Welfare in Production Economies with Heterogeneous Beliefs*

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Abstract

We study welfare, savings, and asset price distortions in economies with disagreement and production, extending previous literature by our focus on settings in which there are real effects of disagreement. We introduce a novel welfare measure, IK-efficiency, and show its usefulness in a work-horse production economy. Aggregate savings may be significantly distorted under disagreement, as may asset prices—especially for those assets about which there is little disagreement. Overall, our results highlight the feedback effects of speculation in financial markets on real investments, and relate the model to several stylized facts and puzzles of capital markets.

Keywords: *Heterogeneous beliefs, welfare, savings puzzle, speculation, underdiversification, idiosyncratic volatility puzzle, efficiency.*

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1 Introduction

In the aftermath of the financial crisis, welfare analysis in economies with heterogeneous beliefs has received attention in several recent studies, see Brunnermeier et al. (2014), Gilboa et al. (2014); Gayer et al. (2014), and Blume et al. (2018). These studies suggest that speculative trade between agents create inefficiencies when some agents end up poor after speculating away much of their wealth. Speculation in the heterogeneous beliefs economy, instead of increasing welfare by allowing agents to hedge risks and smooth consumption, generates volatility of consumption and wealth at the individual level (see Yan 2008, and Fedyk et al. 2013). It also distorts asset price dynamics, increasing volatility and potentially leading to mispricing (see Buraschi and Jiltsov 2006, David 2008, Dumas et al. 2009, Xiong and Yan 2010, Kubler and Schmedders 2012, Simsek 2013, Buss et al. 2016, and Ehling et al. 2018a, for recent contributions), effects that may be viewed as negative in their own right.[1]

The potential policy implications are, of course, huge, since the friction-free complete market equilibrium—even when it is implementable—actually may be inefficient under such a view on the welfare effects of speculation. For example, in a simple calibration, Blume et al. (2018) find that restrictions on the traded asset span, such as borrowing limits and transaction taxes, offer substantial welfare gains relative to the complete market benchmark. Understanding the general conditions under which such inefficiencies arise is therefore important.

Previous literature has mainly focused on the exchange economy setting, in which disagreement leads to redistribution and inefficient risk sharing. These issues are important, but the role of financial markets in allocating resources to real productive assets is overlooked in exchange economies. When there is speculation, the allocation of productive capital, i.e., real investments, may also be impacted. Hence, understanding the feedback to the real economy becomes important once stepping outside the exchange economy.

Studying welfare in production economies with disagreement raises new challenges. For example, in an exchange economy the actual probabilities for different outcomes do not matter for whether

[1]See also Detemple and Murthy (1994), Zapatero (1998), Basak (2000), Basak (2005), Berrada (2006), Jouini and Napp (2007), J.Cvitanic and Malamudand (2011), J.Cvitanic et al. (2012), Bhamra and Uppal (2014), Buraschi et al. (2014), Cujean and Hasler (2017), Ehling et al. (2018b), Andrei et al. (2018).

an allocation is efficient. In contrast, in economies with production, actual probabilities typically *do* matter for whether an allocation is efficient. Identifying efficient outcomes that respect agents' beliefs is therefore more challenging.

To see the new challenges that arise when production is introduced, consider the following example. In the late spring, Ann an Bob each own one unit of a consumption good they plan to consume in the fall. Having realized that they disagree about the prospects for a sunny summer—Ann thinks that the chance, q, is high that the amount of rainfall will be above a certain threshold, whereas Bob thinks the chance is low—they decide to bet against each other and let the winner consume both units of the good, come fall. If Ann and Bob are risk neutral, it follows that they both increase their expected utilities through this bet, as long as Ann believes that $q > 50\%$ and Bob that $q < 50\%$. If Ann and Bob are moderately risk averse, they will enter the bet. If they are very risk averse, they can still find a less aggressive bet, that allows for some consumption in case the bet is lost and that increases their expected utilities. An outcome that is efficient in the ex ante sense, in that it maximizes Ann's and Bob's respective expected utilities, is called an Arrow optimum.[2] Such an outcome will thus induce betting in this case. In the subsequent argument, we assume that both Anne and Bob are somewhat risk averse.

The bet does not affect the aggregate endowment in the economy—which is so far an exchange economy—and it does not arise because of hedging demands. Rather, the resulting transfer is purely speculative. Ann and Bob cannot both be correct, and ex post one of them will surely end up with low consumption, so the bet actually introduces risk where there previously was none. A social planner may therefore be unconvinced about the bet's benefits. Brunnermeier et al. (2014), building on this argument, introduce the concept of belief-neutral inefficiency. In their terminology, Anne's and Bob's bet leads to a belief-neutral inefficient allocation, since regardless of q there exists a Pareto improvement with no risk. For an allocation to be belief-neutral efficient, it must be Pareto efficient for a whole range of reasonable beliefs, q, that include Anne's and Bob's beliefs. Importantly, for each q, both Ann and Bob's utilities should be evaluated using the *same* q under this argument, in contrast to the Arrow optimum. It is straightforward to show that belief-neutral efficient allocations

[2]See Starr (1973) and Harris (1978).

in this setting coincide with Pareto efficient allocation in the economy without disagreement, namely those allocations for which Ann's and Bob's consumption do not depend on the amount of rainfall. Speculative outcomes are ruled out as inefficient regardless of the actual chance of a rainy summer, and regardless of Anne's and Bob's beliefs.

Now consider an extension of the example that includes production. For simplicity, assume that Ann and Bob have access to a field that can be used for farming. Instead of saving the two units of the consumption good for consumption in the fall, they may sell them in the market and use the proceeds to grow crops. Two crop technologies are available, an A-technology (as in "Almonds") that requires a lot of water to grow, and a B-technology (as in "Beans") that requires little water. The two technologies are mutually exclusive, i.e., the field can only be used to grow one of the crops. In case of a rainy summer, Anne and Bob will be able to sell crops from the A-technology in the fall, for eighteen units of the consumption good, whereas the B-technology produces nothing. In case of a sunny summer, the outcome is the opposite: No A-technology crops will grow, whereas the B-technology crops can be sold for eighteen units of the consumption good. Clearly, Ann, who believes that the summer will be rainy, prefers investing in the A-technology over the B-technology in this example, whereas Bob prefers the B-technology. Moreover, if Ann and Bob are only moderately risk averse, they both prefer investing over saving (although they prefer different technologies).

Identifying efficient allocations is clearly more challenging in this setting than in the exchange economy, since the benefits of the production technologies depend on q. For some values of q the A-technology is superior, whereas for others the B-technology is better. As a consequence, the set of belief-neutral efficient allocations is empty in this example. Moreover, for no value of q is saving optimal, but for some values saving is better than investing in the A-technology, and for other values saving is better than investing in the B-technology. A social planner who knows the actual value of q may therefore rule out saving as inefficient, and consequently saving is belief-neutral inefficient in the terminology of Brunnermeier et al. (2014).[3] We argue that assuming such knowledge by the social planner about q is to set the bar too high in many situations. Indeed, a crucial role of markets is to

[3] This does not immediately follow from the fact that the set of belief-neutral efficient allocations is empty. In general, there may be a "gap" between belief-neutral efficiency and inefficiency, i.e., there may be allocations that are neither belief-neutral inefficient, nor belief-neutral efficient.

allow for the dissemination and aggregation of information, beyond that available to any individual agent, and it is therefore unclear how the social planner would obtain such superior knowledge. When the social planner has incomplete knowledge about the actual value of q, one may argue for saving as a reasonable middle-ground approach compared with betting on one of the technologies. Finally, so far in our example we have assumed that Anne and Bob only consume in the fall, after saving or investing in crops. If Ann and Bob also choose how much to consume before the summer versus how much to save or invest, the social planner's problem becomes even more challenging, since the optimal consumption choice typically depends on how good the production technologies are (i.e., on q).

In this paper, we address these challenges, and analyze the effects of heterogeneous beliefs on welfare, consumption and savings in economies with production. We also study how heterogeneous beliefs lead to asset pricing distortions. A general implication of our analysis is that the heterogeneous beliefs economy may behave quite differently than under homogeneous beliefs—even though each individual behaves in line with standard subjective expected utility theory.

Our main contribution, as described in more detail below, is three-fold. First, we introduce the concept Incomplete Knowledge (IK) efficiency, a welfare measure that is suited for production economies. Second, we analyze the real inefficiencies and distortions that arise in the production economy with heterogeneous beliefs and, third, we show that under general conditions there will also be financial distortions, i.e., mispricing.

Loosely speaking, an allocation, a, is IK-inefficient if there is an alternative allocation that dominates a for some belief among a whole set of reasonable beliefs, and is not dominated by a for any such reasonable belief. This approach is consistent with the view that the social planner has incomplete knowledge about the correct beliefs in the economy. Our analysis has similarities with, and builds upon, the concepts of belief-neutral inefficiency introduced in Brunnermeier et al. (2014), no-betting Pareto dominance in Gilboa et al. (2014), and with unanimity efficiency in Gayer et al. (2014), but also has significant differences. We also discuss the concept of U-efficiency, which is related to unanimity efficiency. As suggested already by our previous example with Ann and Bob, the analysis is far from trivial, and our framework is necessarily quite technical.

We next analyze a competitive production economy that satisfies the technical conditions, with focus on the real effects of heterogeneous beliefs. We show that agents' savings—and therefore real investments—in general are distorted. Especially, when agents' elasticities of intertemporal substitution are lower than one, agents save too little compared with what is socially optimal. We relate this result to the under-savings puzzle, and in a simple numerical illustration show that the effect can be substantial. The undersavings effect is especially severe in markets with large belief dispersion, in markets that are complete, and/or in markets with a low degree of inequality, providing potentially testable implications of our theory. The analysis of the effects on consumption and savings provides our second contribution.

Finally, we study the effect of heterogeneous beliefs on asset prices. Equilibrium price distortions are generically present in the heterogeneous beliefs economy, and is, perhaps a priori surprising, easiest identified in assets for which there are *low* disagreement. This follows from the fact that for an asset's price to be identified as distorted, it must be so under all agents' beliefs. In addition, we show that disagreement may lead to under-diversification of agents' portfolios, as well as to mispriced idiosyncratic volatility.

The rest of the paper is organized as follows: In the next section, we analyze efficiency in the general production economy. In Section 3 we, we study efficiency in a competitive market with production, and analyze the effects of disagreement on aggregate consumption-savings and in generating mispricing. Finally, concluding remarks are made in Section 4. Several additional examples are delegated to an Appendix, and all proofs to an Internet Appendix.

2 Efficiency

In this section, we introduce IK efficiency and relate this concept to other efficiency measures. A main takeaway from our analysis is that new challenges arise when disagreement has real effects on the economy, but that it is possible to define efficiency in a robust manner when some natural technical conditions are satisfied.

The IK-efficiency concept has two components that we take as given. The first is a view, in line with several recent papers and motivated in the introduction, that welfare should not be measured based on individuals' subjective ex ante expected utilities. All agents cannot be correct in their different beliefs (a fact that agents agree about), and welfare gains from speculation, as measured by aggregating ex ante expected utilities, may therefore be spurious.[4,5]

The second, incomplete knowledge, component is based on a sober view of the social planner's ability to identify the "true" probability distribution. Such a view may be considered quite pessimistic, but we note that there are plenty of historical examples when it would have been appropriate. For example, it is known that the presence of overoptimistic investors in the market together with short-sale constraints may give rise to price and investment bubbles (see, e.g., Scheinkman and Xiong 2003, and Gilchrist et al. 2005). However, although bubbles and overoptimism may be easy to identify in hindsight, there is often considerable ex ante uncertainty about their presence. There was no consensus, and relatively few warnings about there being a bubble in the run-up years before the crash of the U.S. housing market in 2007, for example. Even former Chairman of the Federal Reserve Board, Alan Greenspan, admitted to not "getting" that there was trouble on the horizon until very late.[6] During the New Economy boom and the associated dot-com bubble between 1997-2000, warnings about a bubble were issued by some, whereas others argued that discontinuous technological transition made "Old Economy" valuation formulas out-of-date. Shiller (2000) warned about a bubble generated by irrational exuberance, borrowing the term from Alan Greenspan. Pastor and Veronesi (2006), in contrast, argue that the uncertainty about future growth rates in the late 1990's may well have justified the valuation of the NASDAQ. Moreover, there are many examples throughout history of technological innovations that turned out to be transformative, but that were originally dismissed by many as fads, including the automobile, personal computers, mobile phones and the Internet.

[4]We note that the expected utility decomposition of agent preferences into a utility part and a probabilistic part is interpreted literally in this argument. Without such a literal interpretation, e.g., if the expected utility specification is viewed as merely a way of representing preferences that are linear over probabilities, in line with the axiomatic approach in Savage (1954), the argument against using ex ante expected utility is not as straightforward.

[5]It has been argued that policies that are based on ex post measures, if they restrict the actions of agents, could be viewed as paternalistic (see, e.g., Harris and Olewiler (1979), and Fleurbaey (2010)). For further discussion, we refer to the extensive existing literature, see Starr (1973), Harris and Olewiler (1979), Hammond (1981), Harris (1978), Portney (1992), Hausman and McPherson (1994), Pollak (1998), Salanie and Treich (2009), and also recent studies by Gilboa et al. (2014), Brunnermeier et al. (2014), Gayer et al. (2014), and Blume et al. (2018). Our focus is not on this issue.

[6]*CBS 60 Minutes*, interview, September 13, 2007.

2.1 IK-efficiency

To formalize the IK-efficiency concept, we consider a general economy with $T+1$ dates, $t = 0, \ldots, T$, $M \geq 2$ states, and $N \geq 2$ agents. A nonempty compact set, $\mathcal{A} \subset \mathbb{R}_+^{M\times N\times T}$, determines the feasible joint production and allocation of a consumption good among the agents in the economy, and will be denoted the *set of feasible allocations.* Here, with a slight abuse of notation, we have denoted by T the set of dates $\{0, \ldots, T\}$. Specifically, for $a \in \mathcal{A}$, $a_{m,n,t}$ represents the allocation of the good in state m to agent n at time t.[7]

Homogeneous beliefs: We first study the benchmark case when there is no disagreement about the probabilities for different states. These probabilities are represented by a probability vector $q \in S^M$, where S^M is the interior of the unity simplex in $\mathbb{R}^M$,

$$S^M = \left\{ x \in \mathbb{R}^M : x_m > 0, \sum_{m=1}^{M} x_m = 1 \right\}.$$

Note that we assume that the probability for each state to occur is strictly positive. We denote the closure of S^M by $\bar{S}^M$.

Agents are expected utility maximizers. Specifically, agent n's expected utility under allocation a given probability vector q is

$$U^n(a|q) = \sum_{m=1}^{M} U_m^n(a) q_m, \qquad where \qquad U_m^n(a) = \sum_{t=0}^{T} u_{m,t}^n(a_{m,n,t}). \tag{1}$$

Here, we assume that each agent-, state-, and time-specific utility function, $u_{m,t}^n : \mathbb{R}_+ \to \mathbb{R}$, is strictly increasing, continuously differentiable, and weakly concave. With each allocation, we associate the utility matrix, $V = \mathcal{V}(a) \in \mathbb{R}^{M\times N}$, through the mapping $V_{m,n} = U_m^n(a), 1 \leq m \leq M, 1 \leq n \leq N$, and define the *utility possibility set* $\mathcal{U} = \mathcal{V}(\mathcal{A}) \subset \mathbb{R}^{M\times N}$ (see Mas-Colell et al. 1995).

A social planner has a Bergson (1938) welfare function over feasible allocations, $U(a|q, \lambda)$, defined

[7] Dynamic revelation of information over time can be incorporated into the model by introducing a filtration over the states, and requiring feasible allocations to be adapted to that filtration. The results are identical in that setting, so we exclude this extra step for the sake of parsimony.

by

$$U(a|q,\lambda) = \sum_{n=1}^{N} \lambda^n U^n(a|q), \tag{2}$$

where $\lambda \in S^N$ are Pareto weights in the planner's welfare function. Using the rules of matrix-vector multiplication, and denoting the transpose of the vector q by q^T, it then follows that

$$U(a|q,\lambda) = q^T \mathcal{V}(a)\lambda, \tag{3}$$

i.e., given an allocation, a, the Bergson welfare function is a bilinear mapping from the probability vector, $q \in S^M$, and vector of Pareto weights, $\lambda \in S^N$, to a real number.

If $U(b|q,\lambda) > U(a|q,\lambda)$ for two allocations, $a, b \in \mathcal{A}$, we write $b >_q^\lambda a$, and if $U(b|q,\lambda) \geq U(a|q,\lambda)$, we write $b \geq_q^\lambda a$. It is straightforward to verify that standard Pareto efficiency within this setting can be defined as follows:

Definition 1 (Pareto dominance and efficiency)**.**

(i) Allocation b Pareto dominates *a, given q, $b \succ_q a$, if $b >_q^\lambda a$ for all $\lambda \in S^N$.*

(ii) Allocation b is not Pareto dominated by a, given q, $b \succeq_q a$, if $b \geq_q^\lambda a$ for some $\lambda \in S^N$.

(iii) Allocation b is Pareto efficient*, given q, if $\forall a \in \mathcal{A} : b \succeq_q a$.*

Note that an equivalent definition of $b \succ_q a$ is that $b \geq_q^\lambda a$ for all $\lambda \in \bar{S}^N$, with the inequality being strict for at least one such λ. Moreover, an equivalent definition of allocation b being Pareto inefficient given q is that

$$\exists a \in \mathcal{A}, \forall \lambda \in S^N : a >_q^\lambda b. \tag{4}$$

As pointed out, the homogeneous belief economy serves as a benchmark and it is therefore useful to characterize the set of all Pareto efficient allocations under homogeneous beliefs.

Definition 2. *We denote by E_q the set of all Pareto efficient allocations given probability vector q, and note that this set is nonempty.*

Heterogeneous beliefs: Let agent n's belief be denoted by q^n. We introduce an *Arrow optimum* in the economy with disagreement in an analogous way as in Definitions 1 and 2. We define the ex ante welfare function[8]

$$U(a|\mathbf{q}, \lambda) = \sum_{n=1}^{N} \lambda^n U^n(a|q^n), \tag{5}$$

where the belief vector $\mathbf{q} = (q^1, q^2, \ldots, q^N) \in \prod_{n=1}^{N} S^M$ summarizes all agents' beliefs. If $U(b|\mathbf{q}, \lambda) > U(a|\mathbf{q}, \lambda)$ for two allocations, we write $b >_{\mathbf{q}}^{\lambda} a$, and if $U(b|\mathbf{q}, \lambda) \geq U(a|\mathbf{q}, \lambda)$, we write $b \geq_{\mathbf{q}}^{\lambda} a$. Further in accordance with Definition 1, we also write $b \succeq_{\mathbf{q}} a$, if $b \geq_{\mathbf{q}}^{\lambda} a$ for some $\lambda \in S^N$.

Definition 3 (Arrow optimum)**.** *Allocation b is an* Arrow optimum*, given heterogeneous beliefs* $\mathbf{q}$*, if* $\forall a \in \mathcal{A} : b \succeq_{\mathbf{q}} a$*. The set of Arrow optima given beliefs* $\mathbf{q}$ *is denoted by* $E_{\mathbf{q}}^{A}$*.*

From Definition 3, we see that an allocation is an Arrow optimum (ex ante efficient) if it is Pareto efficient with respect to the ex ante expected utilities of agents, based on their *own* respective beliefs. Intuitively, being based on individual agents' ex ante expected utilities, Arrow optima allow for speculative outcomes, in which there may be significant variation in agents' allocations across states, since agents put lower weight on consumption in states they subjectively believe are highly unlikely. Speculative allocations for which it is objectively known that many agents will end up poor may therefore qualify as ex ante efficient, as discussed extensively in Brunnermeier et al. (2014), Gilboa et al. (2014), and Blume et al. (2018). The example with Ann and Bob in the introduction exemplifies such a situation. The novel efficiency measures are designed to rule out such allocations, by forcing the same probability measure to be used across agents when comparing allocations.

Which probability measure should be used in the comparison? A sober view is that the planner cannot identify the correct beliefs. Such a planner with incomplete knowledge (about probabilities) views a whole nonempty set, $\mathcal{Q}_R \subset S^N$ of beliefs as "reasonable." The special case when $q^n = q$ for all agents, n, reduces to the homogeneous beliefs setting, in which case we require the planner's reasonable beliefs set to be $\mathcal{Q}_R = \{q\}$. We are agnostic about the choice of $\mathcal{Q}_R$ in the general case. Brunnermeier et al. (2014) suggests using the convex hull of agents' individual beliefs. Specifically, for a set $X \subset \mathbb{R}^K$, define $CH(X) = \left\{\sum_{k=1}^{K} \rho_k x_k : \rho \in \bar{S}^K, x_k \in X, 1 \leq k \leq K\right\}$. The set suggested in

[8]See Starr (1973) and Harris (1978).

