Journal of Financial Economics 128 (2018) 344–362

Contents lists available at ScienceDirect

Journal of Financial Economics

journal homepage: www.elsevier.com/locate/jfec

Interest rate volatility, the yield curve, and the macroeconomy☆

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ARTICLE INFO

Article history:
Revised 30 June 2016
Accepted 26 July 2016
Available online 18 December 2017

JEL classification:
G12
E43
C58

Keywords:
Macro-finance term structure model
Interest rate volatility
No-arbitrage model
Macroeconomy

ABSTRACT

This paper provides theory and evidence that a low-dimensional term structure model can simultaneously price bonds and related options. It shows that a component of volatility risk largely unrelated to the shape of the yield curve is a determinant of expected excess returns for holding long maturity bonds. It also finds evidence for this return relationship both in the model and directly in the data through regression analysis. The paper also identifies a link between corporate earnings performance and interest rate volatility, providing a channel driving interest rate volatility. The structure of risk in the model that gives rise to these features of volatility is distinct from that inherent in recent models with unspanned stochastic volatility.

1. Introduction

This paper examines the joint properties of the risks underlying bond and bond option markets, and the relationship between these risks, the underlying corporate sector, and the macroeconomy. The paper first develops a model that captures the cross-section of bond and bond option prices. The model shows that the volatility of the yield curve—a major macroeconomic factor—is an important predictor of future bond returns and that this volatility is identified through interest rate options. Through the model, we also show that stochastic convexity effects are small and that a volatility factor unrelated to the level, slope, and curvature can create nearly unspanned stochastic volatility through a mechanism where a component of volatility has no effect on risk-neutral expectations. We validate our results (in- and out-of-sample) by demonstrating the ability of the model to match conditional first and second moments in the data, as well as by demonstrating the relationship between volatility and bond risk premium through a model-free regression analysis. To conduct the analysis, the paper also develops a number of Fourier analytic techniques for pricing options and computing exact likelihood functions. These techniques are applicable more broadly both in reduced form and general equilibrium asset pricing models. The paper also shows that times series variation in cross-sectional dispersion of earnings information is an important driver of interest rate volatility. In particular, a major effect on interest rate volatility stems

☆ This paper has greatly benefited from helpful discussions, comments, and suggestions from Caio Almeida, Snehal Banerjee, Mary Barth, Eli Bartov, Darrell Duffie, Jeremy Graveline, Haitao Li, Jun Pan, Kenneth Singleton, Ilya Strebulaev, an anonymous referee, and seminar participants at at Caltech, Chicago, Federal Reserve Board, Georgia Tech, LSE, McGill, MIT, NYU, Penn State, Stanford, University of California at Berkeley, Vanderbilt, Washington University, and WFA. Any remaining errors are our own. The paper has been previously distributed with the title "Pricing and Hedging Volatility Risk in Fixed Income Markets."

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https://doi.org/10.1016/j.jfineco.2017.12.004
0304-405X/© 2017 Published by Elsevier B.V.

from the aggregation of uncertainty about corporate performance.

We study these questions through the lens of a four-factor affine term structure model. Although dynamic models with a small number of risk factors (e.g., two or three) have had considerable success at pricing bonds across a broad spectrum of maturities, they typically generate large errors when pricing options on these bonds.[1] There are two critical features of our model that underlie its relative success in simultaneously pricing bonds and bond options. First, we focus on members of the affine family of term structure models (Duffie and Kan, 1996) that are known to be successful in pricing bonds and allow flexibility in the conditional covariances of the risk factors. In particular, we use an identified version of the affine process specification given in Duffie, Filipovic, and Schachermayer (2003) which allows for a richer covariance structure among risk factors than the commonly used specification of Dai and Singleton (2000).[2] The second feature of our analysis is the dependence of the market price of risk on the state of the macroeconomy. We follow Cheridito, Filipovic, and Kimmel (2007) in parameterizing the market prices of risk, which extends the specification of Dai and Singleton (2002) and Duffee (2002). This extended specification for the market price allows for time-variation in the premium associated with volatility risks, an element that we find critical for matching the data.

