *None of the "recovered" probability distributions when $\lambda < \rho$ satisfy Condition (23).*

Thus, knowledge of $\kappa$, $r$, and $D$ alone is never sufficient to ensure that recovery is possible: One needs to that the real process is recurrent. Note that this circularity in the recovery argument is weak though, since Condition (23) restricts the asymptotic behavior of $\mu(x)$ as $x$ tends to infinity and, given that it is satisfied, the whole function $\mu(x)$ can then be recovered. As discussed, a sufficient condition for recovery to be possible is that the process is stationary. In a discrete setting, standard unit root tests, see Phillips (1987) and Phillips and

Xiao (1998), can be used to test for stationarity. Stationarity tests for continuous time diffusion processes are further analyzed in more recent papers, see, for example, Bandi and Phillips (2010), Aït-Sahalia and Park (2012), and Kim and Park (2015, 2016).

Bandi and Corradi (2014), see also Hamrick and Taqqu (2009), develop nonparametric tests for stationarity that are robust for nonlinear dynamic processes. Their test is based on the different divergence rates of occupation times for stationary and nonstationary processes, and is therefore directly related to the recurrence properties of the underlying process that determine whether recovery is possible.

Stationarity is a sufficient but not necessary condition for recovery, and as we have seen there are nonstationary process for which recovery is possible. Specifically, processes that are null recurrent do not have stationary distributions but allow for recovery. Transient processes on the other hand are explosively nonstationary, and do not allow for recovery. Severely explosive processes are typically easy to detect, and recently developed methods may be used to separate null recurrent processes from those that are moderately explosive, see Phillips and Magdalinos (2007) and Phillips, Wu, and Yu (2011).

### 3.8 Backing out $R$, $\kappa$, and $D$ from Option Prices

At a specific point in time, $t$, we only observe $p^s(x_t, y)$ for general $s > t$ and $y \in \mathbb{R}$. However, in our previous derivation of $r$, $D$, and $\kappa$ we needed $p^t(x, y)$ for general $x \in \mathbb{R}$. The following proposition shows that it is sufficient to know $p^t(x_0, y)$.[10]

**Proposition 6.** *Assume that at time 0, the prices $p^t(x_0, y)$ are observed for all $y$, for all $t \in (0, T)$, for some $T > 0$. Define $V(t, y) = p^t(x_0, y)$. Then, for each $y$ and $t > 0$, $V$ satisfies the PDE:*

$$V_t = D(y)V_{yy} + \alpha_1(y)V_y + \alpha_0(y)V, \tag{35}$$

where

$$\alpha_1(y) = 2D'(y) - \kappa(y), \tag{36}$$

$$\alpha_0(y) = D''(y) - \kappa'(y) - r(y). \tag{37}$$

Thus, by observing $V(t, y)$, we can calculate $V_t$, $V_y$, and $V_{yy}$, and use Equation (35) to solve for $D(y)$, $\alpha_1(y)$, and $\alpha_0(y)$. Since there are three unknowns, for each $y$, $V$, $V_t$, $V_y$, and $V_{yy}$ need to be known for three different $t$, to calculate $D(y)$, $\alpha_1(y)$, and $\alpha_0(y)$. Once $D(y)$ is known in a neighborhood of $y$, $\kappa(y)$ can be calculated, using Equation (36), and given that $\kappa(y)$ is known in a neighborhood of $y$, $r(y)$ can be calculated, using Equation (37).

We note that the prices of AD securities, $V(t, K)$, can be inferred from the prices, $C^t(K)$, of call options with strike price $K$ and maturity $t$, $0 < t < T$, $K \in \mathbb{R}$,

$$C^t(K) \stackrel{\text{def}}{=} \int_K^\infty (y - K)\frac{m(y)}{m(x_0)} f^t(x_0, y)\mathrm{d}y. \tag{38}$$

The price $V(t, K) = C^t_{KK}(K)$ is the second derivative of the price of the call option, with respect to the strike price, so $D$, $r$, and $\kappa$ can thus equivalently be calculated from call option prices. Proposition 6 can either be shown using the risk neutral measure, or equivalently by following similar lines as in Dupire's method for backing out volatility in the local volatility

10 A somewhat related problem is that of deriving state prices from observed option prices, see Breeden and Litzenberger (1978) and the large subsequent literature. In what follows, we assume that state prices have already been identified from existing derivative prices.

model (see Dupire, 1994). We note that the method is local in the sense that to back out $D$, $\kappa$, and $r$ at $x$, only option prices with strike prices around $x$ are needed.

As an example, consider the BM $dX = \mu dt + \sigma d\omega$, $m(x) = 1 + x^2$, and assume that $x_0 = 0$. We then get

$$V(t,y) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\rho t}(1+y^2)e^{-\frac{y^2}{2\sigma^2 t}}.$$

Calculating $V_t$, $V_y$, and $V_{yy}$, and backing out $D$, $\alpha_1$, and $\alpha_0$ from Equation (35) leads to

$$\begin{aligned} D &= \frac{\sigma^2}{2}, \\ \alpha_1 &= -\left(\mu + \frac{2y\sigma^2}{1+y^2}\right), \\ \alpha_0 &= -\rho + \mu\frac{2y}{1+y^2} - \sigma^2\frac{1-3y^2}{(1+y^2)^2}. \end{aligned}$$

Since $D' \equiv 0$, from Equation (36) we get $\kappa = -\alpha_1$ in line with Equation (27), and from Equation (37) $r = \alpha_{1'} - \alpha_2 = -2\sigma^2\frac{1-y^2}{(1+y^2)^2} - \left(-\rho + \mu\frac{2y}{1+y^2} - \sigma^2\frac{1-3y^2}{(1+y^2)^2}\right) = \rho - \frac{2y\mu+\sigma^2}{1+y^2}$ in line with Equation (26).

## 4. Approximate Recovery

The conditions for recovery in the unbounded case, introduced in Proposition 2, also determine how well recovery works on bounded and on discrete domains, as shown in the following discussion.

### 4.1 Bounded Domains

In practice, option prices are not available for arbitrarily large states, so we would not be able to observe $p^t(x,y)$ for all $x$ and $y$. An open question is then whether "approximate" recovery of $m$ is possible given that $p^t(x,y)$ is only known on some domain, $-N \leq x$, $y \leq N$, for some $N > 0$. Of course, in the case when recovery is not possible even if $p^t(x,y)$ is known for all $x$, $y$, that is, even if $N = \infty$, recovery can never be possible when $N < \infty$. We therefore focus on the case when recovery is possible when $N = \infty$.

The question of approximate recovery is important: if few inferences about $m$ can be drawn even for arbitrarily large but finite $N$, then for all practical purposes, recovery in the case with unbounded diffusion processes will not work. An example of such a situation is given in Dubynskiy and Goldstein (2013), where additional information about the representative agent's preference parameters is needed for recovery to work. But their example is exactly one for which recovery does not work even if $N = \infty$, and is therefore of limited use for us.

The following result shows that as long as recovery is possible when $N = \infty$, strong inferences can be drawn about $m$ in the case when $N < \infty$, without any additional information.

**Proposition 7.** *The following two conditions are equivalent:*

1. *When $N = \infty$, $m$ and $\rho$ can be uniquely recovered.*
2. *Given a finite $N > 0$, $m(x)$ and $\rho$ can be approximated by functions $\hat{m}_N(x)$, defined on $(-N, N)$, and $\hat{\rho}_N$, such that*

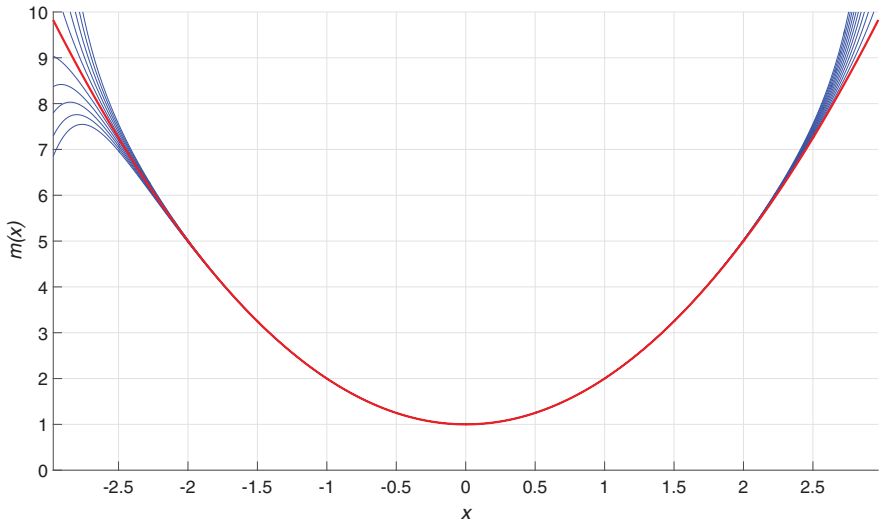


**Figure 5.** Candidate functions, $m(x)$, $c = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.8, 1$ in the OU economy. The solid (red) line is the true $m(x) = 1 + x^2$. Given that $r$, $\kappa$, and $D$ are observed on $x \in [-3, 3]$, the candidate functions are those consistent with positivity of $z$ on $[-3, 3]$, for the estimated discount rate, $\hat{\rho}$. For $x \in [-2, 2]$, $m$ is well approximated. The parameter values are: $\theta = 0.01$, $\rho = 0.01$, $\sigma = 0.1$. The estimated discount rate is $\hat{\rho} = 0.010002$.

3. $\hat{\rho}_N$ *is nonincreasing in* $N$, *and* $\lim_{N\to\infty} \hat{\rho}_n = \rho$,
4. *for each* $x$, $\lim_{N\to\infty} \hat{m}_N(x) = m(x)$.

Thus, as long as recovery is possible on the unbounded domain, approximate recovery is possible on a bounded subdomain.

The argument behind the result is as follows. When $N < \infty$, we can solve for all candidate functions $z_{\rho,c}$, which satisfy Equation (21) on $[-N, N]$, and are positive. Any candidate $z_{\lambda,c}$ for $\lambda > \rho$ will eventually become negative, and can therefore be ruled out if we have a large enough domain of observation. It follows from standard theory of ODEs that the larger $\lambda > \rho$ is, the faster $z$ will become negative, so for large domains, only $z_{\lambda,c}$ for $\lambda$ very close to $\rho$ stay positive on the whole observed domain. However, these candidate $z_{\lambda,c}$s are then also close to the true $z$, because of continuity. Therefore, as $N$ increases, tighter and tighter bounds on both $m = \frac{1}{z}$, and $\rho$ can be inferred.