Brunnermeier et al. (2014) is then

$$\mathcal{Q}_R^{CH} = CH\left(\{q^1, q^n, \ldots, q^N\}\right). \tag{6}$$

This choice may be appropriate in many applications, but other choices may be too. For example, the planner may wish to exclude some agents' beliefs that are obviously incorrect. Moreover, if agents self-report their beliefs, they may for strategic reasons report beliefs that are different than the ones they actually hold, and the social planner may therefore wish to filter out some reported beliefs. Finally, under more complex information structures, nonlinear aggregation of probabilities may be appropriate, rather than the linear aggregation that the convex hull corresponds to. We provide an example where this is the case in Appendix A. The only restriction on the nonempty set $\mathcal{Q}_R$ we impose is that the set contains a single element if and only if agents have homogeneous beliefs, i.e., if and only if $q^n = q$ for some q, for all $1 \leq n \leq N$, and that $\mathcal{Q}_R = \{q\}$ in that case.

Our novel efficiency measure takes into account the welfare associated with all reasonable beliefs.

Definition 4 (IK-dominance)**.** *Allocation b* IK-dominates *a with respect to Pareto weights λ, $b \succ^\lambda a$, if:*

$$(\forall q \in \mathcal{Q}_R : b \geq_q^\lambda a) \;\; \textit{and} \; (\exists q \in \mathcal{Q}_R : b >_q^\lambda a).$$

From Definition 4 it follows that an allocation dominates another if, given Pareto weights, it is never strictly dominated under any reasonable belief, and there exist a reasonable belief under which it strictly dominates the other allocation. We also define weak IK-dominance, $a \succeq^\lambda b$ if $\neg(b \succ^\lambda a)$. Here, "$\neg$" denotes the logical negation symbol. IK-efficiency is now defined as follows:

Definition 5 (IK-efficiency)**.** *Allocation a is* IK-inefficient *if $\forall \lambda \in S^N, \exists b \in \mathcal{A}$: $b \succ^\lambda a$. Equivalently, a is IK-inefficient if*

$$\forall \lambda \in S^N, \exists b \in \mathcal{A}, \forall q \in \mathcal{Q}_R : b \geq_q^\lambda a, \tag{7}$$

where the inequality is strict for at least one q.

An allocation that is not IK-inefficient is called IK-efficient. We denote the set of IK-efficient allocations by IKE, and the set of IK-inefficient allocations is then the complement $IKE^c = \mathcal{A} \backslash IKE$. An IK-inefficient allocation is thus one for which whatever are the Pareto weights in the welfare function, there exists another allocation that is not dominated by the first allocation regardless of q in the set of reasonable beliefs, and that dominates the first allocation for some reasonable q.

Example 1. We revisit the example in the introduction with Ann and Bob, with no production, under the assumption that Ann and Bob are risk averse expected utility maximizers, both with utility of consumption $u(c) = \sqrt{c}$. They each have one unit of the consumption good. There are $M = 2$ states, state 1 representing a rainy summer and state 2 a sunny summer, and Ann and Bob disagree about the probability, q, of state 1 occurring. Ann believes that $q = 0.9$ and Bob believes that $q = 0.1$. As in Brunnermeier et al. (2014), let the set of reasonable beliefs be $\mathcal{Q}_R = \{(q, 1-q) : q \in [0.1, 0.9]\}$.

Consider the winner-takes-it-all bet between Ann and Bob. It is straightforward to show that the resulting speculative allocation is IK-inefficient. Specifically, consider the social planner with welfare function defined by (2) and Pareto weights $\lambda^1 = \lambda$, $\lambda^2 = 1 - \lambda$, who regardless of the realized state allocates the fixed fraction α of the consumption good to Ann and the remaining fraction $1-\alpha$ to Bob. It follows from the fist order conditions that the planner optimally chooses the fraction $\alpha = \frac{\lambda^2}{1-2\lambda+2\lambda^2}$, leading to total welfare $\sqrt{2}(\lambda\sqrt{\alpha} + (1-\lambda)\sqrt{1-\alpha})$. Is is easy to verify that regardless of reasonable belief, this fixed fraction allocation dominates the the speculative allocation, which is associated with welfare $\lambda(q\sqrt{2} + (1-q)\sqrt{0}) + (1-\lambda)(q\sqrt{0} + (1-q)\sqrt{2}) = \sqrt{2}(\lambda q + (1-\lambda)(1-q))$. Hence, betting is IK-inefficient in this example.

It will be useful to introduce the following equivalent definition for allocation a to be IK efficient, namely that

$$\exists \lambda \in S^N, \forall b \in \mathcal{A} : \ a \succeq^{\lambda} b,$$

or equivalently,

$$\exists \lambda \in S^N, \forall b \in \mathcal{A} : \ (\exists q \in \mathcal{Q}_R : a >_q^{\lambda} b, \text{ or } \forall q \in \mathcal{Q}_R : a \geq_q^{\lambda} b). \tag{8}$$

As discussed in the introduction, IK-inefficiency is operational for a planner who is not able to decide which q is correct among the set of reasonable beliefs, in that whatever the planner's Pareto weights are, there is another allocation b that improves upon an IK-inefficient allocation a regardless of the reasonable belief, $q \in \mathcal{Q}_R$. The planner can thus switch from a to b without regret. The IK efficiency concept thus requires the social planner to have a well-defined λ in the welfare function, but not to take a stand on a unique q among the set of reasonable beliefs.

2.2 Alternative efficiency measures

The concepts in Brunnermeier et al. (2014) of belief-neutral efficiency and inefficiency are in our setting defined as follows:

Definition 6 (Belief-neutral efficiency).

(i) Allocation a is belief-neutral inefficient, *$a \in BNI$, if $\forall q \in \mathcal{Q}_R, \exists b \in \mathcal{A} : b \succ_q a$, i.e., if*

$$\forall q \in \mathcal{Q}_R, \exists b \in \mathcal{A}, \forall \lambda \in S^N : b >_q^\lambda a. \tag{9}$$

(ii) Allocation a is belief-neutral efficient, *$a \in BNE$, if $\forall q \in \mathcal{Q}_R, \forall b \in \mathcal{A} : a \succeq_q b$, i.e., if*

$$\forall q \in \mathcal{Q}_R, \forall b \in \mathcal{A}, \exists \lambda \in S^N : a \geq_q^\lambda b. \tag{10}$$

In words, an allocation, a, is belief-neutral inefficient if for every reasonable belief, q, there is another allocation, b, that is strictly better regardless of the Pareto weights, λ.

The set of belief-neutral inefficient allocations is in general a strict subset of the complement of the set of belief-neutral efficient allocations, $BNI \subsetneq BNE^c$. Thus, there may be allocations that are neither belief-neutral efficient, nor belief-neutral inefficient. To avoid the cumbersome terminology of "not belief-neutral inefficient" allocations, we call such allocations "weakly belief-neutral efficient":

Definition 7 (Weak belief-neutral efficiency). *Allocation a is* weakly belief-neutral efficient, *$a \in WBNE$, if $a \notin BNI$, i.e., if*

$$\exists q \in \mathcal{Q}_R, \forall b \in \mathcal{A}, \exists \lambda \in S^N : a \geq_q^\lambda b. \tag{11}$$

We stress that this terminology is not used in Brunnermeier et al. (2014). It follows immediately that equivalent definitions of $WBNE$ and BNE are:

$$WBNE = \cup_{q \in \mathcal{Q}_R} E_q, \tag{12}$$

$$BNE = \cap_{q \in \mathcal{Q}_R} E_q, \tag{13}$$

One verifies easily that the speculative outcome in Example 1 is belief-neutral inefficient.

Note that the roles of the Pareto weights, λ, and probabilities, q, are dual in the definitions of IK-efficiency and belief-neutral efficiency. Specifically, under the IK-efficiency concept, the alternative allocation is allowed to vary with λ but not q, whereas under belief-neutral efficiency it is allowed to vary with q but not λ. Both concepts are reasonable, and if an allocation is efficient with respect to one measure but not the other, there is an issue of robustness. For an allocation that is IK-efficient but belief-neutral inefficient, it is unclear to a planner with incomplete knowledge about q what allocation constitutes an improvement. For an allocation that is weakly belief-neutral efficient but IK-inefficient, a planner who knows q may not be able to find a strict improvement,since the improvement need not be strict for all q.

Finally, we introduce a concept that is related to unanimity Pareto efficiency, discussed in Gayer et al. (2014)

Definition 8 (U-efficiency)**.** *Allocation a is* U-inefficient *if $\exists b \in \mathcal{A}, \forall q \in \mathcal{Q}_R : b \succ_q a$, i.e., if*

$$\exists b \in \mathcal{A}, \forall q \in \mathcal{Q}_R, \forall \lambda \in S^N : b >_q^\lambda a. \tag{14}$$

An allocation, a, that is not U-inefficient is called U-efficient, i.e.,

$$\forall b \in \mathcal{A}, \exists q \in \mathcal{Q}_R, \exists \lambda \in S^N : \ a \geq_q^\lambda b. \tag{15}$$

We denote the set of U-efficient allocations by UE. U-inefficiency is thus a strong form of inefficiency, since it requires the existence of a unique allocation that dominates a current allocation regardless of both Pareto weights, $\lambda \in \mathcal{S}^N$, and probabilities, $q \in \mathcal{Q}_R$. This is in contrast to IK-inefficiency and belief-neutral inefficiency, which both allow the alternative allocation to vary with one of these parameters. It follows that $WBNE \subset UE$, and $IKE \subset UE$.

As mentioned, U-efficiency has similarities with the unanimity efficiency-concept introduced in Gayer et al. (2014), but there are also differences: First, Gayer et al. (2014) focus on agents involved in a transaction, and require all those agents to be strictly better off. Second, they also require each agent to be better off, given his or her own beliefs, for a reallocation to be identified as an improvement. In a speculative Walrasian equilibrium outcome, in which all agents have strictly concave preferences and participate voluntarily, these additional conditions will typically automatically be satisfied. The aforementioned general relationships between the different efficiency sets are summarized in Figure 1.

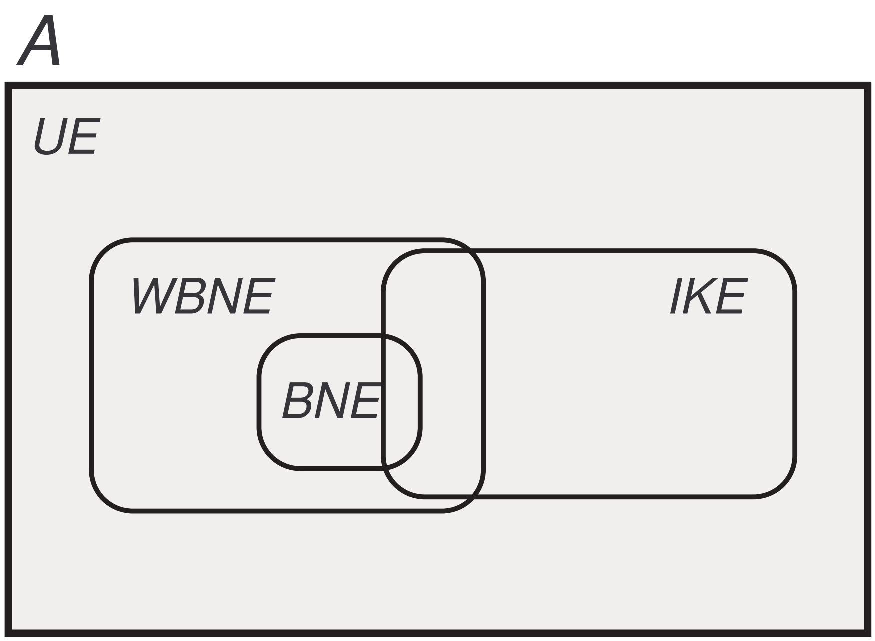

Figure 1: **General relationship between heterogeneous beliefs efficiency concepts.**

2.3 Comparison of measures, and robustness

In line with our previous discussions, for an economy to allow for a *robust* definition of efficiency, we require that IK-inefficiency and belief-neutral inefficiency coincide. It follows immediately that such an economy neither requires the social planner to know true probabilities, nor allows for purely speculative outcomes.

We mainly focus on economies that allow for transfers between agents. Specifically, for a given amount of aggregate production, such economies impose no restrictions on how the good may be shared among agents. We introduce the mapping $\mathcal{P} : \mathbb{R}_+^{M\times N\times T} \to \mathbb{R}_+^{M\times T}$, such that $X_{m,t} = \mathcal{P}(a) = \sum_n a_{m,n,t}$ represents aggregate production in state m at time t for allocation a, and the aggregate production set $\mathcal{A}_X = \mathcal{P}(\mathcal{A})$. Formally, we define:

Definition 9 (Economy with transfers)**.** *An economy is said to* allow for transfers *if for all* $a \in \mathbb{R}_+^{M\times N\times T}$ *such that* $\mathcal{P}(a) \in \mathcal{A}_X$, $a \in \mathcal{A}$.

As a benchmark, we first study homogeneous beliefs economies, in which we have:

Proposition 1. *In a homogeneous beliefs economy with probability vector q:*

(i) In general, $IKE \subset E_q = WBNE = BNE = UE$.

(ii) If the economy allows for transfers, $IKE = E_q = WBNE = BNE = UE$.

Thus, in economies that allow for transfers the efficiency concepts are all identical in the homogeneous beliefs setting, as expressed by Proposition 1(ii), and a robust definition of efficiency is therefore possible in this setting. Without transfers, IKE may be a strict subset of the other efficiency sets, even when agents have homogeneous beliefs, as suggested by Proposition 1(i). An example where this occurs is given in Appendix A.4.

We next study the case with heterogeneous beliefs, in which case $q^n \neq q^{n'}$ for at least two agents, and $|\mathcal{Q}_R| \geq 2$. First, going back to Example 1, one can show that betting is U-efficient in that example, i.e., that there is no alternative allocation to betting that is better for all reasonable beliefs and Pareto weights. Hence, in the example U-efficiency is not powerful enough to identify speculative betting as inefficient. The following proposition extends the result to show that in general it is not possible to use U-efficiency to rule out all speculative allocations that are Arrow optimal, whereas the other efficiency concepts typically do rule out such speculative allocations:

Proposition 2. *In the heterogeneous beliefs economy with transfers, such that* $q^n \in \mathcal{Q}_R$ *for all agents,* $1 \leq n \leq N$:

(i) There is an Arrow optimal allocation, $a \in E^A_{\mathbf{q}}$, that is also U-efficient, $a \in UE$.

(ii) If all agents' utility functions are strictly concave and aggregate production is the same for any two Arrow optima, i.e., $\mathcal{P}(a) = \mathcal{P}(b)$ for any $a, b \in E^A_{\mathbf{q}}$, then all *Arrow optimal allocations are U-efficient, $E^A_{\mathbf{q}} \subset UE$.*

(iii) Any Arrow optimal allocation, $a \in E^A_{\mathbf{q}}$, in which two agents who disagree about the relative likelihood of two states to occur are allocated strictly positive amounts of the consumption goods in both those states, is neither IK-efficient, nor weakly belief-neutral efficient, $a \notin IKE \cup WBNE$.

Since competitive equilibria in a Walrasian economy can be identified with the economy's Arrow optima, via the welfare theorems, Proposition 2 implies that there will be at least one such speculative equilibrium outcome that is identified as U-efficient. Hence, whereas both IK-efficiency and weak belief-neutral efficiency rule out all speculative allocations, U-efficiency is under general conditions not a strong enough criterion to do so.[9] In fact, part (ii) of Proposition 2 shows that under additional assumptions, that are satisfied in several workhorse models in the literature, the set of Arrow equilibria is a subset of the U-efficient allocations.

We next relate $WBNE$ and IKE. In general, these efficiency measures differ, but under additional conditions that are satisfied in several work-horse models they coincide. We introduce the following conditions:

C1. The utility possibility set, $\mathcal{U} \subset R^{M \times N}$, is convex.

C2. The set of reasonable beliefs, $\mathcal{Q}_R$, is convex.

C3. Strict dominance: For all a in $WBNE$, $\exists \lambda \in \bar{\mathcal{S}}^N, \exists q \in \mathcal{Q}_R, \forall b \neq a : a >^{\lambda}_{q} b$.

The convexity condition for the utility possibility set, C1, is standard (see Mas-Colell et al. 1995). In an economy that allows for transfers, a sufficient condition for $\mathcal{U}$ to be convex is, e.g., that the aggregate production set, $\mathcal{A}_X$ is convex. Another way of ensuring convexity of $\mathcal{U}$ is by allowing for randomization, see Yaari (1981). Specifically, if the planner uses a randomization device to choose between

[9] Belief-neutral inefficiency of Arrow optima—part (iii) of Proposition 2—actually follows from the analysis in Starr (1973), see Starr's Corollary 3.1, p. 88.

allocations $a_1, \ldots, a_K$ with probabilities $\rho_1, \ldots, \rho_K$, the associated utility matrix is $\sum_{k=1}^{K} \rho_k \mathcal{V}(a_k)$, which consequently belongs to $\mathcal{U}$. This rationale for a convex utility possibility set is of course more subtle in our setting with disagreement than in Yaari (1981), because agents may not disagree about the probabilities of the randomization device for the argument to hold. Machina (2004) discusses how to construct events such that agents agree on the probabilities of those events. The construction is nontrivial and only works under some technical regularity conditions. If such *objective randomization* is used, however, convexity of the utility possibility set follows.

The convexity condition for the set of reasonable beliefs, C2, is satisfied under the assumptions made in Brunnermeier et al. (2014), see equation (6). The strict dominance condition, C3, states that each allocation that is Pareto efficient for some reasonable q is associated with a Pareto weight and reasonable probability vector for which that allocation strictly dominates all other allocations. We now have:

Proposition 3. *In the heterogeneous beliefs economy with transfers:*

(i) If C1 is satisfied and each utility function $u^n_{m,t}$ is strictly concave, then C3 is satisfied.

(ii) If C1 and C2 are satisfied, then $IKE \subset WBNE$.

(iii) If C3 is satisfied, then $WBNE \subset IKE$.

Proposition 3(i) shows that in economies with strictly concave utility functions, condition C3 is redundant. Moreover, it follows immediately from Proposition 3(ii)-(iii) that:

Corollary 1. *In production economies with transfers, that satisfy conditions C1-C3, IK-inefficiency and belief-neutral inefficiency are equivalent. Such economies thus allow for a robust definition of efficiency.*

We note that it follows from (12) that both $WBNE$ and IKE are nonempty sets when conditions C1-C3 are satisfied.

As suggested by (12,13), belief-neutral efficiency puts substantially stronger restrictions on allocations than weak belief-neutral efficiency, and we may therefore expect BNE to be a small set in many cases. Indeed, a belief-neutral efficient allocation must be efficient for *all* reasonable q. In economies for which the optimal aggregate production depends on q, BNE may therefore be empty, as noted in Brunnermeier et al. (2014), and as we now explore further.

We show that belief-neutral efficient allocations can never be associated with smooth boundary points of the aggregate production set. Specifically, a boundary point of the production set, $a \in \partial \mathcal{A}_X$ is said to be *smooth* if it has a unique supporting hyperplane (see Gallier 2011, p. 108).[10] Using the formulation (3) for the planner's optimization problem, it follows that q defines a supporting hyperplane to the set $\{V\lambda : V \in \mathcal{U}\}$ at an efficient point. There can therefore not be multiple q's for which an allocation is efficient where the boundary of the production set is smooth, leading to an empty belief-neutral efficient set. The argument can be made formal as long as the utility possibility set is sufficiently thick, as guaranteed by the following condition:

C4. The aggregate production set has nonempty interior, $\mathrm{Int}(\mathcal{A}_X) \neq \emptyset$.