Our model also shows that volatility plays an important role in determining risk premiums that investors demand for bearing interest rate risk. A number of studies have shown that the shape of the yield curve is related to expected excess returns for holding long maturity bonds. Our results show that volatility, incrementally to the level, slope, and curvature of the yield curve, is an important determinant of expected excess returns for holding interest rate risk, explaining approximately 40% of the variation in expected returns. This result is consistent with the results of Wright (2011) who argues that inflation uncertainty plays an important role in determining bond risk premiums. This phenomenon offers a potential explanation associated with the ‘conundrum’ period where during 2004–2007, the Federal Reserve raised interest rates in 14 straight Federal Open Market Committee (FOMC) meetings while long maturity yields remained relatively constant. As other researchers have noted (e.g., Rudebusch, Swanson, and Wu, 2006), this pattern could be attributed to declining risk premiums, one cause of which our model would attribute to declining volatility.

In support of our results, we also validate our model out-of-sample. We find similar fit in the out-of-sample period. Additionally, we find that the model is able to match conditional first and second moments both in- and out-of-sample. We also provide complementary evidence on the relationship between implied volatility and bond risk premia through a model-free regression analysis.

Although the model incorporates a component of volatility risk that varies independently of the level, slope, and curvature of the yield curve, the mechanism is very different from a model with unspanned stochastic volatility (USV; see Collin-Dufresne and Goldstein, 2002a). In these models, volatility varies independently of the entire yield curve due to a very specific type of cancellation. In general, volatility drives long maturity interest rates through two channels: (i) a convexity effect and (ii) through an expectations effect whereby changes in the level of volatility affect (risk-neutral) expectations of future short rates. Models with the USV property rely on an exact cancellation of these two effects across maturities. We show that the first channel, in fact, generates very little variation in the yield curve because convexity effects are very small for short maturities while mean reversion of volatility implies that the convexity effect at long maturities is nearly constant. The estimation considers the most general model in order to let the data select the necessary ingredients in the model. We find a component of volatility which, while generating small variations in convexity effects across maturities, also has very little effect on risk-neutral expectations of future short rates. Under these conditions, a component of volatility will have very little to no effect on the shape of the yield curve. As elaborated further in Section 7, such a model turns out to be quite different from a model where volatility affects expectations in such a way to exactly cancel (across all maturities) the convexity effects that it generates.

From a methodological perspective, essential to exploring the issues addressed in this paper is an ability to compute the prices of options (for which closed-form solutions do not exist) and the ability to compute the joint conditional likelihood function of a large cross-section of bond yields and option prices. We develop a Fourier analytic quadrature technique for computing option prices. We also extend this technique to develop a feasible method for full information maximum likelihood estimation of affine diffusions. These results are applicable to a wide array of problems beyond those examined in this paper, both in bond and equity markets, and therefore they are potentially of interest in their own right.

This paper also contributes to research in finance and related fields on firms equity valuation and costs of capital/discount rates (e.g., Modigliani and Miller, 1958; Barth, Konchitchki, and Landsman, 2013). Given that a firms equity discount rate comprises a risk-free interest rate plus a risk premium, by shedding light on the time-series dynamics of interest rates this paper provides input for understanding a key determinant of firms valuations. In addition, this paper extends research on links between accounting performance and the macroeconomy by suggesting that information about firms performance is an important determinant of interest rates, through the effect on interest rate volatility which is shown to be incrementally important in pricing bonds and related options (e.g., Konchitchki, 2011).

The remainder of the paper is organized as follows. Section 2 describes the model and estimation procedures.

[1] Mean-squared relative pricing errors for options on the order of 30% are reported in Buhler, Uhrig-Homburg, Walter, and Weber (1999), Driessen, Klaassen, and Melenberg (2003), and Jagannathan, Kaplin, and Sun (2003). Trolle and Schwartz (2009) propose a model that fits both the term structure of interest rates and the cross-section of options, although their preferred model includes a total of 24 factors (18 of which are locally deterministic).

[2] We use the identification scheme in Joslin (2017), and also see Collin-Dufresne, Goldstein, and Jones (2008).