We show how such approximate recovery works for the OU example with $m(x) = 1 + x^2$. In Figure 5, we assume that $r$, $\kappa$, and $D$ are observed on $x \in [-3, 3]$, and calculate the approximate $m$ function, as well as the approximated $\hat{\rho}$. We see that for $|x| \leq 2$, the approximation is very close to the correct solution, whereas the error is larger when we approach the boundary. This is typical: At $x = N$, the upper bound on $m$ is infinity at one of the boundaries, since the only condition we have is that $z > 0$ (i.e., $m < \infty$) on the whole domain. The approximated $\hat{\rho} = 0.010002$ is very close to the true $\rho = 0.01$. We stress that no additional information was needed in this approximation, that is, we imposed no "artificial" boundary conditions.

In the Appendix, we provide Matlab code for approximating the pricing kernel on a finite domain, given $D$, $r$, and $\kappa$ evaluated at $N$ equidistant points, $x_0, x_0 + \Delta x, \ldots, x_0 + (N-1)\Delta x$, where $\Delta x > 0$. The code consists of two parts: the first part calculates the general solutions to the ODE, given a conjectured $\hat{\rho}$, using a standard finite difference method. The second part tests whether a positive kernel can be constructed as a linear

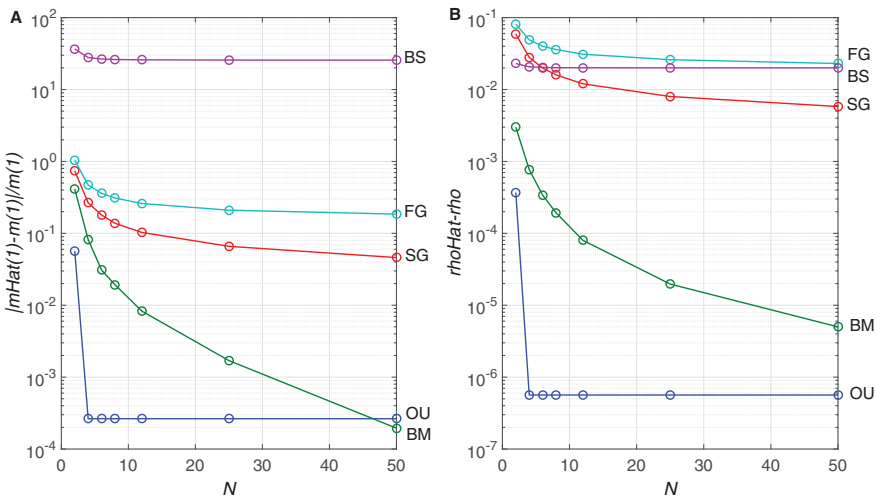


**Figure 6.** Approximate recovery for different diffusion processes. The processes are 1. OU, $dX = -0.01dt + 0.1d\omega$, 2. BM, $dX = 0.01d\omega$, 3. SG, $dX = 0.05Xdt + \sqrt{0.1(1+X^2)}d\omega$, 4. FG, $dX = 0.1Xdt + \sqrt{0.1(1+X^2)}d\omega$, and 5. BS, $dX = 0.01dt + 0.1d\omega$. Processes 1–3 satisfy the conditions for recovery, whereas processes 4 and 5 do not. The left panel (A) shows the relative error of the approximated $\hat{m}$ at $X = 1$, $\frac{|\hat{m}(1)-m(1)|}{m(1)}$, as a function of the interval $-[N, N]$ observed, for the five processes. The right panel (B) shows the error in the approximate personal discount rate, $\hat{\rho} - \rho$. The pricing kernel $m(x) = 1 + x^2$ is used, and the personal discount rate is $\rho = 0.01$. A small step-length of $\Delta x = 10^{-4}$ is used, to focus on the error introduced by bounded observations. In both panels, the convergence to the correct solution is fast for 1 and 2, no convergence occurs for 5, and it is unclear from the figure whether convergence occurs for 3 and 4.

combination of the general solutions, and updates the conjectured $\hat{\rho}$ iteratively. If multiple positive solutions exist, this means that the conjectured $\hat{\rho}$ was too low, and if no positive solutions exist, this means that the conjectured $\hat{\rho}$ was too high.

The code performs well for the examples in this paper, as well as for several other examples. Convergence to $\hat{\rho}$ is typically obtained in 15–30 iterations. In Figure 6, we show the approximation error for several different examples, as the interval, $[-N, N]$ increases. We use the OU process, the classical BS process, and the BM process without drift where $dX = 0.1d\omega$, as previously analyzed. We also introduce two new examples, that are close to the growth threshold that determines whether recovery is possible, but on different sides of this threshold. The first is a slow growth (SG) process, $dX = 0.05Xdt + \sqrt{0.1(1+X^2)}d\omega$, and the second is a fast growth (FG) process, $dX = 0.1Xdt + \sqrt{0.1(1+X^2)}d\omega$. It is easily verified that the first economy satisfies the conditions for recovery, whereas the second does not. In all five examples, we use the nonstandard pricing kernel $m = 1 + x^2$. For robustness, we have also used standard power utility, $m = e^{-\gamma x}$, with similar results (not reported).

The left panel of Figure 6 shows the relative error of the approximated $\hat{m}$, evaluated at $X = 1$, $\frac{|\hat{m}(1)-m(1)|}{m(1)}$, when $N$ varies from 2 to 50. We see that the error decreases quickly as $N$ increases for the OU and BM processes.[11] For the BS process, the error is large and basically

11 For the OU process, the error for $N > 2$ is very small, but constant as $N$ increases. This is because of the error introduced by using a finite difference method to solve the ODE, which is independent of the observation range, $N$. By decreasing $\Delta x$, this error can be decreased.

constant, in line with recovery failing for this process. For the intermediate SG and FG processes, the errors decrease, but it is hard to draw inferences about ultimate convergence. A similar picture emerges for the error in the approximate discount rate, $\hat{\rho} - \rho$, in the right panel of the figure.

From the above results, it is clear that Condition (23) influences how large an interval is needed to get accurate approximate recovery. If $\int_{-N}^{N} Q(y)\mathrm{d}y$ grows quickly as $N$ grows, as in OU and BM, then a small interval is sufficient for good recovery. If $\int_{-N}^{N} Q(y)\,\mathrm{d}y$ converges quickly to a finite value, as in BS, then no convergence occurs. In the intermediate cases, SG and FG, the error decreases very slowly, and very large intervals are needed to draw inferences. The SG example is chosen such that $Q(y) = \frac{1}{\sqrt{1+y^2}}$, leading to $\int_{-N}^{N} Q(y)\mathrm{d}y = 2\sinh^{-1}(N) \approx 2\ln(1+N)$. It is difficult to draw inferences about whether this function is ultimately bounded or unbounded from its behavior on a finite domain. Thus, even though approximate recovery works for this function in theory, stronger constraints on $Q$ may be needed for the method to work in practice. We leave a detailed analysis of the convergence properties of the numerical method for future work.

### 4.2 Discrete Domains

Since recovery is always possible when the number of states is finite, one may argue that the best approach is to simply work with finite state spaces and thereby avoid the issue of recoverability.

It turns out, however, that the conditions for recoverability of the continuous problem are also important for the discrete problem, in that they determine whether the discrete recovery problem is sensitive to perturbations, for example, generated by observation errors in the state prices. We show how such sensitivity manifests itself in a specific example, and carry out a more general analysis in the Appendix that shows that whether the discrete problem is sensitive to perturbations or not is closely related to the conditions of Proposition 2.

Consider the diffusion process

$$\mathrm{d}X_t = \frac{A}{4} G(4X_t)\mathrm{d}t + \mathrm{d}\omega, \qquad A = \pm 1.$$

where

$$G(x) = \begin{cases} \operatorname{sgn}(x), & |x| \geq 1, \\ \operatorname{sgn}(x)(1 - (1 - \operatorname{sgn}(x)x)^2), & |x| < 1 \end{cases}$$

is a smooth, nondecreasing, antisymmetric function which is constant when $|x| \geq 1$. It follows immediately that when $A = -1$, the process is mean reverting and thus satisfies the conditions for recovery, whereas when $A = 1$, neither of the conditions in Proposition 2 are satisfied and thus recovery is not possible.

We discretize this problem using the binomial tree method, with coefficients chosen as in Cox, Ross, and Rubinstein (1979), and step-length $\Delta X = \frac{1}{4}$. Under the assumption that the representative agent is risk neutral with discount factor $\rho = 0$ (corresponding to $\delta = 1$ in the discrete case), the risk neutral and true probabilities then coincide, $P = F$. The state price matrices with seven states are shown in Equations (39) and (40), where the borders of the tridiagonal matrices are chosen so that $X_t$ stays within the state space. Note that the matrices are such that for $P^{A=-1}$, the process tends to revert back to state 4 (corresponding

to $X = 0$), whereas for $P^{A=1}$, it tends to move toward the border states, 1 and 7 (corresponding to $X = -3/4$ and 3/4, respectively).

In general, when the discretized interval goes from $-(N-1)\Delta X$ to $(N-1)\Delta X$, $P$ is tridiagonal of dimension $(2N+1) \times (2N+1)$, and when $N$ increases, so does the number interior points (with identical elements) of $P^{A=1}$ and $P^{A=-1}$.

$$P^{A=-1} = \begin{bmatrix} 0.47 & 0.53 & 0 & 0 & 0 & 0 & 0 \\ 0.47 & 0 & 0.53 & 0 & 0 & 0 & 0 \\ 0 & 0.47 & 0 & 0.53 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.53 & 0 & 0.47 & 0 \\ 0 & 0 & 0 & 0 & 0.53 & 0 & 0.47 \\ 0 & 0 & 0 & 0 & 0 & 0.53 & 0.47 \end{bmatrix}, \tag{39}$$

$$P^{A=1} = \begin{bmatrix} 0.53 & 0.47 & 0 & 0 & 0 & 0 & 0 \\ 0.53 & 0 & 0.47 & 0 & 0 & 0 & 0 \\ 0 & 0.53 & 0 & 0.47 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.47 & 0 & 0.53 & 0 \\ 0 & 0 & 0 & 0 & 0.47 & 0 & 0.53 \\ 0 & 0 & 0 & 0 & 0 & 0.47 & 0.53 \end{bmatrix}. \tag{40}$$

Now, both $P^{A=-1}$ and $P^{A=1}$ satisfy the conditions for (discrete) recovery, and both have the unique positive eigenvector 1 with associated eigenvalue $\delta = 1$, which thus identifies the correct risk neutral pricing kernel. However, the sensitivity properties of the two matrices are very different. Consider the recovered eigenvector in the discretized economy with 201 states, when we perturb the matrix by replacing $P_{11}$ with $P_{11} + \epsilon$, $\epsilon = 0.001$. The effect on the recovered pricing kernel in the mean reverting case, $A = -1$, is marginal, as shown in the left panel of Figure 7, whereas it is drastic when $A = 1$ as shown in the right panel of the same figure.