We then have:

Proposition 4. *In a heterogeneous beliefs economy with transfers that satisfies C1 and C4, BNE contains no allocations that are associated with smooth boundary points of $\mathcal{A}_X$.*

Corollary 2. *In a heterogeneous beliefs economy with transfers that satisfies C1 and C4, that has has a smooth efficient frontier, there are no belief-neutral efficient allocations, $BNE = \emptyset$.*

Corollary 2 rules out the existence of belief-neutral efficient allocations in several standard models with production, for example, models with disagreement about the production technology in a stochastic AK model. Only boundary points in the aggregate production set that are so "pointy" that they allow for a whole set of supporting beliefs may be belief-neutral efficient. An extreme example for which this holds is the exchange economy, in which the production set is a singleton, and in which the belief-neutral and IK-efficiency concepts coincide. Indeed, in the exchange economy, belief neutral efficiency, weak belief-neutral efficiency, and IK-efficiency are all equivalent, as shown by Proposition 5 below.

[10] In one dimension this condition reduces to the efficient frontier being differentiable at the point a.

Definition 10 (Exchange economy)**.** *An* exchange economy *is an economy that allows for transfers, in which the aggregate production set is a singleton, $\mathcal{A}_X = \{X\}$.*

Proposition 5. *In the exchange economy,*

(i) $E_q = E_{q'}$ *for all* $q, q' \in S^M$*,*

(ii) $IKE = WBNE = BNE$*.*

To summarize, the exchange economy allows for a robust definition of efficiency under heterogeneous beliefs,[11] because the exchange economy setting avoids the challenges that arise when there are real aggregate effects of heterogeneous beliefs. Conditions C1-C3, jointly ensure robustness in the production economy, i.e., IKE and WBNE are equivalent under these conditions, whereas U-efficiency and BNE, are usually too weak and strong concepts, respectively, even under these conditions.

We conclude this section with an example that highlights challenges that arise when we move away from economies that satisfy conditions C1-C3:

Example 2. We revisit the example with Ann and Bob from the introduction, with production. Specifically, Ann and Bob are risk averse expected utility maximizers with square root utility. There are three mutually exclusive production technologies (safe, almonds, and beans), one consumption date (the fall), and two states (rainy summer and sunny summer). The expected utility of agent $n \in \{\text{Ann}, \text{Bob}\}$ is

$$U^n = q\sqrt{c_1^n} + (1-q)\sqrt{c_2^n}, \tag{16}$$

where c_m^n is the consumption of agent n in state $m \in \{1,2\}$, state 1 represents a rainy summer, state 2 a sunny summer, and q is the probability of state 1. Ann and Bob disagree about q. Ann believes that the probability is $q^{Ann} = 0.9$, whereas Bob believes it is $q^{Bob} = 0.1$. The social planner, not knowing which beliefs are correct, views any probability in $\mathcal{Q}_R = \{(q, 1-q) : q \in [0.1, 0.9]\}$ as reasonable.

The first production technology is the safe technology, which generates a total output of 2 units of the consumption good in either state. The second technology is production of Almonds, which

[11] Note that since an exchange economy always fails condition C4, Proposition 5(b) and Corollary 2 are mutually consistent.

generates 18 units after a rainy summer, and 0 units after a sunny summer. The third technology is beans, which generates 18 units after a sunny summer and 0 after a rainy one. The economy neither permits transfers, nor objective randomization over events.

There are four possible allocations, captured by the set $\mathcal{A} = \{a_1, a_2, a_3, a_4\}$. Under allocations $a_1 - a_3$, Ann and Bob are each allocated one half of the total production of the safe, almond, and bean technology, respectively. In allocation a_4, the safee production technology is used, as in allocation a_1, but Ann receives 1.9 after a rainy summer and Bob receives 0.1, whereas Bob receives 1.9 after a sunny summer and Ann receives 0.1. Allocation a_4 is thus speculative. This is reminiscent of Example 1. Table 1 shows the consumption by the two agents for different allocations and states.

Allocation,	a_1		a_2		a_3		a_4	
Agent	Ann	Bob	Ann	Bob	Ann	Bob	Ann	Bob
Rain	1	1	9	9	0	0	1.9	0.1
Sun	1	1	0	0	9	9	0.1	1.9
Ex ante utility	1	1	2.7	0.3	0.3	2.7	1.27	1.27

Table 1: **Four allocations in economy with two agents and two states. Allocation a_1 and a_4 are based on investments in the safe technology, with equal (a_1) and unequal (a_4) sharing between agents in the two states. Allocation a_2 and a_3 both have equal sharing, but invest in risky technologies that pay off in the rainy and sunny state, respectively.**

It is easy to verify that the only ex ante inefficient (i.e., not Arrow optimal) allocation is a_1, which both agents agree is dominated by a_4, based on their different beliefs. Of course, both agents also agree that the welfare improvement is speculative, and that whatever the true q is, any allocation in which individual consumption shares vary across states can be improved upon by risk sharing. Thus, a_4 is inefficient whenever the planner uses the same q for both agents' utilities. This is the speculative inefficiency that is captured by all the discussed welfare measures that are based on ex post realizations.

In Figure 2, the right panel compares allocation a_1 with a_4. The horizontal (black) line represents the (same) utility of the two agents under the safe allocation a_1, whereas the sloped (blue) lines represent the utilities of the two agents under the risky allocation a_4. For low and high q's, one of the agents is better off under a_4 than under a_1, whereas the other is worse off, and for q close to 1/2 both agents are worse off. Now, the reason one agent is better off for extreme q's is exactly because of speculative redistributions. Regardless of q, allocation a_4 is therefore inferior when using any measure that forces q to be the same for the two agents. So the planner can always improve upon a_4.

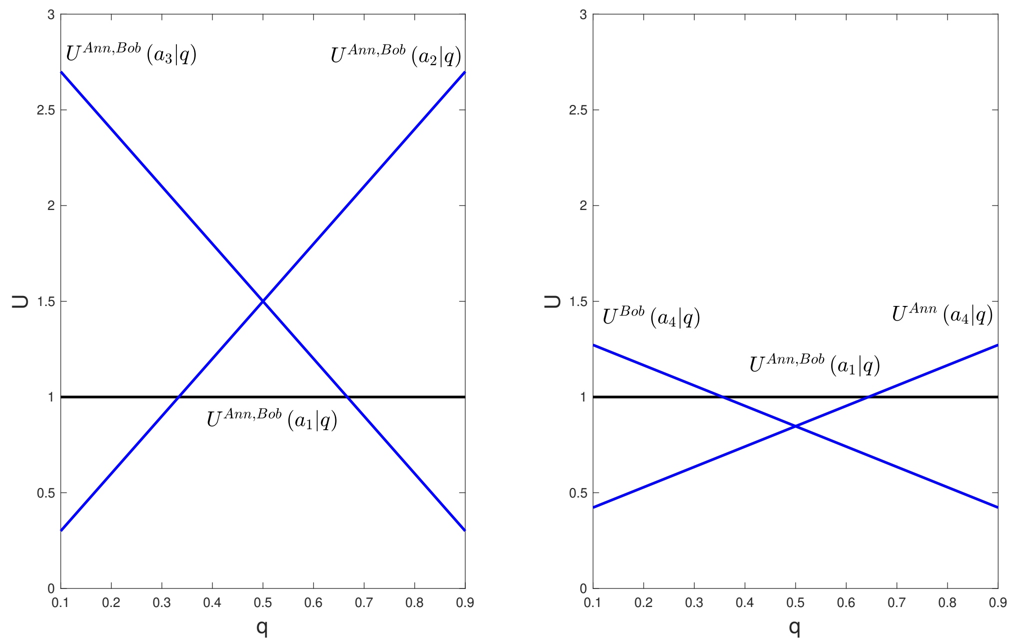

Figure 2: **Expected utilities of agents as a function of q. Left panel: Comparing utilities of allocations a_1, a_2 and a_3. Right panel: Comparing utilities of allocations a_1 and a_4.**

It follows from the efficiency definitions in the previous section that a_1, a_2, and a_3 are all IK-efficient ($IKE = \{a_1, a_2, a_3\}$), that a_1 and a_4 are belief-neutral inefficient ($WBNE = \{a_2, a_3\}$), that there are no belief-neutral efficient allocations ($BNE = \emptyset$), and that all four allocations are U-efficient ($UE = \{a_1, a_2, a_3, a_4\}$). Also, it follows that conditions C2 and C3 are satisfied, whereas C1 is not. These results remain the same even if we allow for transfers between agents, as discussed in the appendix.

The results are consistent with our analysis in the previous section. Specifically, $BNE \subset WBNE \subset IKE \subset UE$, with each inclusion being strict, in line with Figure 1 and Proposition 3. Importantly, since the set of feasible allocations is not convex, it is possible for an allocation to be IK-efficient but not weakly belief-neutral efficient (allocation a_1 in this example), because the conditions for Proposition 3(ii) are not satisfied.

We argue that it is reasonable to view allocation a_1 as efficient in this example. As shown in the left panel of Figure 2, a_1 is dominated by some other allocation for every $q \in \mathcal{Q}_R$, and is thereby belief-netural inefficient. However, since the planner with incomplete knowledge about q is unable to determine which allocation of a_2 and a_3 improves efficiency, allocation a_1 is actually a reasonable choice.

If objective randomization is possible, a_1 no longer remains IK-efficient, because a randomization of a_2 and a_3 with equal probability leads to expected utility of 1.5 for both agents in both states, regardless of q. It follows that in this case $WBNE = IKE = \{a_2, a_3\}$. This is in line with Proposition 3, Parts (ii) and (iii), since the utility possibility set is convex when objective randomization is possible, making condition C1 satisfied. As discussed, such objective randomization may be nontrivial.

3 A competitive market

We study the efficiency of market outcomes in an equilibrium model with production. This also allows us to identify how an inefficient equilibrium leaves footprints in the economy. In other words, we identify quantities that behave differently for efficient and inefficient equilibrium outcomes. We refer to such quantities as being distorted when they differ from the values that could be reached under efficient allocations. Identifying such distortions can potentially serve as a tool for the planner in identifying inefficiencies and moving towards efficient allocations.. A general conclusion of our analysis is that equilibrium outcomes are in general inefficient, and that in addition to speculation and price distortions, aggregate consumption and savings are typically also distorted. We define distortions formally in Definition 13 below, but simply put distortions look like anomalies when seen from a the lens of representative agent models.

We consider a simple production economy with one linear production technology and two dates, $T = \{0, 1\}$, that allows for transfers. There are $M > 1$ possible states, and $N > 1$ agents with heterogeneous beliefs. The state is revealed at $t = 1$, so we require that $a_{m,n,0} = a_{m',n,0}$ for $1 \leq m, m' \leq M$, for all $a \in \mathcal{A}$. Agents have strictly concave utility, i.e., the functions $u^n_{m,t}$ are strictly concave for all n and m, and $t \in \{0, 1\}$. The set of reasonable beliefs is described by the convex hull of agents' beliefs (6), as in Brunnermeier et al. (2014). Condition C2 is therefore satisfied.

Each agent has access to the production technology and is endowed with a strictly positive initial amount of a divisible capital good, $K^n > 0$. We normalize the total supply of the good to unity, $\sum_{n=1}^{N} K^n = 1$. Agents can use the capital to consume today or invest for the next period. Each unit invested yields a random strictly positive amount at time $t = 1$ at which point it is consumed. The vector $R \in \mathbb{R}_+^M$ summarizes the output yielded at $t = 1$ by one unit of investment, in the M states. The initial endowment is represented by a vector, $K = (K^1, \ldots, K^N) \in S^N$. This leads to the following set of feasible allocations and aggregate investments:

Definition 11 (Feasible allocation)**.** *An allocation is feasible, $a \in \mathcal{A}$, with aggregate investment, $I \in [0, 1]$, if*

(i) $I = \sum_{n=1}^{N} (K^n - a_{1,n,0}) = 1 - \sum_{n=1}^{N} a_{1,n,0}$,

(ii) $\sum_{n=1}^{N} a_{m,n,1} \leq I R_m$, $m = 1, \ldots, M$.

It follows that the aggregate production set, $\mathcal{A}_X \subset \mathbb{R}_+^{M \times 2}$ is convex, and therefore that Condition C1 is satisfied. Since utility functions are strictly concave, Condition C3 then also holds. From Proposition 3 it follows that IK-efficiency and weak belief-netural efficiency coincide in this economy,

$$IKE = WBNE,$$

so the economy allows for a robust definition of efficiency.

Henceforth, we focus on the case in which all agents have separable power utility across states and time

$$u_{m,t}^n(c) = \rho^t u(c) = \rho^t \frac{c^{1-\gamma}}{1-\gamma},$$

for all n, m, and t, with $\gamma > 0$, $\rho > 0$, and with logarithmic utility as the special case when $\gamma = 1$. We also define the elasticity of intertemporal substitution (EIS), $\psi = \frac{1}{\gamma}$. For simplicity assume a personal discount rate of zero, so that $\rho = 1$. As in the previous section, agent beliefs are summarized by $\mathbf{q} \in \prod_{n=1}^{N} S^M$. The primitives of the competitive economy is summarized by the quadruple $\mathcal{E} = (\gamma, \mathbf{q}, K, R)$.

3.1 Equilibrium

At $t = 0$, agents trade in a market for state-contingent Arrow Debreu (AD) claims on each state, with the mth security paying off 1 at time $t = 1$, conditioned on state m occurring. The price of the mth AD security, measured in time-0 units of the good, is p_m. The absence of arbitrage implies that

$$\sum_m p_m R_m = 1,$$

since agents would otherwise form arbitrage portfolios in AD securities and the real asset that represents the production technology.[12] The optimization problem for agent n is then

$$\max_{c_{11}^n,\ldots,c_{1M}^n} \quad u(c_0^n) + \sum_{m=1}^{M} u\left(c_{1m}^n\right) q_m^n, \qquad \text{s.t.,} \tag{17}$$
$$c_0^n = K^n - \sum_{i=m}^{M} c_{1m}^n p_m.$$

Here, c_{1m}^n is agent n's demand for the mth AD security, and we define the demand vector $d^n = (c_{11}^n, \ldots c_{1M}^n) \in \mathbb{R}^M$. We restrict our attention to the case with strictly positive AD security prices, since arbitrage opportunities would arise if some prices were nonpositive, inconsistent with equilibrium. We define the state-price vector $p = (p_1, \ldots, p_M) \in \mathbb{R}_{++}^M$. The strict concavity of agents' preferences implies that d^n is a unique and smooth function of K^n, q^n, and p, so we can write

$$d^n = D(K^n, q^n | p),$$

for a smooth function, D. We now have:

Definition 12 (Equilibrium)**.** *Given the economy, $\mathcal{E}$, a competitive (Walrasian) equilibrium is defined by a state-price vector, p, such that*

$$\sum_m p_m R_m = 1, \qquad \textit{and}$$

[12]Strictly speaking, this arbitrage argument only works for interior allocations, such that $0 < I < 1$. This condition will always hold in equilibrium, since the utility specification satisfies the Inada conditions.

$$\sum_n (D(K^n, q^n|p))_m = IR_m, \qquad m = 1, \dots, M,$$

for some $I \in (0,1)$.

We say that p is an equilibrium state price vector, and we then have the following existence and uniqueness result:

Proposition 6. *In the economy,* $\mathcal{E} = (\gamma, \mathbf{q}, K, R)$*, there exists an equilibrium state-price vector,* p*. Moreover, there exists a* $\underline{\gamma} < 1$*, such that* p *is unique when* $\gamma \geq \underline{\gamma}$*.*

In our subsequent analysis, we assume that risk aversion is sufficiently high so that equilibrium is unique, $\gamma \geq \bar{\gamma}$.[13]

The following result shows that equilibrium is always IK-inefficient under heterogeneous beliefs:

Proposition 7. *Competitive equilibrium is efficient if and only if beliefs are homogeneous.*

Thus, just as in the exchange economy setting, as discussed in Brunnermeier et al. (2014) and Blume et al. (2018), the competitive equilibrium outcome is always inefficient under heterogeneous beliefs in this economy.

An immediate reaction to Proposition 7 may be that the social planner should shut down the market for contingent claims, since its main role is to generate speculation. We caution against such an interpretation. To keep the model simple, we have assumed that agents are not exposed to idiosyncratic endowment shocks, but similar results arise when this assumption is relaxed, as discussed in Appendix A.2. When agents are exposed to idiosyncratic shocks, the market also facilitates risk hedging, and shutting it down may be associated with significant welfare costs.

3.2 Distortions

We introduce a concept of distortion that does not depend on knowledge of q among the set of reasonable beliefs. We then use the concept to detect over- and under-investment, and mispricing.

[13] Our uniqueness proof, which uses an approach that to the best of our knowledge is novel, does not rely on the gross substitutes property of demand functions (see Arrow et al. 1959), which would restrict risk aversion to be less than unity in this setting. Instead, based on our argument, uniqueness is not guaranteed if risk aversion is very low. It is commonly believed that $\gamma > 1$ in practice. Hence, uniqueness is guaranteed for reasonable levels of risk aversion in our model, which we view as a strength.

Such distortions may be of interest on own merits. They may also be used to infer that equilibrium is inefficient, because they never arise under efficiency.

Our approach is similar to that taken when defining efficieny. A real-valued equilibrium characteristic of the economy is defined as distorted if it takes on a value that is inconsistent with any equilibrium outcome under homogeneous beliefs in $\mathcal{Q}_R$.[14] We formalize the concept as follows: Let $\Pi(\mathcal{Q}_R) = \left\{\mathbf{q} \in \prod_{n=1}^N S^M : \mathbf{q} = (q, q, \dots, q), q \in \mathcal{Q}_R\right\}$ denote the set of reasonable homogeneous belief vectors.

Now, consider a real-valued characteristic, v of the equilibrium outcome, which in general can be viewed as a function of the equilibrium allocation, a, and price, p. Since a and p in general depend on initial endowments, K, and beliefs, $\mathbf{q}$, we can write $v = v(K, \mathbf{q})$. The set of unanimously reasonable values of v is now defined as $F_v^U = \left\{v(K, \mathbf{q}) : K \in S^N, \mathbf{q} \in \Pi(\mathcal{Q}_R)\right\}$, leading us to:

Definition 13 (Distortion)**.** *The real-valued characteristic v is said to be* distorted *in equilibrium if $v \notin F_v^U$. It is* too low *if $v < \inf\ F_v^U$, and* too high *if $v > \sup\ F_v^U$.*

It follows that distortions never arise in homogeneous beliefs economies, which in turn, via Proposition 7, implies that a distortion is always associated with an inefficient outcome. A planner could therefore potentially identify inefficiencies by detecting distortions, for example, in aggregate savings. This approach may be easier in practice than directly detecting inefficiencies from agent preferences.

3.3 Consumption-savings distortions

We consider consumption-savings distortions, in a special case of the equilibrium model. Specifically, we assume that there is no aggregate uncertainty, so that $R_m = 1$ for all states $m = 1, \dots, M$. In this case, investments in the production technology corresponds to risk-free savings (in a storage technology), allowing a consumption-savings interpretation of aggregate consumption at times 0 and 1.[15] We call this special case the savings economy.

[14] As the the set of competitive equilibria in the homogeneous beliefs economy, via the welfare theorems, is equivalent to the set of efficient outcomes, we could equivalently have defined an outcome to be distorted with respect to the set of efficient outcomes.

[15] The results in this section can easily be extended to economies with aggregate uncertainty, as long as agents only disagree about idiosyncratic uncertainty. In Appendix A.5 we show in an example that similar results may also arise when there is disagreement about aggregate uncertainty.

It is easy to show that under homogeneous beliefs, there is a unique optimal level of aggregate savings, I^*, regardless of agent beliefs and initial endowments, which is achieved in equilibrium. Under heterogeneous beliefs, in contrast, we have:

Proposition 8. *Under heterogeneous beliefs:*

(i) If the elasticity of intertemporal substitution is less than one, $\psi < 1$, there is undersaving (overconsumption) in equilibrium, $I < I^*$.

(ii) If the elasticity of intertemporal substitution is greater than one, $\psi > 1$, there is oversaving (underconsumption) in equilibrium, $I > I^*$.

Thus, aggregate consumption and savings are always distorted when agents disagree, with the exception of for log utility. It may be a priori surprising that such unanimous agreement about the presence of a distortion arises in the model. Indeed, one may conjecture that agents would disagree about the optimal amount of savings.