Section 3 provides the data. Section 4 summarizes the estimation results. Section 5 analyzes the impact of residual variance (yield variance unrelated to the level, slope, and curvature of the yield curve). Section 6 discusses the role of volatility in determining bond risk premia. Section 7 considers the role of convexity in bond yields. Section 8 describes the out-of-sample analysis and validation of the model results through analysis of the first and second conditional moments as well as model-free regression analysis. It also investigates a firm-related driver of interest rate volatility. Finally, Section 9 concludes.

2. Model

We consider four-factor affine short-rate models.[3] The short rate, r_t, is driven by a state variable, X_t, such that

$$r_t = \rho_0 + \rho_1 \cdot X_t\,, \tag{1}$$

and

$$\begin{aligned} dX_t &= \mu_t^{\mathbb{P}} dt + \sigma_t dB_t^{\mathbb{P}}, \\ \mu_t^{\mathbb{P}} &= K_0^{\mathbb{P}} + K_1^{\mathbb{P}} X_t, \end{aligned} \tag{2}$$

where $\rho_1, K_0^{\mathbb{P}} \in \mathbb{R}^4, K_1^{\mathbb{P}} \in \mathbb{R}^{4\times 4}$, and $B_t^{\mathbb{P}}$ is a standard four-dimensional Brownian motion under $\mathbb{P}$, the historical measure. Duffie, Filipovic, and Schachermayer (2003) provide conditions for (2) to give a well-defined process on $\mathbb{R}_+^M \times \mathbb{R}^{N-M}$. Here the conditional covariance is given by $\sigma_t\sigma_t' = \Sigma_0 + \sum_{i=1}^M \Sigma_i X_t^i$. We consider $A_M(4)$ models where either $M = 1$ or $M = 2$ factors drive volatility. For example, in the $A_2(4)$ case this means that, $\sigma_t\sigma_t' = \Sigma_0 + \Sigma_1 X_t^1 + \Sigma_2 X_t^2$, a 4×4 matrix. The constraints in Duffie, Filipovic, and Schachermayer (2003) require that (i) each Σ_i is positive semi-definite, (ii) $\Sigma_{1,22} = \Sigma_{2,11} = 0$, (iii) $K_{1,ij}^{\mathbb{P}} = 0$ for $i \le 2$ and $j \ge 2$, (iv) $K_{1,12}^{\mathbb{P}}, K_{1,21}^{\mathbb{P}} \ge 0$ and (v) $K_{0,1}^{\mathbb{P}}, K_{0,2}^{\mathbb{P}} \ge 0$. These conditions ensure that the covariance is always positive semi-definite and the first two factors, which drive volatility, always remain positive. As Joslin (2017) notes, in the $A_2(4)$ case, this specification allows for greater flexibility in the correlation structure among the risk factors than the normalization of Dai and Singleton (2000).

The dynamics of the macroeconomy are linked to the pricing measure by the market prices of risk. We use the completely affine market price of risk specification in Cheridito, Filipovic, and Kimmel (2007). This specification allows the expected excess returns for exposure to each risk factor to be affine in the state. As elaborated further in Section 7, a flexible market price of risk is critical in matching observed risk premia for holding both bonds and bond options. Under this market price of risk specification, the dynamics of the state variable X_t are affine under $\mathbb{Q}$ as well and satisfy

$$\begin{aligned} dX_t &= \mu_t^{\mathbb{Q}} dt + \sigma_t dB_t^{\mathbb{Q}}, \\ \mu_t^{\mathbb{Q}} &= K_0 + K_1 X_t, \end{aligned} \tag{3}$$

where $B_t^{\mathbb{Q}}$ is a four-dimensional standard Brownian motion under $\mathbb{Q}$ and K_0 and K_1 satisfy the same conditions as before. The absence of arbitrage is then guaranteed by assuming that the Feller condition is satisfied under both measures so that $K_{0,i}^{\mathbb{P}} \ge \frac{1}{2}\Sigma_{i,ii}$ and $K_{0,i}^{\mathbb{Q}} \ge \frac{1}{2}\Sigma_{i,ii}$ for $i \le M$.

To ensure that the parameters are econometrically identified, we impose the normalization constraints given in Joslin (2017). See Appendix A for further details.