That the effects are so different in the two cases can be seen by studying the difference equations corresponding to the eigenvector problems of the two matrices. Indeed, for the interior rows, the eigenvector problem $z = Pz$ corresponds to the difference equation

$$(0.5 - \alpha)z_{j+1} = z_j - (0.5 + \alpha)z_{j-1},$$

where $\alpha = 0.03$ when $A = 1$, and $\alpha = -0.03$ when $A = -1$. The general solution to this difference equation is

$$z_j = ar_1^j + br_2^j,$$

where—as is easy to confirm—the characteristic roots are $r_1 = 1$ and $r_2 = 1 + 2\alpha$. The root $r_1$ corresponds to the correct, constant, eigenvector, whereas $r_2$ is a spurious solution which gets some weight in the perturbed problem. When $r_2 < 1$, corresponding to a mean reverting process, this solution quickly dies out as $j$ increases, as shown in the left panel of Figure 7. In contrast, when $A = 1$, $r_2 > 1$, and the spurious solution completely contaminates the correct solution as shown in the right panel of the same figure.

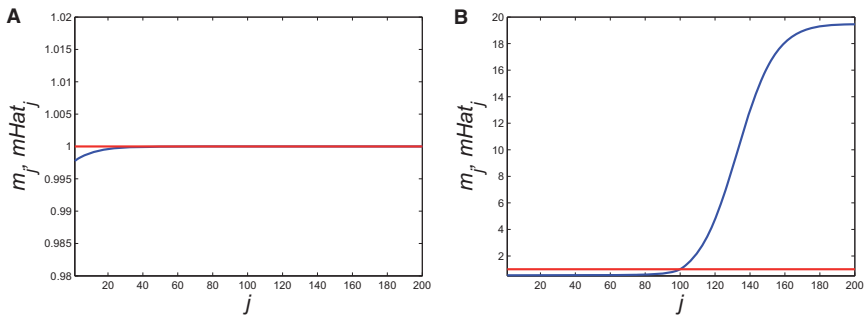


**Figure 7.** Discrete state price matrix with small perturbation, when the underlying diffusion process is recoverable (Panel A, left) and not recoverable (Panel B, right). In each panel, the straight (red) line represents the true kernel, and the (blue) curve represents the recovered kernel. The kernel is not sensitive to perturbations in the recoverable case, whereas it is sensitive in the unrecoverable case. Number of states is $2N + 1 = 201$.

In this example, the process for $X$ leads to a finite difference equation with constant coefficients. In Appendix D, we show that in the general case, with variable coefficients, the behavior of $Q(x)$ for large (positive or negative) $x$ governs whether the spurious solution is dampened or blows up in the discretized problem, providing a link to the continuous problem.

Thus, to conclude, our analysis in this section shows that the continuous recovery conditions also have important implications for the practical feasibility of the recovery method with bounded and discrete domains.

## 5. Concluding Remarks

We have provided a general characterization of when recovery of the pricing kernel and real probability distribution is possible in a model with a time homogeneous diffusion process on an unbounded domain. The existence of a stationary distribution, for example, is a sufficient but not necessary condition for recovery. With further restrictions on marginal utility, long-term growth can be incorporated. When recovery works on the unbounded domain, then even if prices are only observed on a bounded subdomain, the kernel and probability distribution on this subdomain can be approximated well without imposing additional boundary conditions.

Altogether, our results suggest that recovery is possible for many interesting cases, but that it will not work in economies that are "too close" to the standard setting with positive long-term growth and unbounded marginal utility.

## Appendix A: State Prices, Parameters, and Fundamental ODE

We further explore how the parameters, $D$, $\kappa$, and $r$ are related with state prices, and how they lead to the fundamental ODE. Using Ito's lemma and differential notation, given that $X_0 = x$, the price of an asset that pays 1 at d$t$ is $P^r = 1 - r\mathrm{d}t$, where

$$r(x) = \rho - q(x)\mu(x) - (q'(x) + q(x)^2)D(x)$$

is the short risk-free rate, which in general is a function of $x$. We have

$$r(x) = \lim_{t \searrow 0} \frac{1 - \int p^t(x,y)\mathrm{d}y}{t}. \tag{41}$$

Thus, the short rate at any $x$ can be recovered from knowledge of $p^t$. Similarly, the price of the AD security $p^{\Delta t}(x, x + \Delta t)$ is approximately $\frac{1}{\sqrt{2\pi\sigma(x)^2\Delta t}}$, so we can back out

$$D(x) = \lim_{t \searrow 0} \frac{1}{4\pi t} \times \left( \frac{1}{p^t(x, x+t)} \right)^2. \tag{42}$$

Finally, consider

$$\kappa(x) \stackrel{\text{def}}{=} \mu(x) + 2q(x)D(x).$$

The price of a security that pays off $X_{\mathrm{d}t}$ at $\mathrm{d}t$, given that $X_0 = x$, is

$$\begin{aligned} P^x &= E\left[ \mathrm{e}^{-\rho \mathrm{d}t} \frac{m(X_{\mathrm{d}t})}{m(x)} X_{\mathrm{d}t} \right] \\ &= E\left[ x + \frac{\mathrm{d}(Xe^{-\rho t} m(X))}{m(x)} \right] \\ &= x - \rho x \mathrm{d}t + xq(x)\mu(x)\mathrm{d}t + x(q'(x) + q(x)^2)D(x)\mathrm{d}t + \mu(x)\mathrm{d}t + 2q(x)D(x)\mathrm{d}t \\ &= x(1 - r(x)\mathrm{d}t) + \kappa(x)\mathrm{d}t. \end{aligned}$$

In risk neutral terminology, $\kappa(x)$ is the drift of the state variable, $x$, in the risk neutral measure. We can therefore back out $\kappa(x)$ as

$$\kappa(x) = r(x)x + \lim_{t \searrow 0} \frac{\int p^t(x,y) y \,\mathrm{d}y - x}{t}. \tag{43}$$

To summarize, if the prices of AD securities are observable for all $t > 0$, $x$, and $y$, then $r$, $D$, and $\kappa$ are directly observable from Equations (41), (42), and (43).

## Appendix B: Relationship between Fundamental ODE and Integral Equation Formulation

There is a close relationship between the fundamental ODE and the eigenvalue problem (6) in Ross (2015). The relationship also provides an alternative derivation of the fundamental ODE. In the diffusion process setting, the eigenvalue problem turns into a linear integral equation. Specifically, we have

$$\int f^t(x,y)\mathrm{d}y = 1, \qquad \forall x, \tag{44}$$

which is the continuous version of Equation (1). We rewrite this on operator form as

$$f^t[1] = 1, \qquad \text{where} \qquad f^t[s](x) \stackrel{\text{def}}{=} \int f^t(x,y)s(y)\mathrm{d}y,$$

for an arbitrary function $s(y)$.

From Equation (15), we have $f^t(x,y) = \mathrm{e}^{\rho t} m(x) p^t(x,y) m(y)^{-1}$, similar to Equation (4), which when plugged into Equation (44) yields $\int \mathrm{e}^{\rho t} m(x) p(x,y) m(y)^{-1} \mathrm{d}y = 1$, or

$$\int p^t(x,y) m(y)^{-1} \mathrm{d}y = \mathrm{e}^{-\rho t} m^{-1}(x).$$

or, for $z = \frac{1}{m}$,

$$p^t[z](x) \stackrel{\text{def}}{=} \int p^t(x,y)z(y)\mathrm{d}y = \mathrm{e}^{-\rho t}z(x),$$

similar to Equation (6). On operator form, this reads

$$p^t[z] = \mathrm{e}^{-\rho t}z, \tag{45}$$

which is an integral equation eigenfunction problem. This is the continuous time diffusion process equivalent of the eigenvalue problem in Ross (2015).

For small $\Delta t$, the Fokker–Planck equation implies that

$$f^{\Delta t}(x,y) \approx \frac{1}{\sqrt{2\pi\sigma(x)^2\Delta t}}\mathrm{e}^{-\frac{(x-y-\mu(x)\Delta t)^2}{2\sigma(x)^2\Delta t}},$$

which implies that for a smooth function, $s(y)$, that is bounded by $C_\epsilon \mathrm{e}^{\epsilon y^2}$ for large $y$ and any $\epsilon > 0$,

$$f^{\Delta t}[s](x) = s(x) + \Delta t\left(\frac{\sigma^2(x)}{2}s''(x) + \mu(x)s'(x)\right) + \text{h.o.t.},$$

where "h.o.t." denotes higher order terms in $\Delta t$. We define the infinitesimal operator $\mathcal{L}s \stackrel{\text{def}}{=} \frac{\sigma^2(x)}{2}s''(x) + \mu(x)s'(x)$, so that $\mathcal{L}$ is the adjoint of $\mathcal{L}^*$, and we can then write the relation as

$$f^{\mathrm{d}t}[s] - s = \mathrm{d}t \times \mathcal{L}s. \tag{46}$$

Thus, an eigenfunction to $f^{\mathrm{d}t}$ must satisfy $\lambda s = \mathcal{L}s$. Clearly, $s \equiv 1$ is such a function, with $\lambda = 0$, leading to $f^{\mathrm{d}t}[1] = 1$, in line with Equation (44).

Using Equation (15), we get that for an arbitrary function, $v$,

$$\begin{aligned} p^{\mathrm{d}t}[v](x) &= (1 - \rho\mathrm{d}t)m(x)^{-1}\int f^{\mathrm{d}t}(x,y)m(y)v(y)\mathrm{d}y \\ &= (1 - \rho\mathrm{d}t)m(x)^{-1}f^{\mathrm{d}t}[mv](x), \end{aligned}$$

which, using Equation (46), leads to

$$\begin{aligned} p^{\mathrm{d}t}[v](x) &= (1 - \rho\mathrm{d}t)m(x)^{-1}(1 + \mathrm{d}t \times \mathcal{L})[mv](x) \\ &= (1 - \rho\,\mathrm{d}t)v(x) + \mathrm{d}t \times m(x)^{-1}\mathcal{L}[mv](x) \\ &= v(x) - \rho v(x)\mathrm{d}t + \mathrm{d}t \times \mathcal{Q}[v](x), \end{aligned}$$

where $\mathcal{Q}[v](x) \stackrel{\text{def}}{=} (m^{-1}\mathcal{L}[mv])(x)$.

We rewrite

$$\begin{aligned} \mathcal{Q}[v](x) &= \frac{1}{m(x)}\left(\mu(x)\frac{\mathrm{d}}{\mathrm{d}x}[m(x)v(x)] + D(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}[m(x)v(x)]\right) \\ &= \frac{1}{m(x)}\left(\mu(x)(m'(x)v(x) + m(x)v'(x)) + D(x)(m''(x)v(x) + 2m'(x)v'(x) + m(x)v''(x)\right) \\ &= (\rho - r)v + \kappa v' + Dv'', \end{aligned}$$

and for $z = \frac{1}{m}$ we then have

$$0 = m(x)^{-1}\mathcal{L}[1] = \mathcal{Q}[z](x) = (\rho - r)z + \kappa z' + Dz''.$$

The fundamental ODE (21) is thus the differential form of the integral equation eigenvector problem (45).