The reason why there is a unanimous consensus about the savings distortion is because of a wealth effect: When agents disagree, they all feel wealthier because of their anticipated gains from speculation. They therefore readjust their consumption and savings at $t = 0$, decreasing savings if the elasticity of intertemporal substitution $\psi < 1$ and increasing savings if $\psi > 1$. The effect of increased wealth on savings is known to be intimately connected to ψ, and it is therefore not surprising that the direction of the savings distortion also depends on ψ. The distortion is a consequence of "spurious unanimity" (see Mongin 1995): Agents believe the distorted savings outcome to be optimal for themselves, although they base their conclusions on different—mutually exclusive—beliefs. We note that although wealth effects on investments are of course well known from previous literature, what is special in our framework is that all agents agree that there is no aggregate wealth increase, but still jointly behave as if there is one. In a production economy, this typically impacts the aggregate investment in the economy. To the best of our knowledge this impact on the aggregate investment has not been made previously, and creates a wedge between the behavior of a representative agent and agents with heterogeneous beliefs.[16]

[16] The income and substitution effect have recently been studied in relation to disagreement for exchange economies;

The aggregate investments distortions arise because of the speculation in financial markets. The mechanism highlights the role of financial markets in channeling resources to productive assets, and the associated real feedback effect of speculation that is not present in the exchange economy.

The results in Proposition 8 relate our heterogeneous beliefs model to the consumption-savings puzzle. The personal savings rate is lower than predicted by the standard representative agent consumption-savings model and has, moreover, decreased significantly over the last four decades, see Parker (1999) and Guidolin and Jeunesse (2007). A recent suggested explanation is given in Han et al. (2018), namely that individuals overestimate the consumption of their peers because of so-called *visibility bias*. They therefore infer that the risk for adverse shocks in the future is low, and consequently also the need for precautionary savings. The visibility bias mechanism is quite different from our model, in which undersaving arises because of subjective wealth effects, and is driven by disagreement and speculation. In our model, agents believe that other agents are not saving enough, but that their own saving is optimal, an effect not present in Han et al. (2018).

We derive comparative statics, focusing on a tractable symmetric setting with $M = N \times J$ states, where each agent $n = 1, \ldots, N$ is relatively optimistic about $J \geq 1$ states and relatively pessimistic about the rest of the states. There are no two agents that are optimistic about the same two states. Every agent is, in the baseline version, endowed with $\frac{1}{N}$ unites of the capital. Agent $n = 1, \ldots, N$ believes that the probability is $\frac{\Delta}{M}$ for his/her "own states" $m \in \{J(n-1)+1, \ldots, Jn\}$ and $\frac{1-J\frac{\Delta}{M}}{M-J}$ for all other states. In order to ensure that probabilities are strictly positive and that agents are relatively optimistic about their own states, we impose the condition that $1 \leq \Delta < N$. A possible interpretation is that the production technology corresponds to an investment in an index fund covering the whole economy, whereas the states corresponds to individual firms—firms that agents disagree about the prospects for.

Effect of varying disagreement: With $\Delta = 1$, agents believe that all states are equally likely, and therefore there is no disagreement. A higher Δ implies that agents are relatively more optimistic about their own states relative to all the other states. Hence, in this setting, disagreement can be

in Ehling et al. (2018a), who look at the effect on the risk-free rate; in Guzman and Stiglitz (2016), who focus on the sum of individual wealth expectations being higher than total wealth; and in Iachan et al. (2017), who study the effect of introducing new speculative assets.

measured by Δ. Define the aggregate savings to consumption ratio as $Z = \frac{I}{1-I}$. It can be shown that $Z = 1$ for $\Delta = 1$, and for arbitrary $\Delta \in (1, N)$, that $\frac{\partial Z}{\partial \Delta} > 0$ when $\gamma < 1$ and $\frac{\partial Z}{\partial \Delta} < 0$ when $\gamma > 1$.[17] Moreover, in line with arguments earlier in this section, the only reasonable savings-to-consumption ratio is $Z = 1$ regardless of probabilities $q \in S^M$. Therefore, an increase in disagreement also increases the magnitude of the equilibrium savings distortion.

Effect of market completeness: The source of the consumption-savings distortion is that agents are speculating on their own beliefs. Hence, one would expect that as one restrict the access to financial markets, consumption-savings distortions are reduced. Clearly, if agents only have access to the investment technology, there is no savings distortion and consequently $Z = 1$ regardless of beliefs. We study how the distortions are impacted by shutting down a subset of the markets. Specifically, we assume that every agent can trade in $J - l$ of the assets they are relatively optimistic about and the $(N-1)(J-l)$ assets that they are relatively pessimistic about for $\ell = (0, \ldots, J)$, which makes the analysis tractable. Let Z_ℓ denote that investment-consumption ratio when ℓ markets are shut down. It follows that $Z_\ell < Z_{\ell+1} \leq 1$ for $\gamma > 1$ and $Z_\ell > Z_{\ell+1} \geq 1$ for $\gamma < 1$, for $\ell \in \{0, \ldots, J-1\}$. Hence, distortions are reduced as we reduce the number of traded assets.

Effect of inequality: To study the effect of inequality, we vary the initial consumption share of the agents. Specifically, we let the consumption share of the first agent be $1 - \delta \frac{N-1}{N}$ and $\frac{\delta}{N}$ for all the other agents, with $\delta \in [0, 1]$. Hence, when $\delta = 0$ everything is consumed by the first agent and when $\delta = 1$ the consumption share is the same for every agent in the economy. So, as we increase δ the initial consumption distribution is equalized. It can be shown that $\frac{\partial Z}{\partial \delta} < 0$ for $\gamma > 1$ and $\frac{\partial Z}{\partial \delta} > 0$ for $\gamma < 1$ and $\delta \in [0, 1]$. Moreover, Z is maximized (minimized) for $\delta = 0$ when γ is greater (less) than one. Hence, as we increase δ, distortions increase. The reason for this is that the more equal the initial consumption distribution is, the more room there is for speculative trade among agents and, consequently, the more severe are the equilibrium savings distortions.[18] In the extreme case, in which one agent has all the wealth, there is no one for that agent to speculate against.

[17] With $\gamma = 1$ there is no investment distortions and $Z = 1$ for all values of disagreement.

[18] This result is robust to alternative ways of changing the degree of inequality. In the proof in the Appendix we also show the case when there are three groups of agents and we increase the initial consumption of the first group at the expense of the third group, keeping the initial consumption of the middle group constant. In that example, agents within each group also disagree and therefore we are not concentrating all consumption in the hands of a single agent type as in the above example.

We summarize the comparative statics results on savings in:

Proposition 9. *In the symmetric economy, the savings distortion is increasing in*

(i) Disagreement,

(ii) Market completeness, and

(iii) Equality.

Proposition 9 provides potentially testable implications. The first two results suggest that in markets and times of high disagreement, and in well developed financial markets, the consumption-savings puzzle will be more severe. As proxies for belief heterogeneity, analyst forecast disagreement or short interest may, for example, be used, at the individual asset level. Measures of financial development and inequality are harder to measure at the disaggregated level, but cross-country comparisons may be possible.

The predictions separate our model from Han et al. (2018). The third prediction, that the puzzle is more severe when inequality is low, also arises in Han et al. (2018). We note that the rapid development of new financial markets in the U.S. over the last four decades is consistent with increased undersaving over the time period.

To illustrate the size of consumption-savings numerically, we consider an economy with ten agent types ($N = 10$), which could, for example, represents investor groups focusing on different sectors, and with ten states per agent ($J = 10$), which could, for example, represent different firms within each sector. As the effect is qualitatively different when $\gamma < 1$ and $\gamma > 1$ we consider both cases, $\gamma \in \{0.8, 2\}$. We set $\Delta = 9$, so that the probability of "own" states are believed to be 0.09 and for all other states the probability is believed to be 0.0011. There is thus significant disagreement among the agents about which industries will prevail. From Figure 3, we see that the degree of consumption-savings distortions can be significant.

We also note that when agents disagree in this symmetric economy, they adjust their portfolio holdings away from the fully diversified investment portfolio with portfolio weight $\frac{1}{M}$ in each asset (here represented by AD securities), to a portfolio with disproportionately large holdings in a few

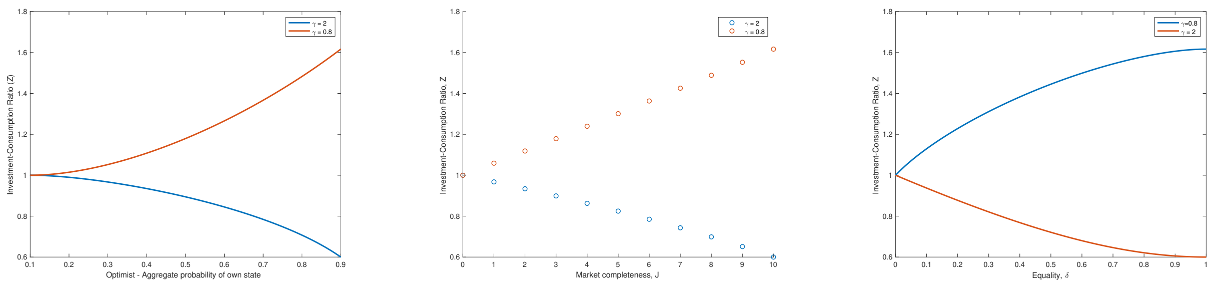

Figure 3: **Equilibrium Consumption-Savings Distortions: The plots show the investment-consumption ratio, Z. The left plot shows the effect of increasing disagreement, Δ. The middle plot shows the effect of increasing the number of traded assets, $J - \ell$. The right plot shows the effect of increasing equality, by increasing δ. Parameters:** $\gamma \in \{0.8, 2\}$.

assets. That is, the more agents disagree, the more under-diversified their portfolios become, relating disagreement to a major stylized fact of capital markets—the under-diversification puzzle. This effect disagreement on under-diversification is consistent with Anderson (2013), who argues that under-diversification is an outcome of agents being overoptimistic. Within our framework, disagreement—whether overoptimisic or not—is sufficient for under-diversification to occur.

Proposition 10. *In the symmetric economy, the more agents disagree, the more underdiversified portfolios they hold in equilibrium.*

A recent literature relates individual investors' beliefs to their network position in the economy, e.g., represented by geography (e.g., in Bhamra et al. 2019). In line with the previous results in this section, our model predicts that an agent's position should be related to the degree of diversification of his/her investment portfolio, and also that the geographical distribution of investors in the economy should be related to aggregate investments and savings. An alternative way of testing our model's predictions may therefore be to use data on individual portfolio holdings and positions of the investment population.

3.4 Price distortions

Historically, many have argued for the presence of significant mispricing of assets during so-called bubble periods, see, for example, Shiller (2000). In our model, mispricing corresponds to price distortions.

As was the case with consumption-savings distortions, it is a priori unclear whether price distortions may arise in our model, since all agents agents need to agree about their presence. Our analysis in this section shows that in markets with disagreement and rich state spaces, presence of price distortions is the rule rather than exception, especially for assets about which there is *low* disagreement.

We proceed by defining price distortion as follows: Let the vector $a \in \mathbb{R}^M$ represent the asset that pays a_m in state m at $t = 1$, and $P^a = \sum_m a_m p_m$ its corresponding equilibrium price, and similarly for the vector $b \in \mathbb{R}^M$ with price P^b.

Definition 14 (Price distortion)**.** *There is an equilibrium* price distortion*, if there exist assets, a and b, such that* $P^b \neq 0$*, and* $\frac{P^a}{P^b}$ *is distorted, as defined in Section 3.2. We denote this by an* (a, b) *price distortion.*

The asset b has the natural interpretation of being the numeraire, i.e., the price distortion is defined in terms of the price of asset a measured in number of units of asset b. When $b = R \in \mathbb{R}^M$, this numeraire is in terms of $t = 0$ consumption, since $P^R = 1$. The (gross) risk-free interest rate is $\frac{1}{\sum_m p_m}$.

Price distortions occur generically, as long as the economic environment is rich enough to contain many states and agents with sufficiently different beliefs, as shown in the following proposition:

Proposition 11. *Consider an economy in which there are more states than agents,* $M > N$*, and define the matrix* $A \in \mathbb{R}_{++}^{M \times N}$*, with elements* $A_{mn} = (q_m^n)^{1/\gamma}$*. Assume that the rank of* A *is* N*. Then, for all Pareto weights,* $\lambda \in S^N$*, except possibly for a subset of Lebesgue measure zero, there exist equilibrium price distortions.*

The proof of Proposition 11 proceeds by constructing an asset that all agents agree upon the correct price of, and then showing that the equilibrium price of this asset is different from that price, except possibly in some knife-edge cases. That asset's price is thus typically distorted in equilibrium. The result is general, but does not describe which assets will have distorted prices. The following result and its corollary provides further guidance.

Proposition 12. *In the savings economy with elasticity of intertemporal substitution* $\psi \neq 1$*, any*

Arrow Debreu security, m, about which agents agree in that q_m^n is the same for all n, is mispriced in equilibrium. Specifically,

(i) If $\psi > 1$, then the mth Arrow Debrey security is underpriced, $p_m < p_m^*$,

(ii) If $\psi < 1$, then the mth Arrow Debreu security is overpriced, $p_m > p_m^*$,

where p_m^* is the unique undistorted equilibrium price.

The intuition behind this result is that price distortions are difficult to identify in states that agents disagree about, since such states have a wide range of reasonable prices. On the other hand, for a state m, about which there is no disagreement, all agents also agree on the correct price and any deviation from this agreed-upon price is then identified as a distortion. In the savings economy, aggregate investments are distorted and hence the relative marginal utility between date zero and one is also distorted. Specifically, when the EIS is less than one, the investment consumption ratio is lower than in the no disagreement economy and therefore the relative marginal utility is higher, which also implies a too high price. Vice versa, when the EIS is greater than one, Arrow Debreu prices of no-disagreement states are too low.

A lesson to be drawn is that it may be easiest to identify mispricing by studying the prices of assets about which there is the *least* disagreement. Moreover, the excess return depends on the EIS. Specifically, when the EIS is less than one, low disagreement assets have lower return than in an equivalent economy without disagreement. Note that the effect on the no-disagreement assets arises because of the investment distortions, not because of the risk free rate which is the same with or without disagreement. Rather, the price distortion is due to a lower risk premium.

For simplicity, we have stated the result for the savings economy. Generalizations are straightforward, as discussed in the proof of Proposition 12. Specifically, we show there that if there is an idiosyncratic event B (i.e., an event that does not affect aggregate output) with at least three states and agents agree about the probability of state $m \in B$, and if there is disagreement about some other states in B then $p_m/(\sum_{b\in B} p_b)$ is distorted.

The fact that mispricing is easiest found in idiosyncratic assets about which there is low disagreement provides a potential link between disagreement and idiosyncratic mispricing. The idiosyncratic volatility puzzle states that assets with high idiosyncratic volatility underperform in the market. To illustrate how such idiosyncratic mispricing may arise under disagreement, consider the following "idiosyncratic asset economy" with six states, $\Omega = \{L, H\} \times \{1, 2, 3\} = \{L1, L2, L3, H1, H2, H3\}$, where $\{L, H\}$ represents states with different aggregate outcomes, and $\{1, 2, 3\}$ represents idiosyncratic states. The payout of the production technology is R_L in the low states $\{L1, L2, L3\}$ and R_H in the high states $\{H1, H2, H3\}$, where, $R_L < R_H$. There are two agents with equal initial endowment, who agree on the probability, q_L, for the low states and consequently on the probability for the high states, $q_H = 1 - q_L$. They also agree that the conditional probabilities for state 2 is 1/3: $q_{L2}/q_L = q_{H2}/q_H = 1/3$. They disagree, however, about the probabilities for states 1 and 3. Agent n believes the probability for states $L1$, $L3$, $H1$, and $H3$ are $q_L z^n$, $q_L(2/3 - z^n)$, $q_H z^n$, and $q_H(2/3 - z^n)$, respectively.

The traded assets in this economy are $a_w = (1 - w)A_s + wA_i$, where $w \in [0, 1]$, A_s represents a systematic asset that pays R_L in state L and R_H in state H, and A_i represents an idiosyncratic asset with payoff 1 in state 2 and zero otherwise. Asset a_0 is therefore purely systematic, whereas a_1 is purely idiosyncratic. Moreover, the amount of idiosyncratic volatility is increasing in w.

We now have:

Proposition 13. *In the idiosyncratic asset economy with disagreement, $z^1 \neq z^2$, and $EIS \neq 1$:*

- *The systematic asset, a_0 is not mispriced.*
- *All assets with idiosyncratic volatility, a_w, $w > 0$, are mispriced. The degree of mispricing increases in idiosyncratic volatility, w.*
  - *The assets are overpriced if the EIS is less than one, $\psi < 1$,*
  - *The assets are underpriced if the EIS, is greater than one, $\psi > 1$.*

We conclude this section by comparing our production economy results to those in an exchange economy in an example.

Example 3. The setup in this case is similar to the equilibrium production economy, with the exception that agents cannot invest. Instead, they receive endowments e_0^n at time 0, and e_{1m}^n at time 1, $n = 1, \dots, N$, $m = 1, \dots, M$, and the total endowment is $e_0 = \sum_{n=1}^{N} e_0^n$, $e_1 = \sum_{n=1}^{N} e_{1m}^n$, $m = 1, \dots, M$. Note that, just as in the savings economy, there is no aggregate uncertainty, i.e., e_0 and e_1 are known.[19] Given a state price vector, p, the optimization problem of agent n is then

$$\max_{c_{11}^n, \dots, c_{1M}^n} \quad u(c_0^n) + \sum_{m=1}^{M} u\left(c_{1m}^n\right) q_m^n, \qquad \text{s.t.} \tag{18}$$
$$c_0^n + \sum_{m=0}^{M} p_m c_{1m}^n = e_0^n + \sum_{m=0}^{M} p_m e_{1m}^n.$$

Similar arguments as those leading to Proposition 6 implies the existence of a unique equilibrium state-price vector. We then have

Proposition 14. *In the savings economy with disagreement, the risk-free interest rate is never distorted. In the exchange economy with disagreement, the risk-free interest rate is always distorted.*

The reason for the difference is that in the savings economy, agents adjust their investments to smooth out perceived gains from speculation across time. In the exchange economy, such smoothing is not possible. Instead, prices have to adjust to keep aggregate consumption equal to total endowments (see Ehling et al. (2018a)). Note that per definition savings distortions do not arise in the exchange economy, whereas they are always present in the savings economy, except for in the knife-edge case of logarithmic utility.

Altogether, the example highlights how the exchange economy and production economy may behave quite differently under disagreement.

4 Concluding remarks

Our results in this paper show the challenges of defining efficiency in production economies with disagreement, but also that these challenges can be overcome under some natural technical conditions that are satisfied in several standard work-horse models. Disagreement in the production economy feeds back to the real economy, leading to investment distortions and thereby relating heterogeneous

[19] Just as the savings economy can be generalized, the exchange economy can easily be generalized to allow for aggregate uncertainty.

beliefs to the well-documented undersaving puzzle. It may also lead to agents holding underdiversified portfolios (underdiversification puzzle), and to overpricing of assets with high idiosyncratic volatility (idiosyncratic volatility puzzle). In general, under heterogeneous beliefs, there are assets that are objectively mispriced in equilibrium.

Our approach does not rely on objective knowledge about "true" probabilities by a policy maker. This is of practical importance. A potential application may arise during a period of disagreement about whether novel technologies have created a bubble in a market—as in our previously mentioned example when during the late 1990s some argued that there was a tech bubble, whereas others argued that the high valuations of Internet startups and other companies were well justified. Our approach suggests the possibility for a policy maker to take action during such periods, for example, by affecting investors' relative trade-off between investing and consuming. Importantly, such actions could be justified without taking a stand on the future prospects of the novel technologies.

A Further examples

A.1 Example in Section 2, allowing for transfers

When allowing for transfers in the example, $\mathcal{A} = \{b_1^{\tau_1,\tau_2}, b_2^{\tau}, b_3^{\tau}\}$ for $\tau_1, \tau_2, \tau \in [0,1]$, where the allocations are defined as in Table 2. It follows that $a_1 = b_1^{0.5,0.5}$, $a_2 = b_2^{0.5}$, $a_3 = b_3^{0.5}$, and $a_4 = b_1^{0.95,0.05}$.

Allocation,	$b_1^{\tau_1,\tau_2}$		b_2^{τ}		b_3^{τ}	
Agent	1	2	1	2	1	2
State 1	$2(1-\tau_1)$	$2\tau_1$	$18(1-\tau)$	18τ	0	0
State 2	$2(1-\tau_2)$	$2\tau_2$	0	0	$18(1-\tau)$	18τ

Table 2: **Economy with three mutually exclusive technologies, two agents and two states with transfers.**

Let us first focus on E_q for a specific $q \in \mathcal{Q}_R$. For $b_1^{\tau_1,\tau_2}$ with $\tau_1 \neq \tau_2$, there is speculation since individual consumption is state dependent even though aggregate output is not. No such speculative allocation can be in E_q for any q, i.e., $b_1^{\tau_1,\tau_2} \notin \cup_{q \in \mathcal{Q}_R} E_q = WBNE$. The same argument as without transfers implies that for any q, and τ, $b_1^{\tau,\tau}$ is dominated by either b_2^{τ} or b_3^{τ}, which in turn are undominated. So $WBNE = \{b_2^{\tau}, b_3^{\tau} : \tau \in [0,1]\}$. It also follows immediately that $b_2^{\tau} \notin E_{(q_1,1-q_1)}$ for large q_1, and $b_3^{\tau} \notin E_{(q_1,1-q_1)}$ for small q_1, so $BNE = \cap_{q \in \mathcal{Q}_R} E_q = \emptyset$.