## Appendix C: Discrete Example in Section 3.4

The relation

$$\lambda z_{1,j} = P_{1,1} z_{1,j+1} + P_{1,2} z_{2,j},$$
$$\lambda z_{2,j} = P_{2,1} z_{1,j} + P_{2,2} z_{2,j-1},$$

$j \in \mathbb{Z}$, represents a second-order system of difference equation of dimensionality two. We use standard methods for systems of difference equations to analyze this equation, rewriting it as a first order system of dimensionality six. Specifically, we define the vector $\hat{z}_j = (z_{1,j+1}, z_{2,j+1}, z_{1,j}, z_{2,j}, z_{1,j-1}, z_{2,j-1})'$, and after some algebra the system can be written

$$\hat{z}_j = \begin{bmatrix} \frac{8\lambda}{3} & -\frac{2}{3} & 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{4}{3\lambda} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \hat{z}_{j-1} = B\hat{z}_{j-1}. \tag{47}$$

The characteristic equation for the matrix $B$ (the equation defined by $\det(B - \xi I)$) is

$$\xi^4 \left( \xi^2 - \xi \left( \frac{8\lambda}{3} + \frac{4}{3\lambda} \right) + 4 \right)$$

with characteristic roots

$$\xi_{1,2} = \frac{2}{3\lambda} \left( 1 + 2\lambda^2 \pm \sqrt{1 - 5\lambda^2 + 4\lambda^4} \right) \tag{48}$$

and $\xi_3 = \xi_4 = \xi_5 = \xi_6 = 0$.

The inhomogeneous characteristic roots and their associated eigenvectors $\nu_1$, $\nu_2$, define the general solution to Equation (47) on the form

$$\hat{z}_j = \alpha \xi_1^j \nu_1 + \beta \xi_2^j \nu_2.$$

The situation is similar to that for the diffusion process: For $\lambda > \delta$ (corresponding to $e^{-\lambda} > e^{-\rho}$ in the continuous model), there are two positive eigenvectors, and therefore multiple positive solutions to Equation (47). For $\lambda = \delta = 1$, there is a unique (up to scaling) positive eigenvector $\nu = (4, 4, 2, 2, 1, 1)'$ and associated eigenvalue $\xi = 2$ which represents the correct pricing kernel, $\hat{z}_j = \xi^j \nu$. For $\lambda < \delta$, there are no strictly positive solutions. Thus, as for the diffusion case we have studied in this paper, when $\hat{X}$ is used to define the state space in this example, the pricing kernel is transition independent and recovery is possible.

## Appendix D: Discrete Problem in Section 4.2

We study the properties of the discretized problem for the general process $dX = \mu(x)dt + \sigma(x)d\omega$. We discretize the problem with a trinomial tree method, with $2N$

$+1$ points, step-length $\Delta X$ in space, and $\Delta t = \alpha \Delta X^2$ in time, where $0 < \alpha \leq 1$ is a constant, chosen sufficiently small to ensure that all probabilities are nonnegative. The discretized problem then covers the domain $[-\bar{X}, \bar{X}]$, where $\bar{X} = N\Delta X$.

The transition probabilities are $F_{i,i+1} = a$, $F_{i,i-1} = b$, and $F_{i,i} = 1 - a - b$, $X_j = (N + 1 - j)\Delta X$, $\mu_j = \mu(X_j)$, $\sigma_j = \sigma(X_j)$, and to match the drift and volatility term, we choose

$$a_j = \frac{1}{2}\left(-\alpha\mu_j\Delta X + \alpha^2\mu_j^2\Delta X^2 + \alpha\sigma_j^2\right), \tag{49}$$

$$b_j = \frac{1}{2}\left(\alpha\mu_j\Delta X + \alpha^2\mu_j^2\Delta X^2 + \alpha\sigma_j^2\right), \tag{50}$$

$$c_j = 1 - \alpha^2\mu_j^2\Delta X^2 - \alpha\sigma_j^2. \tag{51}$$

Assuming a risk neutral representative agent and discount factor $\delta = 1$, we then have $P_{ij} = F_{ij}$, and the eigenvector problem leads to the finite difference relation

$$z_j = a_j z_{j+1} + c_j z_j + b_j z_{j-1},$$

which when plugging in the coefficients (49–51) leads to

$$\begin{aligned} z_j &= \frac{1}{2}\left(1 - \frac{\alpha\mu_j\Delta X}{\alpha^2\mu_j^2\Delta X^2 + \alpha\sigma_j^2}\right)z_{j+1} + \left(1 + \frac{\alpha\mu_j\Delta X}{\alpha^2\mu_j^2\Delta X^2 + \alpha\sigma_j^2}\right)z_{j-1} \\ &= \frac{1}{2}\left(1 - \frac{\mu_j}{\sigma_j^2}\Delta X\right)z_{j+1} + \left(1 + \frac{\mu_j}{\sigma_j^2}\Delta X\right)z_{j-1} + O(\Delta X^2). \end{aligned}$$

For small step lengths, $\Delta X << 1$, this corresponds to the difference equation

$$z_j = \frac{1}{2}(1 - \alpha_j)z_{j+1} + \frac{1}{2}(1 + \alpha_j)z_{j-1}, \qquad \alpha_j = \frac{\mu_j}{\sigma_j^2}\Delta X,$$

which—again disregarding higher-order terms—leads to the equation

$$z_{j+1} = 2(1 + \alpha_j)z_j - (1 + 2\alpha_j)z_{j-1}. \tag{52}$$

Here, we focus on the domain $X > 0$, corresponding to points $j = 1, \ldots, N$. An identical argument holds for $X < 0$.

By defining $\mathbf{z}_j = (z_j, z_{j-1})'$, we can rewrite the difference formula on one-step form as

$$\mathbf{z}_{j+1} = V_j\mathbf{z}_j, \qquad V_j = \begin{bmatrix} 2 + 2\alpha_j & -1 - 2\alpha_j \\ 1 & 0 \end{bmatrix}.$$

Similar to the constant coefficient case, we are interested in the sensitivity of the mapping

$$\mathbf{z}_N = W_{j,N}\mathbf{z}_j, \qquad W_{j,N} = \prod_{k=j}^{N-1} V_k, \qquad j = 1, \ldots, N-1.$$

It is easy to verify that the eigenvalues of $W_{j,N}$ are $R_{j,N}^1 = 1$ and $R_{j,N}^2 = \prod_{k=j}^{N-1}(1 + 2\alpha_k)$, where the eigenvector 1 corresponds to the eigenvalue 1, and is the correctly recovered eigenvector.

Now, if $R^2_{j,N} >> 1$, then the solution will be sensitive to perturbations at $X = (N+1-j)\Delta X$. For small $\Delta X$, we have

$$\begin{aligned}\log(R^2_{j,N}) &= \sum_{k=1}^{N+1-j} \log\left(1 + 2\frac{\mu(k\Delta X)}{\sigma(k\Delta X)}\Delta X\right) \\ &= \sum_{k=1}^{N+1-j} 2\frac{\mu(k\Delta X)}{\sigma(k\Delta X)^2}\Delta X + O(\Delta X^2) \\ &= -\log(Q((N-k)\Delta X)) + O(\Delta X^2).\end{aligned}$$

Finally, if errors of the order $\epsilon\Delta X$, with $0 < \epsilon << 1$ are introduced in each step of the finite difference equation, which we would expect merely from discretization errors when going from a continuous to a discrete domain, then the total error will be of the order

$$\sum_{k=1}^{N-1} R^2_{j,N}\epsilon\Delta X = \epsilon\sum_{k} Q^{-1}((N-k)\Delta X)\Delta X + O(\Delta X) = \epsilon\int_0^{\bar{X}} Q^{-1}(x)\mathrm{d}x + O(\Delta X).$$

Thus, as the domain covered tends to infinity $\int_0^\infty Q^{-1}(x)\mathrm{d}x < \infty$ is needed for the discrete method not to be sensitive to perturbations, which in turn implies that $\int_0^\infty Q(x)\mathrm{d}x = \infty$. A similar argument applies for negative values of $X$, leading to $\int_{-\infty}^0 Q(x)\mathrm{d}x = \infty$. So, recovery for the continuous space problem is a necessary condition for the discretized problem not to be sensitive to perturbations.

The previous analysis assumed $\rho = 0$, corresponding to $\delta = 1$ in the discrete problem, but an identical argument holds when $\rho > 0$. Moreover, the risk neutrality assumption can also be dropped, since in the general case the mapping is $P = \delta^{-1}M^{-1}FM$, but since eigenvalues are invariant to similarity transformations, the same analysis applies when $M$ is not an identity matrix.

## Appendix E: Proofs

*Proof of Proposition 1*: The result follows from the standard properties of solutions to second-order linear ODEs, see, for example, Simmons (1988, pp. 72–78).

*Proof of Proposition 2*: Recall that under our model assumptions, there is a strictly positive $m$ that solves the fundamental ODE for $\lambda = \rho$, and thus also a strictly positive $z$. However, the issue is that there may be other such positive $z$s, for $\lambda \neq \rho$, and potentially also for $\lambda = \rho$. So, we need to understand when it is possible to single out a unique such $z$.

*Necessity*: Assume that $z_1$ is a strictly positive solution to Equation (21). From Proposition 1, we know that the general solution (up to a multiplication by a constant) is on the form $z_c = z_1 + cz_2$, where $z_2$ is also a solution. It is sufficient to show that any other solution, $z_c$, $c \neq 0$ must be negative at some point.

As discussed in Simmons (1988, pp. 81–83), $z_2$ can be solved for, once $z_1$ is known. The general solution, $z_c$ can then be written as

$$\begin{aligned}z_c(x) &= z_1(x)\left(1 + c\int_0^x \frac{1}{z_1(y)^2}\mathrm{e}^{-\int_0^y\left(\frac{\mu(s)}{D(s)} + 2q(s)\right)\mathrm{d}s}\mathrm{d}y\right) \\ &= z_1(x)\left(1 + c\int_0^x \frac{z_1(y)^2}{z_1(y)^2}\mathrm{e}^{-\int_0^y \frac{\mu(s)}{D(s)}\mathrm{d}s}\mathrm{d}y\right) \\ &= z_1(x)(1 + c\int_0^x \mathrm{e}^{-\int_0^y \frac{\mu(s)}{D(s)}\mathrm{d}s}\mathrm{d}y) \\ &= z_1(x)(1 + cR(x)),\end{aligned} \tag{53}$$

where $R(x) \stackrel{\text{def}}{=} \int_0^x e^{-\int_0^y \frac{\mu(s)}{D(s)} ds} dy$. Of course, the sign of $z_c(x)$ is the same as the sign of $1 + cR(x)$, so strict positivity of $z_c$ is equivalent to strict positivity of $1 + cR(x)$. Now, $R(x)$ is a strictly increasing function such that $R(0) = 0$. If $R(\infty) < \infty$, then for small $c < 0$, $z_c$ is strictly positive, as is the case for small $c > 0$, if $R(-\infty) > -\infty$. In this case, recovery is not possible, even if $\rho$ is known, since there are multiple candidate solutions that are all strictly positive, so necessity follows.

*Sufficiency*: The argument above implies that if $R(-\infty) = -\infty$, and $R(\infty) = \infty$, then recovery is possible, given that $\rho$ is known. If we show that there are no strictly positive solutions to $W[z|\lambda] = 0$ for $\lambda > \rho$ in this case, then recovery follows automatically, since $\rho$ must be the largest $\lambda$ for which the solution to $W[z|\lambda] = 0$ has exactly one strictly positive solution.

We transform the ODE

$$\frac{d^2 s}{dx^2} + \frac{\kappa}{D}\frac{ds}{dx} + \frac{\lambda - r}{D} s = 0 \tag{54}$$

to normal form [see Simmons (1988, pp. 119–120)], to get $s = uv$, where $v(x) = e^{-\frac{1}{2}\int_0^x \frac{\kappa}{D} dy} = z e^{-\frac{1}{2}\int_0^x \frac{\mu}{D} dy}$, and $u$ is the general solution to the ODE

$$u'' + \left(\tau(x) + \frac{\lambda - \rho}{D(x)}\right) u = 0, \qquad \tau \stackrel{\text{def}}{=} -\frac{1}{4}\left(\frac{\mu}{D}\right)^2 - \frac{1}{2}\frac{d}{dx}\left(\frac{\mu}{D}\right). \tag{55}$$

For $\lambda = \rho$, it is easy to see that the strictly positive function $u_\rho(x) = e^{\frac{1}{2}\int_0^x \frac{\mu}{D} dy}$ solves Equation (55), which in turn has $u_\rho(0) = 1$, and $u'_\rho(0) = \frac{1}{2}\frac{\mu(0)}{D(0)}$.

Define the function $u_\lambda(x)$, as the solution to Equation (55), with parameter $\lambda > \rho$, and initial conditions $u_\lambda(0) = u_\rho(0)$, $u'_\lambda(0) = u'_\rho(0)$. Then, if we can show that $u_\lambda$ has at least two roots, that is, that there are two points, $x_1$, and $x_2$, for which $u_\lambda(x_1) = u_\lambda(x_2) = 0$, it follows from the Sturm comparison theorem (see Simon, 2005) that any solution to Equation (55) has at least one root. Moreover, since $s = uv$, and $v > 0$ for all $\lambda$ and $x$, this in turn implies that any solution to Equation (54) with $\lambda > \rho$ has at least one root, and is therefore disqualified as a candidate solution for $1/m$. Therefore, $z$ and $\rho$ can be uniquely recovered. Specifically, in this case, $z$ is the unique positive solution to $W[z|\rho] = 0$, and for no $\lambda > \rho$, is there a positive solution to $W[s|\lambda] = 0$.

To show that $u_\lambda$ has at least two roots for all $\lambda > \rho$, we proceed as follows. We define $w_\lambda(x) = \frac{u'_\lambda(x)}{u_\lambda(x)}$. Since $u$ is continuous and defined on the whole of $\mathbb{R}$, it must be that if $|w_\lambda|$ tends to infinity at some some finite $x$, then $u_\lambda(x) = 0$. Of course, $w_\rho(x) = \frac{u'_\rho(x)}{u_\rho(x)} = \frac{1}{2}\frac{\mu(x)}{D(x)}$.

From Equation (55), it follows that

$$\frac{u''_\lambda}{u_\lambda} = \frac{u''_\rho}{u_\rho} - \frac{\lambda - \rho}{D}. \tag{56}$$

Moreover, since $w'_\lambda = \frac{u''_\lambda}{u_\lambda} - w_\lambda^2$, we can rewrite Equation (56) as

$$w_{\lambda'} + w_\lambda^2 = w'_\rho + w_\rho^2 - \frac{\lambda - \rho}{D},$$

or

$$\begin{aligned} w_{\lambda'} - w'_\rho &= -(w_\lambda^2 - w_\rho^2) - \frac{\lambda - \rho}{D} \\ &= -(w_\lambda^2 + w_\rho^2 - 2w_\rho w_\lambda - 2w_\rho^2 + 2w_\rho w_\lambda) - \frac{\lambda - \rho}{D} \\ &= -(w_\lambda - w_\rho)^2 - 2w_\rho(w_\lambda - w_\rho) - \frac{\lambda - \rho}{D}. \end{aligned}$$

Since $w_\rho(0) = w_\lambda(0)$, this means that if we define $\Gamma(x) = w_\lambda - w_\rho$, $\Gamma$ satisfies the following ODE:

$$\Gamma' = -\Gamma^2 - \frac{\mu}{D}\Gamma - \frac{\lambda - \rho}{D}, \tag{57}$$

$$\Gamma(0) = 0. \tag{58}$$

Of course, regardless of $\mu$ and $D$, the solution must satisfy $\Gamma(x) < 0$ for all $x > 0$, since if $\Gamma$ ever gets close to 0, the term $-\frac{\lambda-\rho}{D}$ dominates the right-hand side of the equation. Thus, we can assume that $\Gamma(x_0) = -\epsilon$ for some $x_0 > 0$, $\epsilon > 0$. Now, consider the ODE

$$\begin{aligned} \hat{\Gamma}' &= -\hat{\Gamma}^2 - \frac{\mu}{D}\hat{\Gamma}, \\ \hat{\Gamma}(x_0) &= -\epsilon. \end{aligned} \tag{59}$$

Clearly, it must be that the differential inequality $\Gamma \leq \hat{\Gamma}$ is satisfied for all $x \geq x_0$, since whenever $\hat{\Gamma} = \Gamma$, $\Gamma' < \hat{\Gamma}'$. Therefore, if $\Gamma$ is defined for all $x_0 \geq x$, then so is $\hat{\Gamma}$. Let us assume that this is the case.

We define $Z = -\hat{\Gamma} \geq 0$, and we can then rewrite Equation (59) as

$$\frac{\mu}{D} = Z - \frac{Z'}{Z},$$

which upon integration yields

$$-\int_{x_0}^{y} \frac{\mu(x)}{D(x)}\mathrm{d}x = -\int_{x_0}^{y} Z(x)\mathrm{d}x + [\ln(Z(x))]_{x_0}^{y},$$

in turn leading to

$$\mathrm{e}^{-\int_{x_0}^{y}\frac{\mu(x)}{D(x)}\mathrm{d}x} = \frac{1}{\epsilon}Z(y)\mathrm{e}^{-\int_{x_0}^{y}Z(x)\mathrm{d}x}.$$

Let us define $Q(y) = \mathrm{e}^{-\int_0^y \frac{\mu(x)}{D(x)}\mathrm{d}x}$, so that we can write

$$\begin{aligned} \int_0^\infty Q(y)\mathrm{d}y &= \int_0^{x_0} Q(y)\mathrm{d}y + Q(x_0)\int_{x_0}^{\infty} \mathrm{e}^{-\int_{x_0}^{y}\frac{\mu(x)}{D(x)}\mathrm{d}x}\mathrm{d}y \\ &= \int_0^{x_0} Q(y)\mathrm{d}y + \frac{Q(x_0)}{\epsilon}\int_{x_0}^{\infty} Z(y)\mathrm{e}^{-\int_{x_0}^{y}Z(x)\mathrm{d}x}\mathrm{d}y \\ &\leq \int_0^{x_0} Q(y)\mathrm{d}y + \frac{Q(x_0)}{\epsilon} \\ &< \infty. \end{aligned}$$

Here, we used the inequality $\int_a^b Z(y)\mathrm{e}^{-\int_a^y Z(x)\mathrm{d}x}\mathrm{d}y = \int_a^b -\frac{d}{\mathrm{d}y}\left(\mathrm{e}^{-\int_a^y Z(x)\mathrm{d}x}\right)\mathrm{d}y = \left[\mathrm{e}^{-\int_a^y Z(x)\mathrm{d}x}\right]_b^a = 1 - \mathrm{e}^{-\int_a^b Z(x)\mathrm{d}x} \leq 1$, since $Z \geq 0$. Thus, to summarize, if $\Gamma$ is defined on the whole of $\mathbb{R}_+$, then it must be that the right integral in Equation (23) is finite. So, if Equation (23) is infinite, it must be that $\Gamma \to -\infty$ for some finite $x > 0$, in turn implying that $u_\lambda(x) = 0$.

An identical argument for $x<0$ shows that $u_\lambda(x)=0$ for some finite $x<0$. Thus, in line with the previous argument, $u_\lambda$ has at least two roots, and any other solution to Equation (54) has at least one root, when $\lambda>\rho$. We are done.

*Proof of Proposition 3*: The existence of a stationary distribution is equivalent to Equation (34). From the condition on $\sigma$,

$$|\sigma(x)-\sigma(y)| \leq c|x-y|,$$

it follows that for $x>0$, $\sigma(x) \leq \sigma(0)+C_3x$, and thus that $\sqrt{D(x)} \leq \frac{1}{\sqrt{2}}(\sigma(0)+C_3x)$.

Now, for $d>0$, $\min_{z>0}(z+\frac{1}{zd})=\frac{2}{\sqrt{d}}$, realized by choosing $z=\frac{1}{\sqrt{d}}$, and it therefore follows that:

$$\begin{aligned}\int_0^\infty Q(x)\,\mathrm{d}x+\int_0^\infty \frac{1}{Q(x)D(x)}\,\mathrm{d}x &= \int_0^\infty Q(x)+\frac{1}{Q(x)D(x)}\,\mathrm{d}x\\ &\geq 2\int_0^\infty \frac{1}{\sqrt{D(x)}}\,\mathrm{d}x\\ &\geq 2\int_0^\infty \frac{\sqrt{2}}{\sigma(0)+C_3x}\,\mathrm{d}x\\ &=\infty.\end{aligned}$$

If the second term is finite, the first term must therefore be infinite, $\int_0^\infty Q(x)\,\mathrm{d}x=\infty$. An identical argument for $x<0$ implies that

$$\int_{-\infty}^0 \frac{1}{Q(x)D(x)} < \infty$$

implies that $\int_{-\infty}^0 Q(x)\,\mathrm{d}x=\infty$. We are done.

*Proof of Proposition 4*: From Proposition 2, we know that general recovery, and therefore recovery within $\mathcal{B}$, is possible if both conditions in Equation (23) are satisfied. We therefore study the case in which exactly one condition is satisfied. We note that $m\in\mathcal{B}\iff z\in\mathcal{B}$.

*Necessity:* From the representation (53) of the general solution, it is clear that if both integrals in Equation (23) are finite, given that $z_1\in\mathcal{B}$, for small enough $|c|$, $z_c\in\mathcal{B}$, so recovery is not possible within $\mathcal{B}$ in this case.