For IK-efficiency, among the $b_1^{\tau_1,\tau_2}$ allocations, for a given λ, the allocation $b_1^{\tau,\tau}$ where τ is chosen such that $(1-\lambda)u'(2(1-\tau)) = \lambda u'(2\tau)$, i.e., such that $\frac{1-\tau}{\tau} = \left(\frac{1-\lambda}{\lambda}\right)^2$, dominates all other allocations, regardless of q. Now, any $b_1^{\tau,\tau}$ will dominate any of b_2^{τ} and b_3^{τ} for specific $q \in \mathcal{Q}_R$, and therefore $IKE = \{b_1^{\tau,\tau}, b_2^{\tau}, b_3^{\tau} : \tau \in [0,1]\}$.

For U-efficiency, we know that $IKE \subset UE$, but there are also other $b_1^{\tau_1,\tau_2}$, allocations with $\tau_1 \neq \tau_2$ that are U-efficient. From (15) it follows that for $b_1^{\tau_1,\tau_2}$ to be U-efficient, it is sufficient for any candidate of a dominant allocation, c, to find a q and a Pareto weight such that $b_1^{\tau_1,\tau_2} \geq_q^{\lambda} c$. It is easy to see that b_2^{τ} candidates will always be dominated for some $q \in \mathcal{Q}_R$ and $\lambda \in S^N$, as will b_3^{τ} candidates.

For candidate allocations, $c = b_1^{\tau_1,\tau_2}$, for reasonable belief sets $\mathcal{Q}_R = \{(q, 1-q) : p \leq q \leq 1-p\}$ where $p < 0.5$ (in our example, $p = 0.1$), the set (τ_1, τ_2) such that $b_1^{\tau_1,\tau_2} \in UE$,= includes all allocations such that $\tau_1 = \tau_2$, but excludes some extreme allocations such that one of τ_1 and τ_2 is close to 0 and the other is close to 1, as shown in Figure 4, for $p = 0.1, 0.25, 0.4$. The figure also shows the allocation $a_4 = b_1^{0.95,0.05}$, which when $p = 0.1$ (i.e., when $\mathcal{Q}_R = \{(q, 1-q) : 0.1 \leq q \leq 0.9\}$) consequently is U-efficient.

Thus, the efficiency properties of a_1, a_2, a_3, and a_4 with respect to IKE, $WBNE$, BNE, and UE are identical as in the original example without transfers.

A.2 Allowing for idiosyncratic endowment shocks

We introduce the additional assumption that agents are exposed to idiosyncratic endowment shocks. Each agent faces a shock, $\tilde{e}^n$, at $t = 1$, where e_m^n is the size of agent n's shock in state m. These

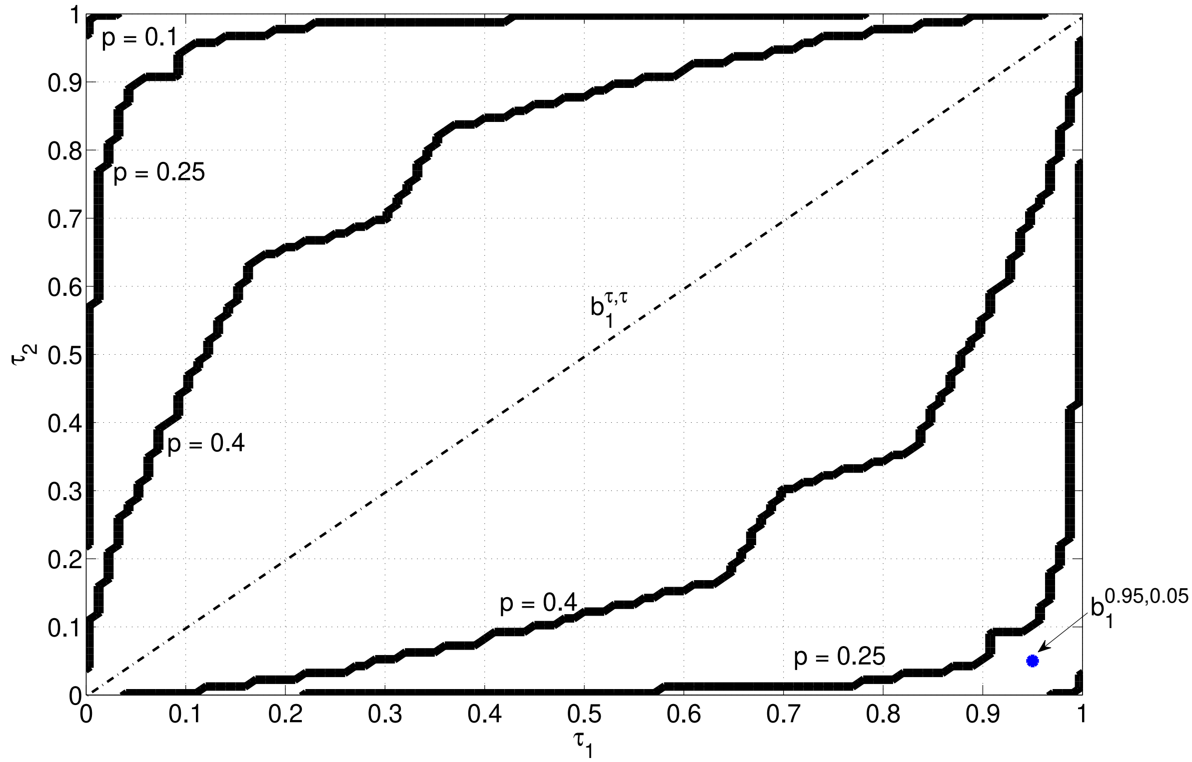

Figure 4: **Region of τ_1, τ_2, such that allocation $b_1^{\tau_1,\tau_2}$ is U-efficient, for different choices if $\mathcal{Q}_R = \{(q, 1-q) : p \leq q \leq 1-p\}$, $p = 0.1, 0.25, 0.4$.**

shocks are idiosyncratic, i.e., they are such that

$$\sum_{n=1}^{N} e_m^n = 0, \qquad m = 1, \ldots, M.$$

The endowment shocks are summarized in the vector

$$e = (e_0^1, e_0^2, \ldots, e_0^N, e_{11}^1, \ldots, e_{11}^N, e_{12}^1, \ldots, e_{1M}^N) \in \mathbb{R}^{N\times(M+1)}.$$

In the complete market equilibrium, these shocks affect agents' wealths. We need to ensure that each agents' total wealth, including the value of endowment shocks, is strictly positive for equilibrium to be well-defined. We do this by the following extension: Given the equilibrium state price vector in an economy with no endowment shocks and initial endowment K, this state price vector also determines an equilibrium in any economy with initial endowments K', and endowment shocks e, such that

$$K^n = (K')^n + \sum_m e_m^n p_m, \qquad n = 1, \dots, N,$$

in which each agent, n, solves the optimization problem

$$\begin{aligned} \max_{d_1^n,\dots,d_M^n} \quad & U(c_0^n, \tilde{c}_1^n | q^n), \qquad \text{s.t.}, \\ c_0^n &= (K')^n - \sum_{i=1}^M d_m^n p_m, \\ c_{1m}^n &= d_m^n + e_m^n, \end{aligned} \tag{19}$$

Our analysis covers this extended set of economies with idiosyncratic shocks, summarized by the tuple $\mathcal{E} = (\gamma, \pi, K, \tilde{R}, e)$. A sufficient condition for agent wealth to be strictly positive in the economy with endowment shocks is that $K^n R_m + e_m^n > 0$ for all n and m.

A.3 An economy with non-convex reasonable belief set, in which $IKE \subsetneq WBNE$

The example in Section 2 explored an economy in which the utility possibility set was not convex, and as a consequence belief-netural inefficiency was distinct from IK-inefficiency. Similarly, the measures may differ when the set of reasonable beliefs is not convex, as illustrated by the following example.

Consider a one-date economy with two agents and three mutually exclusive production technologies that depend on the outcome of three tosses of a coin. If the outcome of the tosses is three tails, production technology a_2 delivers one unit of utility to each agent, otherwise 0. If the outcome is three heads, production technology a_3 delivers one unit of utility to each agent, otherwise 0. If the outcome is neither 3 heads, nor 3 tails, production technology a_4 delivers one unit of utility to each agent, otherwise 0. The two agents agree that the three tosses are independent and identically distributed, but not on the probability, p, for heads in each toss, believing it is p^1 and p^2, respectively, where we assume that $p^1 < p^2$. The economy allows for objective randomization, and the example is also robust to allowing for transfers.

The above probability structure may be viewed as a stylized model for a process where the outcome depends on a multiplicative chain reaction. For instance, the three technologies could for instance represent different types of seeds that a farmer can choose between. Let the states represent rain or sun. Seeds of type 2 (corresponding to a_2 above) require three days of sun, seeds of type 3 (corresponding to a_3) require three days of rain and seeds of type 4 require at least one day of rain and one day of sun.[20]

Since agents agree on the i.i.d. nature of the coin tosses but disagree on p, it is natural for the

[20] Multiplicative processes also arise naturally in biology (in epidemiology, for example, the reproductive ratio represents the number of individuals infected by a single individual, which determines whether a virus spreads) and physics (in nuclear physics, for example, the radioactive decay rate of an atom's nucleus determines the success of fission).

planner to include all probability vectors for the states

$$\{\text{All heads}, \text{All tails}, \text{Both heads and tails}\}$$

on the form

$$(q_1, q_2, q_3) = (p^3, (1-p)^3, 1 - p^3 - (1-p)^3),$$

for $p \in [p^1, p^2]$. Note that this corresponds to a view that either agent can be correct in his or her belief about the probability, and that probabilities in-between the agents' individual beliefs are also reasonable. The corresponding set $\mathcal{Q}_R$ is obviously not the convex hull of the two agents' beliefs, which corresponds to

$$(q_1, q_2, q_3) \in CH\left(\hat{\mathcal{Q}}\right),$$

with $\hat{\mathcal{Q}} = \left\{((p^1)^3, (1-p^1)^3, 1-(p^1)^3-(1-p^1)^3), ((p^2)^3, (1-p^2)^3, 1-(p^2)^3-(1-p^2)^3)\right\}$. In fact, it is easy to check that $\mathcal{Q}_R$ is not even convex.

The convex hull corresponds to the planner taking the view that there is some probability that agent 1 is correct about p, and that otherwise agent 2 is correct, but neglecting the possibility that some p in-between their beliefs for heads may actually be correct. The convex hull represents such mixtures of the two probability vectors.[21]

The utilities associated with the three technologies are shown in Figure 5 (blue lines), as a function of p. In addition to the three risky technologies there is a risk-free technology a_1 that delivers a utility of 0.45 to all agents (the black straight line). We also show the utility of a randomization with equal probabilities for the three risky production technologies (as represented by the dotted red straight line).

Similarly to the example Section 2, we argue that a_1 should be viewed as efficient. Indeed, it is easy to check that a_1 is IK-efficient, since it is above any other technology (including all randomizations) for some p. Indeed $IKE = \{a_1, a_2, a_3, a_4\}$. However, a_1 is belief-netural inefficient, $WBNE = \{a_2, a_3, a_4\}$, since is below some other technology for each p. This is consistent with Proposition 3(iii), since condition C2 is not satisfied in this economy.

A.4 Economies in which $WBNE \subsetneq IKE$

We provide two examples of economies in which there are allocations that are weakly belief-netural efficient but not IK-efficient. The first example is a homogeneous beliefs economy with one state and date, that does not allow for transfers. Thus part (i) of Proposition 1 holds, but not part (ii). There are three allocations and two agents. The first allocation, a_1, provides the utility of 3 to agent 1 and 0

[21] The problem described here of choosing an appropriate set, $\mathcal{Q}_R$, is more generally related to that of defining appropriate sets in dynamic multiple priors models, as, e.g., analyzed in Epstein and Schneider (2003). It is also related to the literature on Bayesian updating. Specifically, Kyburg and Pittarelli (1992) discuss some problems with convex Bayesianism where a convex set of probability functions replaces the single probability function used by strict Bayesians. They argue that in many instances, using a convex set of reasonable beliefs is not always desirable. For instance, if it is known that two random variables are independent, then the set of probabilities that maintain the independence between the two random variables is in general non-convex. This issue is surveyed and further explored in Cozman (2012).

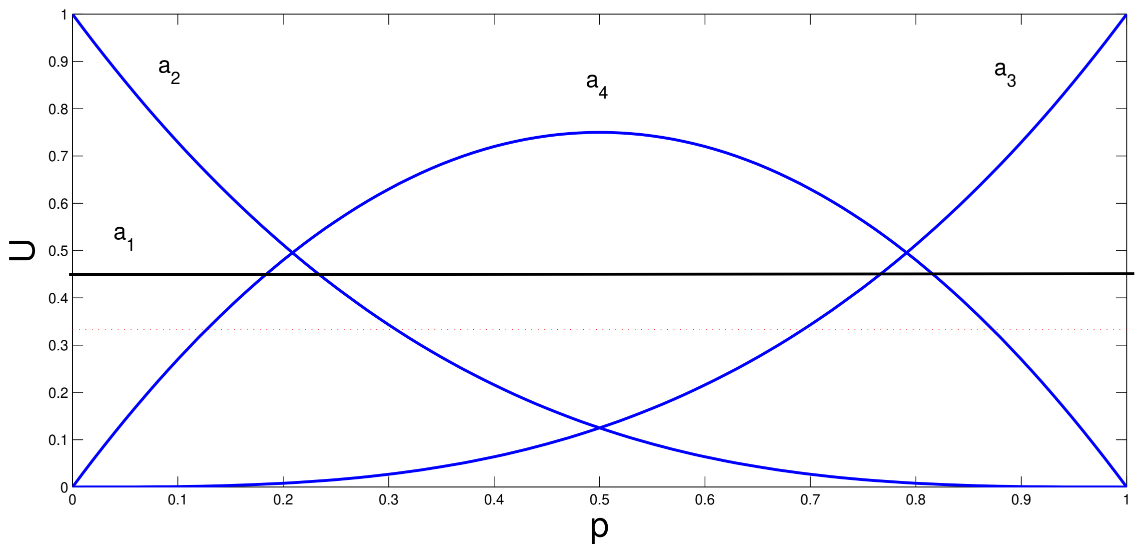

Figure 5: **Utilities associated with production technologies as a function of probability for heads, p. The (blue) curves show utilities for the risky technologies, $a_2 - a_4$, whereas the (black) straight line represents utilities for the risk free technology, a_1. The dotted (red) straight line shows utilities from a randomization of the three risky technologies.**

to agent 2, the second allocation, a_2, 0 to agent 1, and 3 to agent 2, whereas the third allocation, a_3, provides the utility of 1 to both agents. There is thus no uncertainty in this example. The utility possibility set is then $\mathcal{U} = \{(3,0),(0,3),(1,1)\}$.

The different production technologies could, for example, represent different prospective locations of a new airport. Allocation a_3 could represent a location that is very remote from both agents so travel time is long, whereas a_1 and a_2 represent locations that are too close to one of the agents and therefore give raise to noise pollution, but that are at an optimal distance of the other agent.

All three allocations are obviously Pareto efficient and consequently

$$WBNE = \{a_1, a_2, a_3\}.$$

However, for Pareto weights $(\lambda^1, 1-\lambda^1)$, with $\lambda^1 > \frac{1}{3}$, it follows that $a_1 >_q^\lambda a_3$, and for Pareto weights $(\lambda^1, 1-\lambda^1)$, with $\lambda^1 < \frac{2}{3}$, it follows that $a_2 >_q^\lambda a_3$. It is therefore possible to dominate a_3 regardless of Pareto weights, and a_3 is therefore IK-inefficient, $IKE = \{a_1, a_2\}$. The issue is a manifestation of the well-known fact that without transfers, Bergson optimality will not identify all Pareto efficient allocations (see Chapter 8.4 in Kreps 2013). Specifically, without transfers, limited inferences can be drawn about the efficiency of an allocation from the cardinal properties of the expected utility specification, since marginal utilities may not line up across states. If transfers are allowed, the issue disappears, because only the aggregate amount of the consumption good, $(a_i)_{1,1,1} + (a_i)_{1,2,1}$, matters for the ranking of the production technologies, ruling out a_3, and possibly one of a_1 and a_2, from being efficient. Thus, the IK-efficiency concept suffers from the same weaknesses as the Bergson welfare measure in economies without transfers.

Note that if objective randomization is allowed, the two efficiency measures are equivalent in this example. For instance, a policy that randomizes between a_1 and a_2 with equal probability dominates a_3 both with respect to Pareto efficiency and IK-efficiency. Thus, by letting a coin flip determine the position of the airport, a_3 is ruled out as being IK-inefficient in the example, leading to $IKE = WBNE = \{a_1, a_2\}$.

Another example for which there is an allocation that is weakly belief-neutral efficient, but not IK-efficient is given by considering only allocations a_2 and a_3 in the example in Section 2, with reasonable belief set, $\mathcal{Q}_R = \{(q_1, 1-q_1) : q_1 \in [0.2, 0.5]\}$. Both allocations are efficient in the homogeneous beliefs economy with $q_1 = 0.5$, and we therefore have $WBNE = \{a_2, a_3\}$. However, a_3 is IK-inefficient, since it is dominated by a_2 for all q in $\mathcal{Q}_R$ except when $q = (0.5, 0.5)$, and moreover does not dominate a_2 for any q in $\mathcal{Q}_R$, so $IKE = \{a_2\}$. We argue that excluding a_3 from the set of efficient allocations in this example is indeed appropriate under incomplete knowledge. However, since $WBNE = \cup_{q\in\mathcal{Q}} E_q$, there is no possibility to exclude an allocation that belongs E_q for some $q \in \mathcal{Q}_R$ because it is inferior for some (all) other $q' \in \mathcal{Q}_R$.

A.5 Investment distortions with disagreement about aggregate uncertainty

Consider the economy in Section 3. Let there be three agents, $\gamma = 2$, and corresponding EIS, $\psi = \frac{1}{2}$. There are four states in which investment returns are $R_1 = 1$, $R_2 = 1$, $R_3 = 1.4$, and $R_4 = 1.4$, respectively, and no endowment shocks. Note that returns are the same in state 3 and 4, representing purely idiosyncratic risk. Table 3 shows the agents' beliefs about the probabilities for the states. Figure 6 shows equilibrium investments (horizontal axis) and speculative component (vertical axis)

Agent	1	2	3
State 1	0.1	0.25	0.4
State 2	0.1	0.25	0.4
State 3	0.4	0.25	0.01
State 4	0.4	0.25	0.19

Table 3: **The table shows the beliefs of agent $n = 1, 2, 3$ for the probability of each of states $m = 1, 2, 3, 4$ to occur.**

for all equilibrium outcomes, for different initial endowments, $K \in S^N$, where we define the speculative component as in Definition 15 below.

Definition 15. *The speculative component of the complete market equilibrium outcome is*

$$s = \frac{1}{N(M+1)} \sum_{n=1}^{N} \left(\left| \frac{c_0^n}{C_0} - \eta^n \right| + \sum_{m=1}^{M} \left| \frac{c_{1m}^n}{C_{1m}} - \eta^n \right| \right),$$

where $\eta^n = \frac{1}{M+1}\left(\frac{c_0^n}{C_0} + \sum_{m=1}^{M} \frac{c_1^n}{C_{1M}}\right)$ *is agent n's mean consumption share across states and time.*

Several observations are in order: The speculative component approaches zero at three points: $I = 0.4694$, $I = 0.4847$ and $I = 0.4984$, corresponding to initial endowments where all capital is given to agent 1, 2, and 3, respectively. Since we assume strictly positive initial capital for each agent, these limit points are not part of the set of equilibrium outcomes.