*Sufficiency:* Without loss of generality, assume that the left integral in Equation (23) is infinite, the right integral is finite, and that $z_1\in\mathcal{B}$ in Equation (53). Then, because $R(-\infty)=-\infty$ for any $c>0$, $z_c\notin\mathcal{C}_+^2$ as the function eventually turns negative for negative $x$. Moreover, for $c<0$, $z_c$ is everywhere positive but unbounded, $\limsup_{x\to-\infty} z_c(x)=\infty$, so $z_c\notin\mathcal{B}$. Thus, given $\lambda=\rho$, the only function in $\mathcal{B}$ that is a candidate for the inverse of $m$ is $z_1$.

For $\lambda>\rho$, we proceed as follows. Recall that $Q(x)=\mathrm{e}^{-\int_0^x \frac{\mu(y)}{D(y)}\mathrm{d}y}$ is positive, $R(x)=\int_0^x Q(y)\,\mathrm{d}y$ is increasing, and define the limit $K=R(\infty)<\infty$. Moreover, define the function $z_{\lambda,\alpha}$, as the solution to $\mathcal{W}[z|\lambda]=0$, given initial conditions $z_{\lambda,\alpha}(0)=1$, $z'_{\lambda,\alpha}(0)=\alpha$. Given that $z=1/m$ is the correct reciprocal of $m$, normalized such that $z(0)=1$, and that $z'(0)=\beta$, it is easy to verify the relationship with $z_c$ in Equation (53), $z_{\rho,c+\beta}=z_c$. Now, $0<C_1\leq z\leq C_2<\infty$, since $z\in\mathcal{B}$. Defining $\beta^*=\beta-\frac{1}{K}$, it then follows immediately that for $\alpha>\beta$, for $x\geq 0$, $C_1\leq z_{\rho,\alpha}\leq C_2(1+(\alpha-\beta)K)$, that for $\beta^*<\alpha\leq\beta$, $0<C_1(1-(\beta-\alpha)K)\leq z_{\rho,\alpha}\leq C_2$, $x\geq 0$, and that for $\alpha\leq\beta^*$, $\limsup_{x\to\infty} z_{\rho,\alpha}\leq 0$.

Now, similar to the approach in the proof of Proposition 2, we can write

$$z_{\lambda,\alpha}(x) = z(x)\sqrt{Q(x)}\,u_{\lambda,\alpha}(x), \tag{60}$$

where

$$u''_{\lambda,\alpha} + \left(\tau(x) + \frac{\lambda-\rho}{D(x)}\right)u_{\lambda,\alpha} = 0, \qquad u_{\lambda,\alpha}(0) = 1, \qquad u'_{\lambda,\alpha}(0) = \alpha + \frac{1}{2}\frac{\mu(0)}{D(0)} - \beta. \tag{61}$$

It is easy to verify that for $\lambda = \rho$, the solution is $u_{\rho,\alpha} = e^{\frac{1}{2}\int_0^x \frac{\mu}{D}dy}(1 + (\alpha-\beta)R(x))$.

Following the proof of Proposition 2, we define $w_{\lambda,\alpha}(x) = \frac{u'_{\lambda,\alpha}(x)}{u_{\lambda,\alpha}(x)}$, which is well defined as long as $u_{\lambda,\alpha}(x) > 0$. We then have $w_{\rho,\alpha}(x) = \frac{u'_{\rho,\alpha}(x)}{u_{\rho,\alpha}(x)} = \frac{1}{2}\frac{\mu(x)}{D(x)} + \frac{\mathrm{d}}{\mathrm{d}x}(\ln(1 + (\alpha-\beta)R(x))) = \frac{1}{2}\frac{\mu(x)}{D(x)} + \frac{(\alpha-\beta)Q(x)}{1+(\alpha-\beta)R(x)}$. Similar steps as in the proof of Proposition 2 lead to

$$\begin{aligned}\Gamma' &= -\Gamma^2 - A(x)\Gamma - \frac{\lambda-\rho}{D},\\ \Gamma(0) &= 0,\end{aligned}$$

where $\Gamma(x) =^{\text{def}} w_{\lambda,\alpha}(x) - w_{\rho,\alpha}(x)$, and $A(x) = \left(\frac{\mu(x)}{D(x)} + 2\frac{\mathrm{d}}{\mathrm{d}x}(\ln(1 + (\alpha-\beta)R(x)))\right)$.

As before, the solution must satisfy $\Gamma(x) < 0$ for all $x > 0$, since if $\Gamma$ ever gets close to 0, the term $-\frac{\lambda-\rho}{D}$ dominates the right-hand side of the equation. This, means that we can immediately rule out any $z_{\lambda,\alpha}$ for $\alpha \le \beta^*$ as candidate solutions, since as long as $u_{\rho,\alpha} > 0$ and $u_{\lambda,\alpha} > 0$,

$$0 > \int_0^x \Gamma(y)\mathrm{d}y = \int_0^x \left(\frac{u'_{\lambda,\alpha}(y)}{u_{\lambda,\alpha}(y)} - \frac{u'_{\rho,\alpha}(y)}{u_{\rho,\alpha}(y)}\right)\mathrm{d}y = \ln(u_{\lambda,\alpha}(x)) - \ln(u_{\rho,\alpha}(x)),$$

in turn implying that $u_{\lambda,\alpha}(x) < u_{\rho,\alpha}(x)$. As long as both $z_{\lambda,\alpha}$, and $z_{\rho,\alpha}$ are positive, via Equation (60) we have $\frac{z_{\lambda,\alpha}}{z_{\rho,\alpha}} = \frac{u_{\lambda,\alpha}}{u_{\rho,\alpha}}$, so this means that $z_{\lambda,\alpha} < z_{\rho,\alpha}$. Since $\limsup_{x\to\infty} z_{\rho,\alpha} \le 0$ when $\alpha \le \beta^*$, it must either be that $z_{\lambda,\alpha}$ reaches zero for a finite $x$, or approaches zero as $x$ tends to infinity, in both cases disqualifying $z_{\lambda,\alpha}$ as a candidate function in $\mathcal{B}$.

It remains to be shown that $z_{\lambda,\alpha} \notin \mathcal{B}$ when $\alpha > \beta^*$ and $\lambda > \rho$. In this case, $A(x)$ is well defined for all $x > 0$. Of course, if $u_{\lambda,\alpha} = 0$ for some $x > 0$, then $z_{\lambda,\alpha} \notin \mathcal{B}$, so we assume that $u_{\lambda,\alpha} > 0$. As in the proof of Proposition 2, we can assume that $\Gamma(x_0) = -\epsilon$ for some $x_0 > 0$, $\epsilon > 0$. Now, assume that for $x \ge x_0$, $A(x)$ satisfies the bound $A(x) \le Cx$, for some $C < \infty$. Define $\xi = \frac{\lambda-\rho}{\sup_x D(x)} > 0$, and consider the ODE

$$\begin{aligned}\hat{\Gamma}' &= -Cx\hat{\Gamma} - \xi,\\ \hat{\Gamma}(x_0) &= -\epsilon.\end{aligned} \tag{62}$$

It must be that $\Gamma \le \hat{\Gamma}$ for all $x \ge x_0$, since whenever $\hat{\Gamma} = \Gamma$, $\Gamma' < \hat{\Gamma}'$ (similar to the argument in the proof of Proposition 2). Now, the solution to Equation (62) is

$$\hat{\Gamma}(x) = -e^{-\frac{C(x-x_0)^2}{2}}\left(\epsilon + \xi\sqrt{\frac{2\pi}{4C}}\mathrm{Erfi}\left(\sqrt{\frac{C}{2}}(x-x_0)\right)\right),$$

and it is easy to verify that $\hat{\Gamma}(x) = -\frac{\xi}{C(x-x_0)} + O((x-x_0)^{-3})$ for large $x$, and thus that $\int_{x_0}^y \hat{\Gamma}(x)\mathrm{d}x$ tends to $-\infty$ as $y$ grows. Since $\Gamma \le \hat{\Gamma}$, it must be that $\int_{x_0}^y \Gamma(x)\mathrm{d}x$ tends to $-\infty$ too. But, $\int_{x_0}^y \Gamma(x)\mathrm{d}x = \ln\left(\frac{u_{\lambda,\alpha}(y)}{u_{\rho,\alpha}(y)}\right) - \ln\left(\frac{u_{\lambda,\alpha}(x_0)}{u_{\rho,\alpha}(x_0)}\right)$, so this implies that $\frac{u_{\lambda,\alpha}(y)}{u_{\rho,\alpha}(y)} \to 0$, as $y$ grows. Now, since $z_{\lambda,\alpha} > 0$ and $z_{\rho,\alpha} > 0$, $\frac{z_{\lambda,\alpha}}{z_{\rho,\alpha}} = \frac{u_{\lambda,\alpha}}{u_{\rho,\alpha}}$. Moreover, $z_{\rho,\alpha}(x) \le C_2(1 + (\alpha-\beta)K) < \infty$. It must therefore be that $z_{\lambda,\alpha}(x) \to 0$ for large $x$, so $z_{\lambda,\alpha} \notin \mathcal{B}$.

The only part remaining is to show that $A(x) \leq Cx$ for $x \geq x_0$, for some constant $C < \infty$. We have

$$A(x) = \frac{\mu(x)}{D(x)} + 2\frac{(\alpha - \beta)Q(x)}{1 + (\alpha - \beta)R(x)}.$$

Since, per assumption, $D(x) \geq C_2^2/2 > 0$, and $\mu(x) \leq C_1(1+x) \leq C_1(x_0^{-1}+1)x = C_1'x$, it follows that such a bound exists for the first term, $\frac{\mu(x)}{D(x)} \leq Cx$. For the second term, the denominator is bounded below by a strictly positive constant, since $\alpha > \beta^*$. Therefore, as long as $\limsup_{x\to\infty} \frac{Q(x)}{x} < \infty$, the second term can also be bounded by $Cx$ for $x \geq x_0$.

Intuitively, since $\int_0^\infty Q(x)\mathrm{d}x < \infty$, it should not be possible for $\frac{Q(x)}{x}$ to be large infinitely often. This intuition can be formalized as follows. Since the integral of $Q$ is finite, $Q(x) \leq \frac{C'}{x}$ infinitely often for any constant $C' > 0$. Now, assume that also $Q(x) = C'x$ infinitely often, for some $C' > 0$. Then, consider a large $x_1$, such that $Q(x_1) = C'x_1$, and an even larger $x_2 = x_1 + \delta$, such that $Q(x_2) = \frac{C'}{x_1}$. Since $\frac{C'}{x_1} = Q(x_1+\delta) = Q(x_1)\mathrm{e}^{-\int_{x_1}^{x_1+\delta}\frac{\mu}{D}\mathrm{d}s} = C'x_1\mathrm{e}^{-\int_{x_1}^{x_1+\delta}\frac{\mu}{D}\mathrm{d}s}$, it follows that $\int_{x_1}^{x_1+\delta}\frac{\mu}{D}\mathrm{d}s = 2\ln(x_1)$, and since $\frac{\mu(s)}{D(s)} \leq cs$, that

$$\int_{x_1}^{x_1+\delta} cs\mathrm{d}s = \frac{c}{2}\left((x_1+\delta)^2 - x_1^2\right) = \frac{c}{2}\left(2\delta x_1 + \delta^2\right) \geq 2\ln(x_1).$$

The positive root to this second-order equation $\frac{c}{2}\left(2\delta x_1 + \delta^2\right) = 2\ln(x_1)$ is

$$\delta = x_1\left(\sqrt{1 + \frac{4\ln(x_1)}{cx_1^2}} - 1\right),$$

and for large $x_1$, the term within the square root is small, so we can use a Taylor expansion, $\sqrt{1+\epsilon} \geq 1 + \frac{\epsilon}{2} - k_0\epsilon^2$ for small positive $\epsilon$, where $k_0 > 8$ is a constant, to get

$$\begin{aligned}
\delta &\geq x_1\left(\frac{2\ln(x_1)}{cx_1^2} - k\left(\frac{4\ln(x_1)}{cx_1^2}\right)^2\right) \\
&\geq \frac{2\ln(x_1)}{cx_1}\left(1 - \frac{2k_0\ln(x_1)}{c^2x_1^2}\right) \\
&= \frac{2\ln(x_1)}{cx_1}\left(1 - k\frac{\ln(x_1)}{x_1^2}\right),
\end{aligned}$$

where $k = 2k_0/c^2$. We now have

$$\begin{aligned}
\int_{x_1}^{x_2} Q(x)\mathrm{d}x &= \int_{x_1}^{x_1+\delta} Q(x)\mathrm{d}x \\
&= \int_{x_1}^{x_1+\delta} \mathrm{e}^{-\int_0^y \frac{\mu}{D}\mathrm{d}s}\mathrm{d}y \\
&= \int_{x_1}^{x_1+\delta} Q(x_1)\mathrm{e}^{-\int_{x_1}^y \frac{\mu}{D}\mathrm{d}s}\mathrm{d}y \\
&= C'x_1\int_{x_1}^{x_1+\delta} \mathrm{e}^{-\int_{x_1}^y \frac{\mu}{D}\mathrm{d}s}\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&\geq C' x_1 \int_{x_1}^{x_1+\delta} \mathrm{e}^{-\int_{x_1}^{y} cs\mathrm{d}s} \mathrm{d}y \\
&= C' x_1 \int_{x_1}^{x_1+\delta} \mathrm{e}^{-\frac{c}{2}(y^2 - x_1^2)} \mathrm{d}y \\
&= C' x_1 \int_0^{\delta} \mathrm{e}^{-cx_1 p - cp^2/2} \mathrm{d}p \\
&\geq C'' x_1 \int_0^{\frac{2\ln(x_1)}{cx_1}(1 - k\frac{\ln(x_1)}{x_1^2})} \mathrm{e}^{-cx_1 p - cp^2/2} \mathrm{d}p \\
&\geq C''' x_1 \int_0^{\frac{2\ln(x_1)}{cx_1}(1 - k\frac{\ln(x_1)}{x_1^2})} \mathrm{e}^{-cx_1 p} \mathrm{d}p \\
&= C''' x_1 \left[\frac{\mathrm{e}^{-cx_1 p}}{cx_1}\right]_0^{\frac{2\ln(x_1)}{cx_1}\left(1 - k\frac{\ln(x_1)}{x_1^2}\right)} \\
&\geq \frac{C'''}{c}\left(1 - x_1^{-2\left(1-k\frac{\ln(x_1)}{x_1^2}\right)}\right) \\
&\geq C'''' > 0.
\end{aligned}
$$

Thus, every time $Q(x)$ reaches $C'x_1$, the contribution to $R(x)$ on the subsequent interval, $[x_1, x_1 + \delta]$, over which $Q(x)$ decreases to $\frac{C'}{x_1}$ is bounded below by a strictly positive constant, $C''''$, and if there are infinitely many such intervals it must then be that $R(\infty) = \infty$, contradicting the assumption that $R(\infty)$ is finite. Therefore, $\frac{Q(x)}{x} \to 0$, for large $x$, in turn implying that $A(x) \leq Cx$, and that, in extension, $\limsup_{x\to\infty} z_{\lambda,\alpha}(x) \geq 0$ for $\alpha > \beta^*$. This completes the proof.

*Proof of Proposition 5:* We take Equations (57) and (58) as a starting point to construct positive solutions to the ODE $\mathcal{W}[s|\lambda] = 0$ for each $\lambda < \rho$. For $\lambda < \rho$, a similar argument as in Proposition 2 implies that $\Gamma(x) \geq 0$ for all $x$, which in turn implies that

$$
\frac{u_{\lambda'}(x)}{u_\lambda(x)} \geq \frac{u_\rho(x)'}{u_\rho(x)} = \frac{1}{2}\frac{\mu(x)}{D(x)},
$$

for each $x$ such that $u_\lambda(x) > 0$.

W.l.o.g., we focus on the domain $x \geq 0$. Define $x^* = \inf\{x : u_\lambda(x) = 0\}$. Since $u_\lambda(0) = 1$ and $u_\lambda$ is a smooth function, $x^* > 0$. Moreover, if $x^* = \infty$, $u_\lambda$ is positive on the whole of $\mathbb{R}_+$. Assume, to the contrary, that $x^*$ is finite. Then, since $u_\lambda$ is smooth, it must be that $\lim_{x\to x^*} u_\lambda(x) = 0$.

Now, define $R = \inf_{0\leq x\leq x^*} \frac{1}{2}\frac{\mu(x)}{D(x)}$. Then, it follows that

$$
u_{\lambda'}(x) \geq R u_\lambda(x), \qquad 0 \leq x < x^*.
$$

A standard differential inequality then implies that $u_\lambda(x) \geq \xi(x)$, $0 \leq x < x^*$, where $\xi(x) = \mathrm{e}^{Rx}$ solves the ODE $\xi' = R\xi$, $\xi(0) = 1$. It follows that $u_\lambda(x^*) \geq \mathrm{e}^{Rx^*} > 0$, contradicting the assumption that $u_\lambda(x^*) = 0$. Thus, no such finite $x^*$ exists, $u_\lambda$ is strictly positive for all $x \in \mathbb{R}_+$, which then also implies that $s_\lambda(x) = \nu u_\lambda > 0$ (see Proposition 2). An identical argument shows that $s_\lambda$ is also strictly positive for negative $x$.

Thus, for each $\lambda < \rho$, the marginal utility function $m_\lambda(x) = \frac{1}{s_\lambda(x)}$ is strictly positive, which together with $\lambda$ provides an "alternative" pricing kernel consistent with the fundamental ODE. We are done.

Note also that the corollary follows immediately, since if any of these alternative pricing kernels would satisfy the conditions for recovery, Proposition 2 would be violated.

*Proof of Proposition 6*: We use the risk neutral measure to show the result. An earlier version of the proof was based on Dupire's formula (Dupire, 1994).[12]

As noted, the risk neutral dynamics for $X$ is:

$$\mathrm{d}X = \kappa(X)\mathrm{d}t + \sigma(X)\mathrm{d}\omega^{\mathbb{Q}}.$$

Via Fokker–Planck's equation, it then follows that the risk neutral probability density function, $\phi^{\mathbb{Q}}(t, y)$, for $X(t) = y$, satisfies the PDE

$$\phi_t^{\mathbb{Q}} = -\kappa'\phi^{\mathbb{Q}} - \kappa\phi_y^{\mathbb{Q}} + D''\phi^{\mathbb{Q}} + 2D'\phi_y^{\mathbb{Q}} + D\phi_{yy}^{\mathbb{Q}}. \tag{63}$$

In the risk neutral formulation, defining $Z(t) = E^{\mathbb{Q}}\left[\mathrm{e}^{-\int_0^t r(s)\mathrm{d}s}\right]$, the price of the AD security that pays off at $t$ if $X(t) = y$ is

$$V(t, y) = Z(t)\phi^{\mathbb{Q}}(t, y),$$

and taking partial derivatives, we get $V_t = -r(y)V + Z(t)\phi_t^{\mathbb{Q}}$, $V_y = Z(t)\phi_y^{\mathbb{Q}}$, $V_{yy} = Z(t)\phi_{yy}^{\mathbb{Q}}$. We therefore get

$$\begin{aligned} V_t - (DV_{yy} + \alpha_1 V_y + \alpha_0 V) &= Z(-r\phi^{\mathbb{Q}} + \phi_t^{\mathbb{Q}} - D\phi_{yy}^{\mathbb{Q}} - \alpha_1\phi_y^{\mathbb{Q}} - \alpha_0\phi^{\mathbb{Q}}) \\ &= Z(-r\phi^{\mathbb{Q}} + \phi_t^{\mathbb{Q}} - D\phi_{yy}^{\mathbb{Q}} - (2D' - \kappa)\phi_y^{\mathbb{Q}} - (D'' - \kappa' - r)\phi^{\mathbb{Q}}) \\ &= Z(r(\phi^{\mathbb{Q}} - \phi^{\mathbb{Q}}) + (\phi_t^{\mathbb{Q}} - (-\kappa'\phi^{\mathbb{Q}} - \kappa\phi_y^{\mathbb{Q}} + D''\phi^{\mathbb{Q}} + 2D'\phi_y^{\mathbb{Q}} + D\phi_{yy}^{\mathbb{Q}}))), \\ &= 0 \end{aligned}$$

where we used Equation (63) in the last step. We are done.

*Proof of Proposition 7:* 1. $\Rightarrow$ 2.: We will use a specific parametrization of the general independent solutions, $z_\lambda^i$, $i = 1, 2$, $\lambda \geq \rho$, to the ODE $\mathcal{W}[z|\lambda] = 0$. We define $z_\lambda^i$ to be the solution to

$$\begin{aligned} \mathcal{W}[z_\lambda^i|\lambda] &= 0, \\ z_\lambda^i(0) &= z_{\rho,i,}(0), \\ z_\lambda^{i\prime}(0) &= z'_{\rho,i}(0), \end{aligned}$$

where $z_{\rho,1}$, as before, is the strictly positive solution to $\mathcal{W}[z|\rho] = 0$, and $z_{\rho,2}$ is another solution, which given that recovery is possible is chosen to be zero and increasing at $x = 0$. Finally, define the general solution $z_{\lambda,c} = cz_\lambda^1 + (1 - c)z_\lambda^2$, $c \in [0, 1]$, $\lambda \geq \rho$. It follows from standard properties of linear second-order ODEs that for any $x$, $z_{\lambda,c}(x)$ depends continuously on $\lambda$ and $c$ (see, e.g., Simmons, 1988).

The correct $z = \frac{1}{m}$ is then the only positive function, $z_{\rho,1}$, whereas $z_{\rho,c}(x) = 0$ for some $x$, if either $c \neq 1$, or $\lambda > \rho$. We are interested in how strong inferences we can draw about $z$ from observing $D$, $\kappa$, and $r$ on the domain $[-N, N]$. Candidate $z$s are then solutions $z_{\lambda,c}$ that are strictly positive on $[-N, N]$.