The set of reasonable investments is $F_I^U = [0.4694, 0.4984]$, is represented by the dotted (red) line on the horizontal axis. This line also represents the set of IK-efficient allocations. Any value of I outside of this interval is distorted. There is a nonempty set of equilibria with underinvestment ($I < 0.4694$), but no equilibrium with overinvestment. This is in line with Proposition 8, since the EIS, ψ, is less than one. Moreover, there is also a nonempty set of equilibria for which investments are not distorted, which is also in line with Proposition 8, since agents also disagree on aggregate investment opportunities in this example.

We note that there are no belief-neutral efficient allocations, again suggesting that the concept is too strong in this setting. Specifically, any investment level is inefficient for some $q \in \mathcal{Q}_R$, causing the efficiency set to be empty. Moreover, U-efficiency is too weak to rule out any equilibrium allocations—the whole equilibrium region is U-efficient.

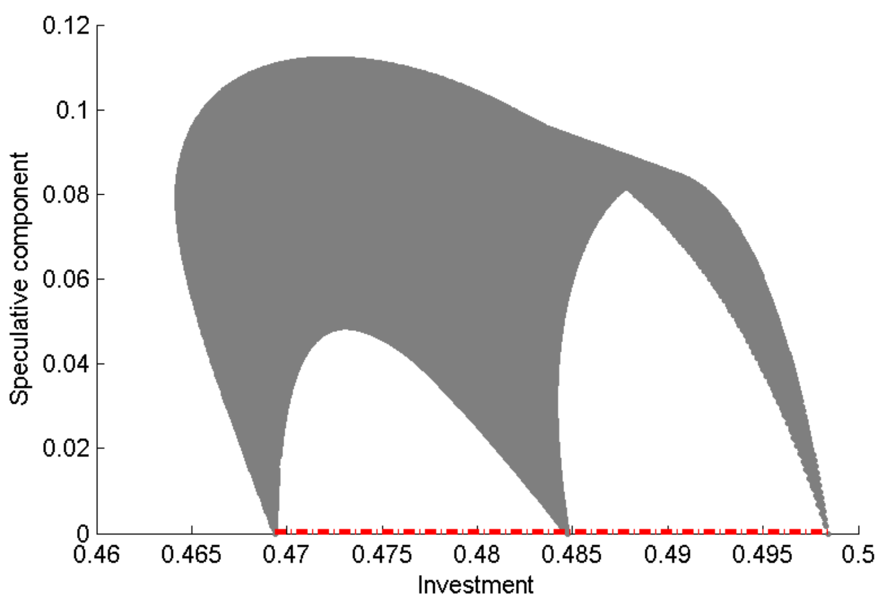

Figure 6: **Equilibrium distortions. The figure shows the possible investment (horizontal) and speculative (vertical) equilibrium outcomes, for all different initial endowments, $K \in S^N$ in the example described in Table 3.**

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Internet Appendix

"Welfare in Production Economies under Heterogeneous Beliefs."

This internet appendix includes all the proofs for the paper "Welfare in Production Economies under Heterogeneous Beliefs" by Christian Heyerdahl-Larsen and Johan Walden.

Proof of Proposition 1:

(i): If a is IK-efficient, from (8) it follows that with homogeneous beliefs $\exists\lambda \in S^N, \forall b \in \mathcal{A}, a \geq_q^\lambda b$, which of course is stronger than (10), since λ is allowed to depend on b in (10), so $IKE \subset WBNE$. Also, $E_q = BNE = WBNE$ trivially follows from (12,13). Finally, (9) is obviously equivalent to (14) when $\mathcal{Q}_R = \{q\}$, so $BNE = UE$.

(ii): From (i), it is sufficient to show that $E_q \subset IKE$, i.e., that $a \notin IKE \Rightarrow a \notin E_q$. From (7), it follows that if a is IK-inefficient, $\forall\lambda \in S^N, \exists b \in \mathcal{A} : b >_q^\lambda a$. However, such an a can clearly not be the solution to the planner's problem for any $\lambda \in S^N$ (recall that $\lambda \in \partial S^N$ can be excluded in economies that allow for transfers), and therefore cannot be in E_q. ∎

Proof of Proposition 2:

(i): First note that UE is non-empty. This follows from the fact that $WBNE \subset UE$, with $WBNE = \cup_{q\in\mathcal{Q}_R} E_q$, and that E_q is non-empty. Assume that $a \in E_{\mathbf{q}}^A$. Then either $a \in UE$ and the result follows, or $a \notin UE$. If a is not in UE, then since UE is non-empty there exists a $b \in UE$ such that b dominates a. It follows that $b >_q^\lambda a$ for all $q \in \mathcal{Q}_R$ and $\lambda \in S^N$, which implies that $U^n(b|q) \geq U^n(a|q)$ for all n and $q \in \mathcal{Q}_R$, with the inequality being strict for at least one n (for each q). Consequently $U(b|\mathbf{q}, \lambda) \geq U(a|\mathbf{q}, \lambda)$, so $b \in E_{\mathbf{q}}^A$.

(ii): Following similar arguments as above, assume that $a \in E_{\mathbf{q}}^A$, with associated λ in (5), but that $a \notin UE$. Then there is an allocation $b \in \mathcal{A}$ such that $b >_q^\lambda a$ for all $q \in \mathcal{Q}_R$ and $\lambda \in S^N$, which implies that $U^n(b|q) \geq U^n(a|q)$ for all n and $q \in \mathcal{Q}_R$, with the inequality being strict for at least one n (for each q). If there is an agent n for which $U^n(b|q^n) > U^n(a|q^n)$, then it immediately follows that $a \notin E_{\mathbf{q}}^A$ which leads to a contradiction. If there is no agent for which $U^n(b|q^n) > U^n(a|q^n)$, i.e., $U^n(b|q^n) = U^n(a|q^n)$ for all n, then we can consider the allocation $b' = \frac{1}{2}b + \frac{1}{2}a$. It follows that $\mathcal{P}(b') = \mathcal{P}(b) = \mathcal{P}(a)$, and as transfers are allowed $b' \in \mathcal{A}$. By strict concavity of the utility function, $U^n(b'|q^n) > \frac{1}{2}U^n(b|q^n) + \frac{1}{2}U^n(a|q^n) = U^n(a|q^n)$ which implies that $a \notin E_{\mathbf{q}}^A$, again leading to a contradiction.

(iii): Without loss of generality, we assume that agents 1 and 2 disagree about the relatively likelihood of states 1 and 2 such that $\frac{q_1^1}{q_2^1} > \frac{q_1^2}{q_2^2}$. Since redistribution is possible, the social planner's first order conditions imply that

$$\frac{q_1^1 u_{1,t}^{1'}(a_{1,1,t})}{q_2^1 u_{2,t}^{1'}(a_{2,1,t})} = \frac{q_1^2 u_{1,t}^{2'}(a_{1,2,t})}{q_2^2 u_{2,t}^{2'}(a_{2,2,t})},$$

(see (21) in the proof of Proposition 4). However, for any $q \in \mathcal{Q}_R$, the planner's first order conditions are

$$\frac{u^{1'}_{1,t}(a_{1,1,t})}{u^{1'}_{2,t}(a_{2,1,t})} = \frac{u^{2'}_{1,t}(a_{1,2,t})}{u^{2'}_{2,t}(a_{2,2,t})},$$

so $a \notin E_q$ for any such q, and thus $a \notin WBNE$. Now, an identical argument as will be used in Proposition 5 (ii), shows that if such improving redistributions are possible without changing the aggregate production, i.e., keeping X the same, then $a \notin IKE$.

∎

Proof of Proposition 3:

(i): Since the economy allows for transfers, and since $u^n_{m,t}$ are differentiable on $[0, \infty)$ and thus $(u^n_{t,m})'(0)$ is finite for all n, m, and t, for small enough strictly positive λ^n it will not be optimal to allocate the good to agent n in any state. It follows that $E_q = \cup_{\lambda \in S^N} E_{q,\lambda}$ in this case, where $E_{q,\lambda} = \arg\max_{a \in \mathcal{A}} U(a|q,\lambda)$, i.e., that Pareto weights on ∂S^N are not needed in determining efficiency. Thus, from (12) it follows that $WBNE = \cup_{q \in \mathcal{Q}_R} \cup_{\lambda \in S^N} E_{q,\lambda}$. Strict dominance is therefore equivalent to $|E_{q,\lambda}| = 1$ for all $q \in \mathcal{Q}_R$, and $\lambda \in S^N$.

Now, for $a \in WBNE$, and associated $q \in \mathcal{Q}_R$, $\lambda \in S^N$, assume that $E_{q,\lambda}$ contains another allocation $b \neq a$, and therefore that $U(a|q,\lambda) = U(b|q,\lambda)$. Since the production set is convex, $c = \frac{1}{2}a + \frac{1}{2}b \in \mathcal{A}$. Moreover, from (3) and the strict concavity of agents' utilities, it follows that $U(\cdot|q,\lambda)$ is strictly concave over allocations, and thus

$$U(c|q,\lambda) > \frac{1}{2}U(a|q,\lambda) + \frac{1}{2}U(b|q,\lambda) = U(a|q,\lambda),$$

contradicting the assumption that $a \in E_{q,\lambda}$. So, no such $b \neq a \in E_{q,\lambda}$ exists, $|E_{q,\lambda}| = 1$, and the result thus follows.

(ii): The planner's optimization problem, given q and λ is to maximize

$$\max_{X \in \mathcal{A}_X} \max_{a|X} \sum_n \lambda^n \sum_t \sum_m u^n_{m,t}(a_{m,n,t}) q_m = \max_{V \in \mathcal{U}} q^T V \lambda. \tag{20}$$

Note that from the first welfare theorem it follows that if we define $E_{q,\lambda} = \{a_{q,\lambda}\}$ as the solutions to the planner's problem given $q \in \mathcal{Q}_R$ and $\lambda \in \mathcal{S}^N$, and $E_\lambda = \cup_{q \in \mathcal{Q}_R} E_{q,\lambda}$, then $E_Q = \cup_{\lambda \in \mathcal{S}^N} E_\lambda$, since transfers are allowed.

The F.O.C., given X is that

$$\lambda^n (u^n_{m,t})'(a_{m,n,t}) = \rho^{m,t} \tag{21}$$

across all m, and t, for all agents for which $a_{m,n,t} > 0$, and

$$\lambda^n (u^n_{m,t})'(0) \leq \rho^{mt} \tag{22}$$

for all agents such that $a_{m,n,t} = 0$, where $\rho^{m,t} > 0$ are the Lagrange multipliers that make total consumption equal to total production in each state.

Let us assume that $a \notin E_Q$. Then, it could either be (i) that (21,22) are not satisfied across states for any $\lambda \in S^N$, or (ii) that an optimal X is not chosen, or both (i) and (ii).

If $a \notin E_Q$ because of (i), then regardless of $\lambda \in S^N$, there is some state and time for which (3,4) are not satisfied for two agents, at least one of which has a strictly positive consumption, which we w.l.o.g. assume are agents 1 and 2. But this then means that a redistribution of consumption between the two agents will improve welfare, regardless of $q \in \mathcal{Q}_R$. Indeed, it is possible to strictly increase

$$\lambda^1 u^1_{m,t}(a_{m,1,t})q_m + \lambda^2 u^2_{m,t}(a_{m,2,t})q_m \tag{23}$$

regardless of $q_m > 0$, since the FOC is not satisfied in the state t, m. But this then implies that for all $q \in \mathcal{Q}_R$, (6) in the paper is satisfied with strict inequality, so the allocation is IK-inefficient.

If (i) is satisfied for some nonempty set $\Lambda \subset S^N$, but $a \notin E_Q$ because of (ii), then for any $\lambda' \notin \Lambda$, a redistribution such that (21,22) holds, with the same X, leads to a strict improvement regardless of $q \in \mathcal{Q}_R$, along the same lines as the previous point when (i) failed. Therefore, (7) holds for any such $\lambda' \notin \Lambda$. It remains to be shown that (7) also holds for $\lambda \in \Lambda$ for which (21,22) are satisfied, given that X is not optimal for any $q \in \mathcal{Q}_R$.

For such a λ, define the mapping $\mathcal{F}_\lambda : \mathcal{U} \to \mathbb{R}^M$, by $\mathcal{F}_\lambda(V) = V\lambda$, and the set $F_\lambda = \mathcal{F}_\lambda(\mathcal{U})$, which is a closed, convex, and bounded subset of $\mathbb{R}^M$, because of Q2. Now, defining $f_a = \mathcal{F}_\lambda(\mathcal{R}(a))$, since $a \notin E_Q$ it follows from (9) that

$$\max_{q \in \mathcal{Q}_R} \min_{f \in F_\lambda} q^T(f_a - f) = s < 0,$$

where $s < 0$ follows from the fact that the optimum is realized for some f^*, q^* (since both $\mathcal{Q}_R$ and F_λ are compact).

Sion's minmax theorem then implies that

$$\min_{f \in F_\lambda} \max_{q \in \mathcal{Q}_R} q^T(f_a - f) = s,$$

where the same, f^*, q^* can be chosen for the maxmin and minmax problems, and thus that for all $q \in \mathcal{Q}_R$,

$$q^T(f_a - f^*) \le s < 0.$$

This, in turn, implies that (7) holds for any such $\lambda \in \Lambda$.

The allocation a is therefore IK-inefficient, so indeed $a \notin E_Q \Rightarrow a \notin IKE$.

(iii): Consider an $a \in WNBE$, i.e., $a \in E_Q$, with associated $q \in \mathcal{Q}_R$, $\lambda \in S^N$ (∂S^N excluded since transfers are allowed). Then, $\forall b \neq a$, $a >^\lambda_q b$, immediately implying (8) for λ and q, not

even depending on b, so $a \in IKE$.

∎

Proof of Proposition 4:

Assume that $a \in BNE$, then $a \in E_q$ and $a \in E_{q'}$ for $q \in \mathcal{Q}_R$, $q' \in \mathcal{Q}_R$, $q \neq q'$, and with associated Pareto weights $\lambda \in S^N$, $\lambda' \in S^N$ (see proof of Proposition 3(ii)), and with associated $X \in \mathcal{A}_X$. Given that (21,22) do not depend on q, the allocation to individual agents can be written $W(X, \lambda)$, and therefore, $W(X, \lambda) = W(X, \lambda')$ (otherwise a would not be in both $E_{q,\lambda}$ and $E_{q',\lambda'}$). Moreover, (21,22) imply that λ and λ' are basically unique, down to possible differences on weights for agents that are not allocated the good in any state. Indeed, there is a $\hat{\lambda} \in S^N$ that puts extremely low weight on any such agents, such that $\lambda_i/\lambda_j = \lambda'_i/\lambda'_j = \hat{\lambda}_i/\hat{\lambda}_j$ for all agents i, j who receive strictly positive allocation in any state, and such that a is in both $E_{q,\hat{\lambda}}$ and $E_{q',\hat{\lambda}}$.

But this then means that $f_a = \mathcal{F}_{\hat{\lambda}}(\mathcal{R}(a))$, as defined in the proof of Proposition 3(ii), is a point in the convex set $F_{\hat{\lambda}}$ that has both the hyperplanes defined by q and q' as supporting hyperplanes. It follows (see, e.g., Gallier (2011)) that f_a must be a non-smooth point on $\partial F_{\hat{\lambda}}$ (as defined on the whole of $\mathbb{R}^M$). Indeed, $F_{\hat{\lambda}}$ does not lie in any proper affine subspace of $\mathbb{R}^M$ (because $\mathcal{A}_X$ has nonempty interior), and must therefore be a nonsmooth point to have multiple supporting hyperplanes (see Definition 3.3.3 in Gallier (2011), page 108). Finally, since $\mathcal{F}_{\hat{\lambda}} : \mathbb{R}^{M\times N} \to \mathbb{R}^M$, $\mathcal{V} : \mathbb{R}^{T\times M\times N} \to \mathbb{R}^{M\times N}$ and $\mathcal{P} : \mathbb{R}^{T\times M\times N} \to \mathbb{R}^{M\times N}$ are all smooth mappings, it follows that $\mathcal{P}(a)$ is also nonsmooth in $\mathcal{A}_X$.

∎

Proof of Proposition 5:

(i) The planner's problem in the exchange economy is simplified since only one X is feasible. The F.O.C., w.r.t. λ, (21,22), are q independent, and the result therefore follows immediately.

(ii) Since $\mathcal{Q}_R = \{q\}$, it follows immediately that $E_Q = \cap_q E_q = \cup_q E_q$, so $BNE = E_Q = WBNE$. Now, Q3 is satisfied in the exchange economy, because efficient allocations are characterized by (21,22), and X is fixed, which implies that whether an allocation is considered efficient does not dependent of q. Thus for all $a \in E_Q$,

$$\exists \lambda \in S^N, \forall b \in \mathcal{A}, \forall q \in S^N : a >_q^\lambda b. \tag{24}$$

It is easy to show that if transfers are allowed, then the definition of Q3 can equivalently be stated using $\lambda \in S^N$, implying that (24) is stronger than Q3 (the argument is identical as in the proof of Proposition 3(i): for any $\lambda \in \partial S^N$, there is a close enough $\lambda \in S^N$ that leads to an identical allocation, that can be chosen instead). It therefore follows that $WBNE \subset IKE$ from Proposition 3(iii).

For $a \notin WBNE$, we follow the same argument as in the proof of Proposition 3(ii). Given that $\mathcal{A}_X = \{X\}$ the reason why $a \notin WBNE$ must be that equality of Pareto weighted marginal utilities across agents and states fails. An identical argument as in the proof of Proposition 3(ii) (which in that part does not depend on convexity of either $\mathcal{Q}_R$ or $\mathcal{U}$) implies that a is IK-inefficient, so $BNI \subset IKI$, i.e., $IKE \subset WBNE$.

∎

Proof of Proposition 6: *Existence*:

We follow Basak 2005 (exchange economy) and Baker et al. 2014 (production economy) to use the corresponding planner problem

$$\begin{aligned} U = \max_{I,c^1,\ldots,c^N} & \quad \sum_{n=1}^{N} \lambda^n U(c_0^n, \tilde{c}_1^n | q^n), \qquad \text{s.t.,} \\ \sum_{n=1}^{N} c_0^n &= K - I, \\ \sum_{n=1}^{N} c_{1m}^n &= R_m I, m = 1, \ldots, M, \end{aligned} \tag{25}$$

to generate equilibria. From the FOC we have

$$\begin{aligned} \lambda^n \left(c_0^n\right)^{-\gamma} &= \lambda^{n'} \left(c_0^{n'}\right)^{-\gamma}, \\ \lambda^n q_m^n \left(c_{1m}^n\right)^{-\gamma} &= \lambda^n q_m^{n'} \left(c_{1m}^{n'}\right)^{-\gamma}. \end{aligned}$$

Using the market clearing conditions we can solve for the optimal consumption share of agent n

$$\begin{aligned} c_0^n &= \frac{\left(\lambda^n\right)^{\frac{1}{\gamma}}}{\sum_{n'=1}^{N} \left(\lambda^{n'}\right)^{\frac{1}{\gamma}}} \left(K - I\right), \\ c_{1m}^n &= \frac{\left(\lambda^n\right)^{\frac{1}{\gamma}} \left(q_m^n\right)^{1/\gamma}}{\sum_{n'=1}^{N} \left(\lambda^{n'}\right)^{\frac{1}{\gamma}} \left(q_m^{n'}\right)^{1/\gamma}} R_m I. \end{aligned}$$

Rather than working with the Pareto weights, λ^n it is convenient to define, $\hat{\lambda}^n = \left(\lambda^n\right)^{\frac{1}{\gamma}}$. In addition, define $a_{mn} = \left(q_m^n\right)^{\frac{1}{\gamma}}$. We can then write the consumption shares as

$$c_0^n = \frac{\hat{\lambda}^n}{\sum_{n'}^{N} \hat{\lambda}^{n'}} \left(K - I\right), \tag{26}$$

$$c_{1m}^n = \frac{\hat{\lambda}^n a_{mn}}{\sum_{n'=1}^{N} \hat{\lambda}_m^{n'} a_{mn'}} R_m I. \tag{27}$$

Let $z_m = R_m^{1-\gamma}$, $T_m = \sum_{n'} \hat{\lambda}^{n'} a_{mn'}$ and $\alpha(\hat{\lambda}) = \sum_{m=1}^{M} z_m T_m^{\gamma}$. Plugging the optimal consumption share into the the social planner's problem, we get

$$U = \max_I \left(\frac{\left(\sum_{n=1}^{N} \hat{\lambda}^n\right)^{\gamma}}{1-\gamma} (1-I)^{1-\gamma} + \alpha(\hat{\lambda}) \frac{1}{1-\gamma} I^{1-\gamma} \right).$$

The first-order condition is

$$-\left(\sum_{n=1}^{N} \hat{\lambda}^n\right)^{\gamma} (1-I)^{-\gamma} + \alpha\left(\hat{\lambda}\right) I^{-\gamma} = 0.$$

Solving for the optimal I gives

$$I = \frac{Z}{\sum_{n=1}^{N} \hat{\lambda}^n + Z},$$

where Z is the investment to consumption ratio given by

$$Z = \frac{\alpha\left(\hat{\lambda}\right)^{\frac{1}{\gamma}}}{\sum_{n'=1}^{N} \hat{\lambda}^{n'}}. \tag{28}$$

We calculate the mapping from the planner's weights, $\hat{\lambda}$ to initial endowments, K. To this end, note that the budget condition for agent $n = 1, \ldots, N$ is given by

$$K^n = c_0^n + E\left[\tilde{\xi}^n \tilde{c}_1^n | q^n\right], \tag{29}$$

where $\xi^n = \left(\frac{c_m^n}{c_0^n}\right)^{-\gamma}$ is the stochastic discount factor. Using the optimal consumption of agent

n we can calculate the expression in Equation (29) as

$$\begin{aligned}
K^n &= c_0^n + E\left[\tilde{\xi}^n \tilde{c}_1^n | q^n\right] \\
&= \frac{\hat{\lambda}^n}{\sum_{n'}^N \hat{\lambda}^{n'}}(1 - I(\hat{\lambda})) + \sum_{m=1}^M q_m^n \left(\frac{\frac{\hat{\lambda}^n a_{mn}}{\sum_{n'=1}^N \hat{\lambda}_m^{n'} a_{mn'}} R_m I(\hat{\lambda})}{\frac{\hat{\lambda}^n}{\sum_{n'}^N \hat{\lambda}^{n'}}\left(1 - I(\hat{\lambda})\right)} \right)^{-\gamma} \frac{\hat{\lambda}^n a_{mn}}{\sum_{n'=1}^N \hat{\lambda}_m^{n'} a_{mn'}} R_m I(\hat{\lambda}) \\
&= \frac{\hat{\lambda}^n}{\sum_{n'}^N \hat{\lambda}^{n'}}(1 - I(\hat{\lambda})) + I(\hat{\lambda}) \frac{\hat{\lambda}^n}{\alpha\left(\hat{\lambda}\right)} \sum_{m=1}^M z_m a_{mn} T_m^{\gamma - 1} \\
&= \frac{\hat{\lambda}^n}{\sum_{n'} \hat{\lambda}^{n'}}(1 - I(\hat{\lambda}) + I(\hat{\lambda}) F^n(\hat{\lambda})) \\
&= Y^n(\hat{\lambda}),
\end{aligned} \tag{30}$$

where we have defined $F^n(\hat{\lambda}) = \frac{L}{\alpha} G^n(\hat{\lambda})$, $G^n(\hat{\lambda}) = \sum_m z_m a_{mn} T_m^{\gamma-1}$ and $L = \sum_{n'} \hat{\lambda}^{n'}$. We note that the function $Y(\hat{\lambda}) = (Y^1(\hat{\lambda}), \ldots, Y^N(\hat{\lambda}))'$ is defined on the whole of $\bar{S}^N$.

Because of the functional mapping $K = Y(\hat{\lambda})$, it is easy to show that the mapping $K = Y(\hat{\lambda})$ is surjective, that is, that for every K, there exists a $\hat{\lambda}$ such that $K = Y(\hat{\lambda})$. Note that $Y(\hat{\lambda}) = \frac{\hat{\lambda}^n}{\sum_{n'} \hat{\lambda}^{n'}}(1 - I(\hat{\lambda}) + I(\hat{\lambda}) F^n(\hat{\lambda}))$ is homogeneous of degree zero. Hence, we can without loss of generality focus on $\hat{\lambda}$ on the unit simplex, $\bar{S}^N$, which is a compact and convex set. It follow immediately that Y is continuous on $\bar{S}^N$, being a product of continuous functions on this domain.

Define the function $f : \bar{S}^N \to \bar{S}^N$, by $f(\hat{\lambda}) = Y(\hat{\lambda}) - K + \hat{\lambda}$. Here, K is treated as a constant parameter. That $f(\hat{\lambda})$ maps the unit simplex into itself follows from the fact that $\sum_n Y^n(\hat{\lambda}) = 1$, and $\sum_n K^n = 1$. Since f is continuous, and $\bar{S}^N$ is compact and convex, Brouwer's fixed theorem point now immediately implies that f has a fixed point, $\hat{\lambda}^*$, and it therefore follows that $Y(\hat{\lambda}^*) = K$.

Thus, Y is a surjective mapping from Pareto weights to capital on the unit simplex, associated with the resulting equilibrium state prices (A.5):

$$p_m = E\left[\tilde{\xi}^n \delta_m | q^n\right] = \left(\frac{c_m^n}{c_0^n}\right)^{-\gamma} q_m^n, \qquad m = 1, \ldots, M,$$

where δ_m is the random variable, such that $\delta_m(\omega_w) = 1$, and $\delta_m(\omega_{m'}) = 0$, $m' \neq m$, representing the payout of the m:th AD security, and any $n = 1, \ldots, N$ can be chosen. We have shown the first part of the theorem.

Uniqueness: We first study the case $\gamma \geq 1$. Define

$$\begin{aligned} G^{nj} &= \sum_m z_m a_{mn} a_{mj} T_m^{\gamma-2}, \\ F^{nj} &= \frac{L^2}{\alpha} G^{nj}. \end{aligned}$$

The mapping from Pareto weights to capital, $K = Y(\hat{\lambda})$, is given in Equation (30). The partial derivatives are

$$\begin{aligned} \frac{\partial I}{\partial \hat{\lambda}^j} &= \frac{1}{\sum_{n'} \hat{\lambda}^{n'}} I(1-I)(F^j - 1), \\ \frac{\partial F}{\partial \hat{\lambda}^j} &= \frac{1}{\sum_{n'} \hat{\lambda}^{n'}} \left(F^n + (\gamma - 1) F^{nj} - \gamma F^n F^j \right). \end{aligned}$$

We define the matrix Q, with elements

$$[Q]_{nj} = -(1-I)^2 - I(1-I)(F^n + F^j) - (\gamma - 1) I F^{nj} - (\gamma + I - 1) I F^n F^j, \qquad 1 \leq n, j \leq N,$$

and then get

$$\frac{\partial Y^n}{\partial \hat{\lambda}^j} = \frac{\hat{\lambda}^n}{L^2} Q_{nj}, \qquad j \neq n.$$

Moreover, we get

$$\frac{\partial Y^n}{\partial \hat{\lambda}^n} = \frac{1}{L}(1 - I + I F^n) + \frac{\hat{\lambda}^n}{L^2} Q_{nn}.$$

We next do a change of coordinate transformation to y, where $y_n = \log(\hat{\lambda}^n)$, i.e., $\hat{\lambda}^n = e^{y_n}$, i.e., we write

$$K^n = \hat{Y}(y) = \frac{e^{y_n}}{L}(1 - I(e^y) + I(e^y) F^n(e^y)). \tag{31}$$

It follows from the chain rule that partial derivatives w.r.t. y will be similar as w.r.t. $\hat{\lambda}$, but with extra e^{y_j} inner derivative terms:

$$\begin{aligned}\frac{\partial \hat{Y}^n}{\partial y_j} &= \frac{e^{y_n} e^{y_j}}{L^2} Q_{nj}, \qquad j \neq n, \\ \frac{\partial \hat{Y}^n}{\partial y_n} &= \frac{e^{y_n}}{L}(1 - I + IF^n) + \frac{e^{y_n} e^{y_n}}{L^2} Q_{nn}.\end{aligned}$$

Now we rewrite this on matrix form, defining the matrix $H(e^y)$, with elements $[H]_{nj} = \frac{\partial \hat{Y}^n}{\partial y_j}$, to get:

$$H(\hat{\lambda}) = \frac{1}{L^2}\Lambda_{\hat{\lambda}} Q \Lambda_{\hat{\lambda}} + \frac{1}{L}\Lambda_{\hat{\lambda}}^{1/2}\Lambda_w \Lambda_{\hat{\lambda}}^{1/2}. \tag{32}$$

Here, for a general vector, x, we use the notation $\Lambda_v = diag(x)$, and we define the vector

$$w = (1 - I)\mathbf{1} + IF,$$

treating F as a vector. Moreover,

$$\begin{aligned}Q &= -(1-I)^2\mathbf{1}\mathbf{1}' - I(1-I)(F\mathbf{1}' + \mathbf{1}F') + (\gamma - 1)IR - (\gamma + I - 1)IFF' \\ &= -\left((1-I)^2\mathbf{1}\mathbf{1}' + I(1-I)(F\mathbf{1}' + \mathbf{1}F') + I^2F'\right) + (\gamma - 1)I(R - FF') \\ &= -ww' + (\gamma - 1)I(R - FF'),\end{aligned} \tag{33}$$

so we have

$$H(\hat{\lambda}) = \underbrace{\frac{1}{L}\Lambda_{\hat{\lambda}}^{1/2}\Lambda_w\Lambda_{\hat{\lambda}}^{1/2} - \frac{1}{L^2}\Lambda_{\hat{\lambda}}(ww')\Lambda_{\hat{\lambda}}}_{C} + (\gamma - 1)I\frac{1}{L^2}\Lambda_{\hat{\lambda}}(R - FF')\Lambda_{\hat{\lambda}}. \tag{34}$$

Here, the matrix R is the matrix with elements $[R]_{nj} = F^{nj}$, and we note that by defining the matrix $a \in \mathbb{R}_+^{M \times N}$ with elements a_{mn}, it can be written on self adjoint matrix form as $R = \frac{L^2}{\alpha} a' \Lambda_v a$, where $v_m = T_m^{\gamma - 2} z_m$. Thus, R is positive semidefinite. Note that homogeneity of Y implies that

$$H(\hat{\lambda})\mathbf{1} = \mathbf{0}. \tag{35}$$

We have $F = \frac{L}{\alpha}a'w$, where $w_m = T^{\gamma-1}z_m$, and thus $FF' = \frac{L^2}{\alpha^2}a'ww'a$, leading to

$$\begin{aligned} R - FF' &= \frac{L^2}{\alpha}a'\left(\Lambda_v - \frac{1}{\alpha}ww'\right)a \\ &= \frac{L^2}{\alpha}a'\Lambda_v^{1/2}\left(E - \frac{1}{\alpha}\Lambda_v^{-1/2}ww'\Lambda_v^{-1/2}\right)\Lambda_v^{1/2}a, \end{aligned}$$

Note that $\Lambda_v^{-1/2}w = (T_1^{\gamma/2}z_1^{1/2}, \ldots, T_M^{1/2}z_m^{1/2})$. Also, recall that the eigenvalues of the 1-rank perturbation of the identity matrix $E - xx'$ are 1 with multiplicity $N-1$, and $1-(x'x)$ with multiplicity 1, where x is the vector with eigenvalue $1-(x'x)$. Now, $(\Lambda_v^{-1/2}w)'(\Lambda_v^{-1/2}w) = \sum_m (T_m^{\gamma/2}z_m^{1/2})^2 = \alpha$, and thus the eigenvalues of $E - \frac{1}{\alpha}\Lambda_v^{-1/2}ww'\Lambda_v^{-1/2}$ are 1 with multiplicity $N-1$, and 0 with multiplicity 1. So, Sylvester's law of inertia (on general rectangular form, see Higham and Cheng (1998)), implies that $R - FF'$ is also positive semidefinite, with number of 0 eigenvalues dependent on N, M, and the rank of a.

It is easy to verify that $R\Lambda_{\hat{\lambda}}\mathbf{1} = R\hat{\lambda} = LF$, which in turn implies that the generic zero eigenvalue is generated by $\mathbf{1}$, since $FF'\Lambda_{\hat{\lambda}}\mathbf{1} = FF'\hat{\lambda} = FL$, so

$$(R - FF')\Lambda_{\hat{\lambda}}\mathbf{1} = \mathbf{0}.$$

Next, consider the remaining term

$$\begin{aligned} C &= \frac{1}{L}\Lambda_{\hat{\lambda}}^{1/2}\Lambda_w\Lambda_{\hat{\lambda}}^{1/2} - \frac{1}{L^2}\Lambda_{\hat{\lambda}}ww'\Lambda_{\hat{\lambda}} \\ &= \frac{1}{\sqrt{L}}\Lambda_{\hat{\lambda}}^{1/2}\Lambda_w^{1/2}\left(E - gg'\right)\Lambda_w^{1/2}\Lambda_{\hat{\lambda}}^{1/2}\frac{1}{\sqrt{L}} \end{aligned}$$

where

$$g_n = \sqrt{\frac{\hat{\lambda}_n w_n}{L}}.$$

It immediately follows that $g'g = \frac{1}{L}\sum_n \hat{\lambda}_n((1-I)+IF_n) = 1$, and therefore a similar argument as for $R-FF'$ implies that there is one eigenvalue of C which is 0, and all the other eigenvalues are 1. It is also easy to check that $\mathbf{1}$ is the eigenvector that corresponds to the eigenvalue 0.

Thus, altogether, since $q'(A+B)q = q'Aq + q'Bq$ for general matrices A and B, it follows that H is positive semidefinite, with exactly one zero eigenvalue and the corresponding eigenvector $\mathbf{1}$, for all $\hat{\lambda}$.

Now, assume that $Y(\hat{\lambda}_1) = Y(\hat{\lambda}_2)$ for distinct $\hat{\lambda}_1$ and $\hat{\lambda}_2$ (i.e., such that it is not the case that $\hat{\lambda}_2$ is proportional to $\hat{\lambda}_1$). Define $y_1 = \log(\hat{\lambda}_1)$ and $y_2 = \log(\hat{\lambda}_2)$. It then follows that

$\hat{Y}(y_2) - \hat{Y}(y_1) = \mathbf{0}$, and thus that $(y_2 - y_1)'(\hat{Y}(y_2) - \hat{Y}(y_1)) = 0$. Define $\Delta y = y_2 - y_1$, to get

$$\hat{Y}(y_2) - \hat{Y}(y_1) = \int_{s=0}^{s=1} H(e^{y_1 + s\Delta y})(\Delta y) ds = 0,$$

and therefore

$$\int_{s=0}^{s=1} (\Delta y)' H(e^{y_1 + s\Delta y})(\Delta y) ds = 0.$$

However, since H is positive semidefinite for all y, with eigenvector $\mathbf{1}$, this is only possible if $y_2 - y_1 = c\mathbf{1}$, for some constant c, i.e., only in the proportional case, $\frac{\hat{\lambda}_2^n}{\hat{\lambda}_n^1} = e^c$ for all n.

For $\gamma < 1$, it follows immediately from the definition of Q that as long as $\gamma - 1 + I > 0$, $\partial Y^n / \partial \hat{\lambda}^j < 0$, which means that the Gross Substitution property holds, and thereby that equilibrium is unique. Since $I > 0$, there is thus always a $\underline{\gamma} < 1$, such that this property holds for all $\gamma > \underline{\gamma}$. This completes the second part of the Proposition.

∎

Proof of Proposition 7: In an economy with homogeneous beliefs q, it follows from the first welfare theorem that any equilibrium allocation, $a \in E_q$, and by Proposition 1, the allocation is therefore IK-efficient.

In the heterogeneous beliefs economy, $q_m^n > q_m^{n'}$ and $q_{m'}^n < q_{m'}^{n'}$ for some n, n', m, m', and the equilibrium condition:

$$\frac{1}{\upsilon^n} c_{1m}^n (q_m^n)^{1/\gamma} = \frac{1}{\upsilon^{n'}} c_{1m}^{n'} (q_m^{n'})^{1/\gamma}, \qquad m = 1, \ldots, M$$

implies that consumption shares are not constant across states. Therefore, the allocation does not belong to E_q for any q. Again using Proposition 6, it follows that the equilibrium allocation is IK-inefficient.

∎

Proof of Proposition 8: We consider the slightly more general case when there is aggregate uncertainty, but agents agree on aggregate uncertainty. They still disagree about idiosyncratic risk. Define $\bar{\lambda}^n = \frac{\hat{\lambda}^n}{\sum_{n'=1}^N \hat{\lambda}^{n'}}$. The equilibrium investment-to-consumption ratio, Z, is given by

$$Z = \left(\sum_{m=1}^M R_m^{1-\gamma} \left(\sum_{n=1}^N \bar{\lambda}^n \left(q_m^n \right)^{\frac{1}{\gamma}} \right)^{\gamma} \right)^{\frac{1}{\gamma}}. \tag{36}$$

By Jensen's inequality we have that

$$\begin{aligned} Z &> \left(\sum_{m=1}^{M} R_m^{1-\gamma}\left(\sum_{n=1}^{N}\bar{\lambda}^n\left(q_m^n\right)\right)\right)^{\frac{1}{\gamma}}, \quad \gamma<1 \\ Z &= \left(\sum_{m=1}^{M} R_m^{1-\gamma}\left(\sum_{n=1}^{N}\bar{\lambda}^n\left(q_m^n\right)\right)\right)^{\frac{1}{\gamma}}, \quad \gamma=1 \\ Z &< \left(\sum_{m=1}^{M} R_m^{1-\gamma}\left(\sum_{n=1}^{N}\bar{\lambda}^n\left(q_m^n\right)\right)\right)^{\frac{1}{\gamma}}, \quad \gamma>1. \end{aligned}$$

As agents agree on aggregate risk we have that $\left(\sum_{m=1}^{M} R_m^{1-\gamma}\left(\sum_{n=1}^{N}\bar{\lambda}^n\left(q_m^n\right)\right)\right)^{\frac{1}{\gamma}} = \left(E[\tilde{R}^{1-\gamma}|q^n]\right)^{\frac{1}{\gamma}}$ for all $n=1,..,N$. Hence, the only case in which the aggregate investment is equal to the unanimously reasonable investments is when agents have log utility.

∎

Proof of Proposition 9: To prove part *(i)* that distortions are increasing in disagreement note that we have the following:

$$\begin{aligned} Z &= \left(\sum_{m=1}^{M}\left(\sum_{n=1}^{N}\bar{\lambda}_n\left(q_m^n\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}} \\ &= \left(M\alpha\left(\Delta\right)^{\gamma}\right)^{\frac{1}{\gamma}}, \end{aligned} \tag{37}$$

where

$$\alpha\left(\Delta\right)=\frac{1}{N}\left(\frac{\Delta}{M}\right)^{\frac{1}{\gamma}}+\frac{N-1}{N}\left(\frac{1-J\frac{\Delta}{M}}{M-J}\right)^{\frac{1}{\gamma}}. \tag{38}$$

Differentiating $\alpha\left(\Delta\right)$ we get

$$\alpha'\left(\Delta\right)=\frac{1}{\gamma NM}\left(\left(\frac{\Delta}{M}\right)^{\frac{1}{\gamma}-1}-\left(\frac{1-J\frac{\Delta}{M}}{M-J}\right)^{\frac{1}{\gamma}-1}\right), \tag{39}$$

and therefore $\alpha'\left(\Delta\right)<0$ for $\gamma>1$ and $\alpha'\left(\Delta\right)>0$ for $\gamma<1$ for $\Delta\in(1,N)$. As the derivative of Z has the same sign as that of α it follows that distortions are increasing in disagreement, Δ.

Next to prove *(ii)*, let the probability of the optimists be q_1 and the pessimists be q_2 with $q_1>q_2$. Due to symmetry we have that the optimal investment-consumption ratio, Z_j, when

j markets per agent are open

$$Z_j = \left(jN\alpha^\gamma + (M - jN)\,\bar{q}\right)^{\frac{1}{\gamma}}, \tag{40}$$

where $\alpha = \frac{1}{N}q_1^{\frac{1}{\gamma}} + \frac{N-1}{N}q_2^{\frac{1}{\gamma}}$ and $\bar{q} = \frac{1}{N}q_1 + \frac{N-1}{N}q_2$. By Jensen's inequality we have that $\alpha^\gamma \leq \bar{q}$ when $\gamma > 1$ and $\alpha^\gamma \geq \bar{q}$ when $\gamma < 1$. Hence, $Z_{j+1} < Z_j$ when $\gamma > 1$ and $Z_{j+1} > Z_j$ when $\gamma < 1$ for $j = \{0, \ldots, J-1\}$. Hence, the distortions are increasing in the number of open markets, j, as long as $\gamma \neq 1$.