12 I thank Ngoc-Khanh Tran for suggesting using the risk neutral formulation.

We define $N_{\lambda,c} =^{\text{def}} \inf\{|x| : z_{\lambda,c} = 0\}$. It follows that if $N_{\lambda,c} \leq N$, $z_{\lambda,c}$ cannot be a candidate $z$, since it is not strictly positive on the observable domain. The following properties of $N_{\lambda,c}$ follow:

1. For $\lambda = \rho$, $N_{\rho,c}$ is continuous and strictly increasing in $c$, for $0 \leq c < 1$. Moreover, $N_{\rho,1} = \infty$. This follows from Proposition 2, and the definition of $z_{\rho,c}$ as a linear combination of the strictly positive $z^1_{\rho,1}$, and $z_{\rho,2}$ which has exactly one root.
2. For $(\lambda, c) \neq (\rho, 1)$, $N_{\lambda,c}$ is continuously differentiable in $\lambda$, and $\frac{\mathrm{d}N_{\lambda,c}}{\mathrm{d}\lambda} < 0$. This follows from the Sturm comparison theorem, see for example, Simon (2005).
3. For $\lambda > \rho$, $N_{\lambda,c}$ is a continuous function of $c \in [0, 1]$, and therefore also bounded, $R_\lambda =^{\text{def}} \sup_c N_{\lambda,c} < \infty$.
4. $R_\lambda$ is nonincreasing in $\lambda$. This follows directly from point 2.

Point 3 follows from the following argument: From the proof of Proposition 2, it follows that $z_{\lambda,1}$ has at least two roots for any $\lambda > \rho$, one for $x$ less than zero, and one for $x$ greater than zero. Let us call these two roots $v_1 < 0$ and $v_2 > 0$. From the Sturm separation theorem (see Simmons, 1988, p. 118), it follows that $z_{\lambda,0}$ has exactly one root in $(v_1, v_2)$, which from the construction of $z_{\lambda,0}$ in Proposition 2, lies at $x = 0$. Moreover, $z_{\lambda,c}$ has exactly one root in $(v_1, 0)$, for $0 < c < 1$. We denote this root by $v_1(c)$. Clearly, if we define $c_1(x) = \frac{z_{\lambda,0}(x)}{z_{\lambda,0}(x) - z_{\lambda,1}(x)}$, for $x \in [v_1, 0]$, we have $c_1(x) z_{\lambda,1}(x) + (1 - c_1(x)) z_{\lambda,0}(x) = 0$, that is, $v_1(c_1(x)) = x$. Now, $c_1$ is continuous, $c_1(v_1) = 1$, $c_1(0) = 0$, and $\frac{\mathrm{d}c_1(x)}{\mathrm{d}x} = \frac{1}{(z_{\lambda,0}(x) - z_{\lambda,1}(x))^2} \left( z'_{\lambda,0}(x) z_{\lambda,1}(x) - z'_{\lambda,1}(x) z_{\lambda,0}(x) \right)$. Since the Wronskian, $z'_{\lambda,0}(x) z_{\lambda,1}(x) - z'_{\lambda,1}(x) z_{\lambda,0}(x) \neq 0$ (see Simmons, 1988), it follows that $c_1(x)$ is strictly decreasing on $[v_1, 0]$, and therefore its inverse, $v_1(c)$ is a continuous function on $c \in [0, 1]$. If $|v_1| \leq v_2$, then clearly $N_{\lambda,c} = |v_1(c)|$ but if $|v_1| > v_2$, we must also consider a potential root to the right of $v_2$ as a candidate for being closest to zero. If $z_{\lambda,0}$ has a root at $x = v_3 > v_2$, then an identical argument as that above can be made to infer that there is a unique root of $z_{\lambda,c}$, $v_3(c) \in (v_2, v_3]$, for all $c \in [0, 1]$, which decreases continuously in $c$. If $z_{\lambda,0}$ has no such root to the right of $v_2$, then neither does $z_{\lambda,1}$ (again by the Sturm separation theorem). In this case, it follows that $c_2(x) \stackrel{\text{def}}{=} \frac{z_{\lambda,0}(x)}{z_{\lambda,0}(x) - z_{\lambda,1}(x)}$, for $x \geq v_2$ is a continuous, strictly decreasing (because of the nonzero Wronskian) function, and that its inverse $v_2(c)$ can be defined on $c \in [c_2(|v_1|), 1]$. The function $v_2(c)$ can then be continuously extended to the domain $c \in [0, 1]$, so that for $0 \leq c < c_2(|v_1|)$, $v_2(c) = v_2(c_2(|v_1|))$. It now follows that $N_{\lambda,c} = \min(v_1(c), v_2(c))$ is also continuous in $c \in [0, 1]$. Since the domain of $c$, $[0, 1]$, is compact, boundedness of $R_\lambda$ follows immediately. We also define $A_\lambda = \{c : N_{\lambda,c} = R_\lambda\}$, and note that $A_\lambda$ must be nonempty, again since $N_{\lambda,c}$ is continuous in $c$.

The results above are sufficient to imply that as $N$ grows, the set of candidate functions both over $c$ and $\lambda$ shrinks so that ultimately only $z = \frac{1}{m}$ remains. Specifically, define $\mathcal{G}_N = \{(\lambda, c) : z_{\lambda,c}(x) > 0, |x| \leq N\}$. This set contains the candidate $z$-functions, given that $D$, $r$, and $\kappa$ are observed on $[-N, N]$. Clearly, $\mathcal{G}_{N'} \mathcal{G}_N$, for $N' > N$, and from Proposition 1, $\mathcal{G}_\infty = \{(\rho, 1)\}$. We wish to show that $\mathcal{G}_N$ converges to $\mathcal{G}_\infty$ as $N \to \infty$.

Define

$$c_N = \inf\{c : N_{\rho,c} \geq N\},$$
$$\lambda_N = \inf\{\lambda : R_\lambda \geq N\}.$$

It then follows immediately that $\mathcal{G}_N[c_N, 1] \times [\rho, \lambda_N]$. Moreover, Proposition 2 implies that $\lim_{N\to\infty} c_N = 1$ and $\lim_{N\to\infty} \lambda_N = \rho$, since otherwise there would be other strictly positive solutions to the fundamental ODE with $\lambda \geq \rho$. Thus, $\lim_{N\to\infty} \cap_{n=1}^{N} \mathcal{G}_n = \mathcal{G}_\infty = \{(\rho, 1)\}$, as claimed.

The results in the proposition follow immediately. By choosing $\hat{\rho} = \lambda_N$ (which of course is observable, given that $D$, $\kappa$, and $r$ are on $[-N, N]$), we get the first result. Next, we choose a $\hat{z}_N = z_{\hat{\rho}_N, w_N}$, and $\hat{m}_N = \frac{1}{\hat{z}_N}$, where $w_N \in A_{\hat{\rho}_N}$. Since $z_{\lambda,c}(x)$ depends continuously on $\lambda$ and $c$, which converge to $\hat{\rho}$ and 1, respectively, as $N$ tends to infinity, it follows that $\lim_N \to \infty$ $z_{\hat{\rho}_N, w_N}(x) = z(x)$ for any $x$, and since $z$ is strictly positive, also that $\hat{m}_N(x) \to m(x)$, completing the proof that 1. $\Rightarrow$ 2.

Finally, 2. $\Rightarrow$ 1. is immediate. We are done.

## Appendix F: Matlab Code for Recovery Algorithm in Section 4

The results in Section 4 are based on the following Matlab code, which approximates the pricing kernel and personal discount rate from $D$, $r$, and $\kappa$.

```
% Filename: Recovery.m
% By Johan Walden, November 7, 2013
% Recovery method for diffusion process
% Described in: Recovery with diffusions on unbounded domains
% Original method with finite state space described in Ross (2013)
%
% Input:
% dx: stepsize (e.g. 1E-4)
% rhomax: Assumed maximum possible personal discount rate
% NoSteps: Number of iterations (e.g., 30)
% D: Vector of D values [D(0), D(dx),. . .,D(N*dx)];
% r: Vector of r values [r(0),r(dx),. . .,r(N*dx)];
% k: Vector of kappa values [kappa(0),kappa(dx),. . .,kappa(N*dx)];
%
% Output:
% rho: Approximate personal discount rate
% m: Vector of approximate marginal utility [m(0),m(dx),. . .,m(N*dx)];
function [rho,m]=Recovery(dx,rhomax,NoSteps,D,r,k)
rhomin = 0; %Lower bound on personal discount rate
N = length(D);
zapp = zeros(N,1);
FoundPositive = 0;
for n = 1:NoSteps % Iterate over conjectured discount rate
  rho = (rhomax + rhomin)/2; %Conjectured rho
  %Solve ODEs
  z = zeros(N,2); %Two solutions
  Mid = floor(N/2);
  z(Mid-1:Mid + 1,1) = [1,1,1]; %Solution with initial condition z'=0;
  z(Mid-1:Mid + 1,2) = [1-dx,1,1 + dx]; %Solution with initial condition z'=1;
```

```
    for j = Mid + 1:N-1
      vj = k(j)*dx/(2*D(j));
      z(j + 1,:) = 1/(1 + vj)*((2-dx^2*(rho-r(j))/D(j))*z(j,:)-(1-vj)*z(j-1,:));
    end
    for j = Mid-1:-1:2
      vj = k(j)*dx/(2*D(j));
      z(j-1,:) = 1/(1-vj)*((2-dx^2*(rho-r(j))/D(j))*z(j,:)-(1 + vj)*z(j + 1,:));
    end
    % Check number of roots of solutions to infer new rho
    Roots = sum(z(2:N,:).*z(1:N-1,:)<= 0);
    if (Roots(1)>1 || Roots(2)>1) % Too high rho, since multiple roots
      rhomax = rho;
    elseif (Roots(1)==0) %Too low rho (weakly), since positive solution
      rhomin = rho;
      zapp = z(:,1); %Update approximate kernel
      FoundPositive = 1;
    elseif (Roots(2)==0)
      rhomin = rho;
      zapp = z(:,2); %Update approximate kernel
    else %No solution with two roots, at least one with one, check for linear combination
      A1 = angle(z(:,1)+i*z(:,2));
      A2 = angle(-(z(:,1)+i*z(:,2))); %Rotate angle by pi
      if ((max(A1)-min(A1) < pi)) %Positive possible
        rhomin = rho;
        A = 1/2*(max(A1)+min(A1));
        zapp = cos(A)*z(:,1)+sin(A)*z(:,2);
        FoundPositive = 1;
      elseif (max(A2)-min(A2) < pi) %Positive possible
        rhomin = rho;
        A = 1/2*(max(A2)+min(A2))+pi;
        zapp = cos(A)*z(:,1)+sin(A)*z(:,2);
        FoundPositive = 1;
      else
        rhomax = rho;
      end
    end;
  end
  if(FoundPositive==0)
  disp('Did not find a positive kernel')
  end;
  m = 1./zapp';
```

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