Finally, to prove *(iii)* note that the consumption share of agent n at time zero is given by the normalized Pareto weight $\bar{\lambda}_n$. Hence, rather than working with the initial consumption share, we can work with $\bar{\lambda}$. We have that $\bar{\lambda}^1 = 1 - \delta\frac{N-1}{N}$ and $\bar{\lambda}^n = \frac{\delta}{N}$ for $n = 2, \ldots, N$. We have the following

$$Z = \left(J\alpha_1(\delta)^\gamma + (M - J)\,\alpha_2(\delta)^\gamma\right)^{\frac{1}{\gamma}}, \tag{41}$$

with

$$\alpha_1(\delta) = \left(1 - \delta\frac{N-1}{N}\right)q_1^{\frac{1}{\gamma}} + (N-1)\frac{\delta}{N}q_2^{\frac{1}{\gamma}} \tag{42}$$

$$\alpha_2(\delta) = \frac{\delta}{N}q_1^{\frac{1}{\gamma}} + \left(1 - \frac{\delta}{N}\right)q_2^{\frac{1}{\gamma}}. \tag{43}$$

Note that $\alpha_1(\delta) \geq \alpha_2(\delta)$ for $\delta \in [0,1]$. Moreover, $\alpha_1'(\delta) = \frac{N-1}{N}\left(q_2^{\frac{1}{\gamma}} - q_1^{\frac{1}{\gamma}}\right) < 0$ and $\alpha_2'(\delta) = \frac{1}{N}\left(q_1^{\frac{1}{\gamma}} - q_2^{\frac{1}{\gamma}}\right) > 0$. Therefore, $Z'(\delta) = Z(\delta)^{1-\gamma}\left(\alpha_1(\delta)^{\gamma-1} - \alpha_2(\delta)^{\gamma-1}\right)\frac{J(N-1)}{N}\left(q_1^{\frac{1}{\gamma}} - q_2^{\frac{1}{\gamma}}\right)$. Hence, we have

$$\begin{aligned} Z'(\delta) &> 0 \; for \; \gamma < 1 \\ Z'(\delta) &= 0 \; for \; \gamma = 1 \\ Z'(\delta) &< 0 \; for \; \gamma > 1. \end{aligned}$$

We can also study the effect of equality by perturbing the weights in a different way. Specifically, we consider three groups where the consumption share at time zero of a member of group one, two and three are given by $\frac{1+\delta}{N}$, $\frac{1-\delta}{N}$ and $\frac{1}{N}$, respectively. Hence, as we increase δ we are making group one's consumption share bigger at the expense of group two. The third group is unaffected by the perturbation. We assume that there are equally many agents in group one and two while the number of agents in group three can be arbitrary. Let k be the number of agents in group one and two. For instance, if $N = 2$ and $k = 1$, then as δ approaches one, the economy moves to the homogeneous agent economy and clearly there are no distortions. It can be shown that $\frac{\partial Z}{\partial \delta} < 0$ for $\gamma < 1$ and $\frac{\partial Z}{\partial \delta} > 0$ for $\gamma > 1$ for $\delta \in [0,1)$. Moreover, Z is minimized (maximized) for γ greater (less) than one. Hence, as we increase δ the distortions are reduced. The reason for this is that as more weight is put on a subset of agents, the less room there is for speculative trade and consequently, the less severe are the equilibrium savings distortions. Let $K \leq N$, K even and $k = \frac{K}{2}$. We can split the agents into three groups with the following normalized Pareto shares

$$\begin{aligned}
\bar{\lambda}_i &= \frac{1+\delta}{N} \quad for\ i = 1, \ldots, k \\
\bar{\lambda}_i &= \frac{1-\delta}{N} \quad for\ i = k+1, \ldots, K \\
\bar{\lambda}_i &= \frac{1}{N} \quad for\ i = K+1, \ldots, N\ if\ K < N.
\end{aligned}$$

We then have the following

$$Z = \left(Jk\alpha_1(\delta)^{\gamma} + Jk\alpha_2(\delta)^{\gamma} + J(N-K)\alpha_3(\delta)^{\gamma}\right)^{\frac{1}{\gamma}}, \tag{44}$$

where

$$\begin{aligned}
\alpha_1(\delta) &= q_1^{\frac{1}{\gamma}}\left(\frac{1+\delta}{N}\right) + q_2^{\frac{1}{\gamma}}\left(1 - \frac{1+\delta}{N}\right) && (45) \\
\alpha_2(\delta) &= q_1^{\frac{1}{\gamma}}\left(\frac{1-\delta}{N}\right) + q_2^{\frac{1}{\gamma}}\left(1 - \frac{1-\delta}{N}\right) && (46) \\
\alpha_3(\delta) &= q_1^{\frac{1}{\gamma}}\left(\frac{1}{N}\right) + q_2^{\frac{1}{\gamma}}\left(1 - \frac{1}{N}\right). && (47)
\end{aligned}$$

Next, note that $\alpha_1'(\delta) = \frac{q_1^{\frac{1}{\gamma}} - q_2^{\frac{1}{\gamma}}}{N} = -\alpha_2'(\delta)$ and $\alpha_3'(\delta) = 0$. It follows that $sign(Z'(\delta)) = sign\left(\alpha_1(\delta)^{\gamma-1} - \alpha_2(\delta)^{\gamma-1}\right)$. Moreover,

$$\alpha_1(\delta) - \alpha_2(\delta) = 2\left(q_1^{\frac{1}{\gamma}} - q_2^{\frac{1}{\gamma}}\right)\delta > 0\ when\ q_1 > q_2, \tag{48}$$

hence we have

$$\begin{aligned}
Z'(\delta) &> 0\ for\ \gamma > 1 && (49) \\
Z'(\delta) &= 0\ for\ \gamma = 1 && (50) \\
Z'(\delta) &< 0\ for\ \gamma < 1. && (51)
\end{aligned}$$

∎

Proof of Proposition 10: Note that due to symmetry we have that the optimal consumption of every agent at time $t = 0$ is $\frac{1}{N}$, and that the normalized Pareto weights are $\bar{\lambda}^n = \frac{1}{N}$. It follows from the derivations in the proof of Proposition 6 that the optimal consumption of agent n in the states he is relatively optimistic about is $\frac{1}{N}\left(\frac{q_1}{q_2}\right)^{\frac{1}{\gamma}}$, with q_1 and q_2 defined as in the proof of Proposition 9. Hence, with $\Delta = 1$, i.e, no disagreement we have that the optimal consumption is $\frac{1}{N}$ in each state. As there are M states, this implies a fraction $\frac{1}{M}$ in each, i.e. perfect diversification. As $\frac{\partial \frac{1}{N}\left(\frac{q_1}{q_2}\right)^{\frac{1}{\gamma}}}{\partial \Delta} > 0$ for $\Delta \in [1, N)$ the agents move further away from the fully diversified portfolio as disagreement, Δ, increases.

Proof of Proposition 11: Define $z_m = R_m^{-\gamma}$, $m = 1, \ldots, M$. In homogeneous beliefs equilibrium for agent n, we have:

$$\frac{z_m}{z_{m'}} \frac{q_m^n}{q_{m'}^n} = \frac{p_m^n}{p_{m'}^n}, \qquad 1 \le m, m' \le M,\ 1 \le n \le N.$$

Consider portfolios w and v. Agent n's value ratio for these two portfolios is:

$$\frac{\sum_m w_m p_n^m}{\sum_m v_m p_m^n} = \frac{\sum_m w_m z_m q_m^n p_1^n/(z_1 q_1^n)}{\sum_m v_m z_m q_m^n p_1^n/(z_1 q_1^n)} = \frac{\sum_m w_m z_m q_m^n}{\sum_m v_m z_m q_m^n}.$$

If $\sum_m w_m z_m q_m^n = 0$, $n = 1, \ldots, N$, all agents therefore agree that the price of asset w should be 0. In matrix form, defining the matrix Π with $\Pi_{mn} = q_m^n = (A_{mn})^\gamma$, we write this as

$$\Pi \Lambda_z w = \mathbf{0}.$$

Now, since $N < M$, it is clear that a nontrivial such asset with unanimous price 0 exists. For convenience, we define $r = \Lambda_z w$, and we then have $\Pi r = \mathbf{0}$. Note that every agent with beliefs in $\mathcal{Q}_R$ also agrees on the reasonable price being zero, since

$$\begin{aligned} \sum_m w_m z_m (\lambda q_m^{n_1} + (1-\lambda) q_m^{n_2}) &= \lambda \sum_m w_m z_m q_m^{n_1} + (1-\lambda) \sum_m w_m z_m q_m^{n_2} \\ &= \lambda \times 0 + (1-\lambda) \times 0 \\ &= 0. \end{aligned}$$

The equilibrium price of w under Pareto weights $\lambda \in S^n$, in terms of time 0 consumption, is

$$\frac{\sum_m (\sum_n A_{mn} \lambda^n)^\gamma r_m}{\sum_m (\sum_n A_{mn} \lambda^n)^\gamma \tilde{R}^{1-\gamma}},$$

so there is intertemporal mispricing of this asset if and only if

$$Q(\lambda) = \sum_m \left(\sum_n A_{mn} \lambda^n \right)^\gamma r_m \neq 0. \tag{52}$$

We wish to show that under the conditions of the proposition, (52) is typically satisfied except for possibly a small set of λ's. In other words, if we define the set $X = \{\lambda \in S^N : Q(\lambda) = 0\}$, we want to show that the Lebesgue measure of X is zero, $\mu_{S^N}(X) = 0$. Of course, since Q is a continuous function, and $X = Q^{-1}(\{0\})$ is the preimage of a Borel set, X is measurable.

We use the following standard lemma:

Lemma 1. *Consider the function $f(x) = \sum_{m=1}^M (1 + c_m x)^\gamma b_m$, where $\gamma > 0$, $\gamma \neq 1$, $c_m > -1$, $b_m \in \mathbb{R}$, $m = 1, \ldots, M$. If there are at least two distinct c's, $c_m \neq c_{m'}$, the number of roots of $f(x)$ in $x \in [0, 1]$ is finite.*

Proof: The proof follows from the Principle of Permanence, given the facts that f is a real analytic function on the interval $x \in [-\epsilon, 1 + \epsilon]$ for some $\epsilon > 0$, and that f is not a constant function as long as $c_m \neq c_{m'}$. ∎

We use Lemma 1 to show that on any ray between two distinct Pareto weights, $\lambda_a, \lambda_b \in S^N$, $Q(\lambda)$ is nonzero, except possibly at a finite number of points. Indeed, consider the function

$$\begin{aligned}
g(x) &= \sum_m \left(x \sum_n A_{mn} \lambda_a + (1 - x) \sum_n A_{mn} \lambda_b \right)^\gamma r_m \\
&= \sum_m \left(\sum_n A_{mn} \lambda_b^n + x \left(\sum_n A_{mn} \lambda_a^n - \sum_n A_{mn} \lambda_b^n \right) \right)^\gamma r_m \\
&= \sum_m \left(1 + x \frac{\sum_n A_{mn} \lambda_a^n - \sum_n A_{mn} \lambda_b^n}{\sum_n A_{mn} \lambda_b^n} \right)^\gamma \left(\sum_n A_{mn} \lambda_b^n \right)^\gamma r_m \\
&= \sum_m (1 + c_m x)^\gamma b_m.
\end{aligned}$$

For all c_m's to be the same, it must be that $A\lambda_a = kA\lambda_b$, but since the rank of A is N, this implies that $\lambda_a = k\lambda_b$, and given that both λ_a and λ_b are in S^n, that $\lambda_a = \lambda_b$, contradicting our assumption that the two vectors are distinct. Thus, Lemma 1 implies that there is a finite number of roots of (52) on any such line.

That $\mu_{S^N}(X) = 0$ now follows immediately from Fubini's theorem. Specifically, consider the characteristic function on X, χ_X. Any line integral of χ_X is zero, $\int_{\lambda_1} \chi_X(\lambda) d\lambda = 0$, since X only contains a finite number of points along any line. Fubini's theorem then implies that the total Lebesgue integral can be viewed as an $N-2$-dimensional integral over such line integrals, each of which integrates to zero, and therefore that

$$\int_{S^N} \chi_X(\lambda) d\lambda = \int_{\lambda_2,\ldots,\lambda_{N-1}} \left(\int_{\lambda_1} \chi_X(\lambda) d\lambda_1 \right) d\lambda_2 \cdots d\lambda_{N-1} = 0.$$

We are done.

∎

Proof of Proposition 12: The unanimously reasonable relative price $\frac{p_m}{p_B}$, where $p_B = \sum_{m' \in B} p_{m'}$ is $\mathbb{P}(\omega_m|B) = \frac{q_m^n}{\sum_{m' \in B} q_{m'}^n} = \frac{q_m^{n'}}{\sum_{m' \in B} q_{m'}^{n'}}$ for all $n, n' = 1, \ldots, N$. The equilibrium price is

$$\begin{aligned}
\frac{p_m}{p_B} &= \frac{\left(\sum_{n=1}^N \bar{\lambda}^n \left(q_m^n\right)^{\frac{1}{\gamma}} \right)^\gamma}{\sum_{m' \in B} \left(\sum_{n=1}^N \bar{\lambda}^n \left(q_{m'}^n\right)^{\frac{1}{\gamma}} \right)^\gamma} \\
&= \frac{\left(\sum_{n=1}^N \bar{\lambda}^n \left(q_m^n\right)^{\frac{1}{\gamma}} \right)^\gamma}{\sum_{m' \in B} \left(\sum_{n=1}^N \bar{\lambda}^n \left(\frac{q_{m'}^n}{q_m^n}\right)^{\frac{1}{\gamma}} \left(q_m^n\right)^{\frac{1}{\gamma}} \right)^\gamma} \\
&= \frac{1}{\sum_{m' \in B} \left(\sum_{n=1}^N \frac{\bar{\lambda}^n (q_m^n)^{\frac{1}{\gamma}}}{\left(\sum_{n=1}^N \bar{\lambda}^n (q_m^n)^{\frac{1}{\gamma}}\right)^\gamma} \left(\frac{q_{m'}^n}{q_m^n}\right)^{\frac{1}{\gamma}} \right)^\gamma} \\
&= \frac{1}{\sum_{m' \in B} \left(\sum_{n=1}^N W_n \left(\frac{q_{m'}^n}{q_m^n}\right)^{\frac{1}{\gamma}} \right)^\gamma},
\end{aligned}$$

where $\sum_{n=1}^N W_n = 1$. Note that the only reasonable price can be written as $\frac{1}{\sum_{m' \in B} \frac{q_{m'}^n}{q_m^n}}$. Hence, only if $\sum_{m' \in B} \frac{q_{m'}^n}{q_m^n} = \sum_{m' \in B} \left(\sum_{n=1}^N W_n \left(\frac{q_{m'}^n}{q_m^n}\right)^{\frac{1}{\gamma}} \right)^\gamma$ will there be no mispricing. Define $H =$

$\sum_{m' \in B} \left(\sum_{n=1}^{N} W_n \left(\frac{q^n_{m'}}{q^n_m} \right)^{\frac{1}{\gamma}} \right)^{\gamma}$. By Jensen's inequality we have

$$
\begin{aligned}
H &> \sum_{m' \in B} \left(\sum_{n=1}^{N} W_n \left(\frac{q^n_{m'}}{q^n_m} \right) \right) = \sum_{m' \in B} \frac{q^n_{m'}}{q^n_m}, \quad \gamma < 1 \\
H &= \sum_{m' \in B} \left(\sum_{n=1}^{N} W_n \left(\frac{q^n_{m'}}{q^n_m} \right) \right) = \sum_{m' \in B} \frac{q^n_{m'}}{q^n_m}, \quad \gamma = 1 \\
H &< \sum_{m' \in B} \left(\sum_{n=1}^{N} W_n \left(\frac{q^n_{m'}}{q^n_m} \right) \right) = \sum_{m' \in B} \frac{q^n_{m'}}{q^n_m}, \quad \gamma > 1.
\end{aligned}
$$

Hence, as long as $\gamma \neq 1$, which we rule out by assumption, there will be mispricing. ∎

Proof of Proposition 13: First, it is useful to show that there are investment distortions in this economy. Specifically, the investment-consumption ratio, Z, is

$$Z = Z^U B^{\frac{1}{\gamma}}, \tag{53}$$

where

$$Z^U = \left(R_L^{1-\gamma} q_L + R_H^{1-\gamma} q_H \right)^{\frac{1}{\gamma}}, \tag{54}$$

and

$$B = \left(\frac{1}{2} \left(z^1 \right)^{\frac{1}{\gamma}} + \frac{1}{2} \left(z^2 \right)^{\frac{1}{\gamma}} \right)^{\gamma} + 1/3 + \left(\frac{1}{2} \left(2/3 - z^1 \right)^{\frac{1}{\gamma}} + \frac{1}{2} \left(2/3 - z^2 \right)^{\frac{1}{\gamma}} \right)^{\gamma}. \tag{55}$$

Note that $B = 1$ when $z^1 = z^2$, i.e., when they agree. Hence, Z^U is the investment-consumption ratio without disagreement. Moreover, by Jensen's inequality $B > 1$ for $\gamma < 1$ and $B < 1$ for $\gamma > 1$ and therefore there are investment distortions when there is disagreement. Next consider the price of the systematic claim, A_s:

$$A_s = \frac{\left(q_L R_H^{1-\gamma} + q_H R_H^{1-\gamma} \right) B}{\left(Z^U \right)^{\gamma} B} = \frac{\left(q_L R_H^{1-\gamma} + q_H^{1-\gamma} R_H \right)}{\left(Z^U \right)^{\gamma}} = 1. \tag{56}$$

Hence, the A_s is not distorted. This also follow directly as A_s is the value of the aggregate investment technology and this cannot be distorted in our production economy. Note also that the risk free asset is not distorted either as

$$\frac{1}{R_f} = \frac{\left(q_L + q_H \right) B}{\left(Z^U \right)^{\gamma} B} = \frac{1}{\left(Z^U \right)^{\gamma}} \tag{57}$$

where R_f is the gross risk free rate. Next, consider the price of the idiosyncratic asset A_i

$$A_i = q_L 1/3 Z^{-\gamma} R_L^{1-\gamma} + q_H 1/3 Z^{-\gamma} R_H^{1-\gamma} = \frac{A_i^U}{B}, \tag{58}$$

where $A_i^U = 1/3 \left(q_L R_L^{1-\gamma} + q_H R_H^{-\gamma}\right) \left(Z^U\right)^{1-\gamma} = 1/3$ is the price of the idiosyncratic asset without disagreement. Consequently, A_i is too low when $\gamma < 1$ and too high when $\gamma > 1$. It immediately follows that $a_0 = A_s$ is not mispriced and that A_w for $w > 0$ is mispriced and that it increases in the idiosyncratic volatility. The direction of the mispricing follows from the EIS with the asset being overpriced when EIS is less than one and underpriced when EIS is greater than one. Finally, as the risk free rate is not distorted, the equilibrium price distortion is a risk premium effect.

∎

Proof of Proposition 14: In the production economy without aggregate uncertainty we have that $R_m = R$ for all $m = 1, \ldots, M$. Hence, the unanimously reasonable value of the risk free rate is R. In equilibrium, the risk free rate is

$$R_f = \left(\frac{R^{1-\gamma}}{R^{-\gamma}}\right) \left(\frac{\sum_{n=1}^{N} T_m^{\gamma}}{\sum_{n=1}^{N} T_m^{\gamma}}\right) = R.$$

Since the equilibrium risk free rate is always equal to the unanimously reasonable value, the risk free rate is thus not distorted.

Exchange economy: Let aggregate consumption at time zero be C_0 and the aggregate consumption at time 1 be C_1. Then, the unanimously reasonable value of the risk free rate is

$$R_f = E\left[\left(\frac{C_1}{C_0}\right)^{-\gamma} \middle| q\right] = \left(\frac{C_1}{C_0}\right)^{-\gamma},$$

for all $q \in \mathcal{Q}_R$. In equilibrium the risk free rate is

$$\begin{aligned} R_f &= E\left[\left(\frac{\tilde{c}_1^n}{c_0^n}\right)^{-\gamma} \middle| q^n\right] \\ &= \left(\frac{C_1}{C_0}\right)^{-\gamma} \left(\frac{\sum_{m=1}^{M} T_m^{\gamma}}{L^{\gamma}}\right). \end{aligned}$$

Since $\left(\frac{\sum_{m=1}^{M} T_m^{\gamma}}{L^{\gamma}}\right) \neq 1$ for $\gamma \neq 1$ due to Jensen's inequality, the risk free rate is always distorted.
∎