Any claim with payoff at time T given by $f(X_T)$ can be priced by the discounted risk-neutral expected value

$$E_t^{\mathbb{Q}}\left[e^{-\int_t^T r_\tau \mathrm{d}\tau} f(X_T)\right]. \tag{4}$$

Duffie and Kan (1996) show that zero coupon bond prices are given by

$$P_t^T(X_t) = e^{A(T-t)+B(T-t)\cdot X_t}, \tag{5}$$

where P_t^T denotes the price at time t for a zero coupon bond paying \$1 at time T. The loadings A and B satisfy the Riccati differential equations

$$\dot{B} = -\rho_1 + \left(K_1^{\mathbb{Q}}\right)' B + \frac{1}{2} B^\top H_1 B\,, \quad B(0) = 0\,, \tag{6}$$

$$\dot{A} = -\rho_0 + \left(K_0^{\mathbb{Q}}\right)' B + \frac{1}{2} B^\top \Sigma_0 B\,, \quad A(0) = 0\,, \tag{7}$$

where H_1 is a tensor in $\mathbb{R}^{4\times4\times4}$ defined (as in Duffie, 2001) so that $B' H_1 B$ is a four-dimensional vector with $\left(B'H_1B\right)_i = B'\Sigma_i B$.

Collin-Dufresne and Goldstein (2002a) show that it is possible that some linear combination of the bond loadings is identically zero for all maturities. In such a case, a volatility factor can affect conditional second moments but not be contemporaneously spanned by bonds. Such unspanned volatility factors will directly affect fixed income derivative prices. Therefore, in addition to the more general specifications, we estimate models with the additional constraints required for unspanned stochastic volatility. In the case of the $A_2(4)$ model, see Joslin (2017) for a list of the constraints.

We also consider interest rate swaptions. An interest rate swaption is an option to enter into a swap, exchanging a fixed interest rate for a floating interest rate. Since the floating side of the swap is always worth par on initiation/reset, a swaption is equivalent to an option on a coupon bond.

An option on a Q-year swap expiring in P-year, referred to as an in P-for-Q (or P years into Q years) swaption, may be priced by

$$S_t = E^{\mathbb{Q}}[e^{-\int_t^{t+P} r_\tau d\tau}(CB(X_{t+P}, Q) - 1)^+]\,, \tag{8}$$

where $CB(X, Q)$ is the price of a Q-year coupon bond with a coupon equal to the strike when the state is X. Singleton and Umantsev (2003) approximate this expectation by replacing the exact exercise region, $\{CB(X_{t+P}, Q) \ge 1\}$, with the region implied by a linearization of the swap rate. Since the coupon bond price is a sum of coupons whose prices are exponential affine functions of the state, this reduces the problem of pricing the swaption to that of computing forward probabilities which may be evaluated by the transform method in Duffie, Pan, and Singleton (2000).[4]

[3] See Dai and Singleton (2000) for a summary of affine term term structure models. They classify affine term structure models into non-nested families denoted $A_M(N)$. N is the total number of factors and M is the number of factors driving volatility.

[4] Collin-Dufresne and Goldstein (2002b) suggest computing swaption prices using an Edgeworth expansion using the cumulants of the price of

In estimation of the models, computation is required for a large number of swaption coupons. This involves evaluations of many transforms each of which is an integral whose integrand is defined as the solution of ordinary differential equations similar to (6) and (7) which must be solved numerically for the general models that we consider. Because of this difficulty, we develop an adaptive integration scheme to compute the required forward probabilities. This scheme provides accurate prices using only three or four quadrature nodes. See Appendix B for details.

After computing security prices using the dynamics under the risk-neutral measure, it remains to estimate the parameters governing the evolution of the economy under the physical measure. Ideally, one would like to estimate the affine diffusion in (2) by maximum likelihood. Although the exact transition likelihood for an affine diffusion is known in terms of Green's functions of the Feynman-Kac partial differential equation, a direct computation is intractable. There is an extensive literature dealing with alternative estimation methods. Some alternative approaches to maximum likelihood include moment-based estimators e.g. quasi-maximum likelihood, generalized method of moments, or characteristic-function based methods as in, e.g., Singleton (2001), Carrasco, Chernov, Florens, and Ghysels (2007), simulation methods (e.g., Duffie and Singleton, 1993; Brandt and Santa-Clara, 2002), and approximate methods (e.g., Duffie, Pedersen, and Singleton, 2003; Ait-Sahalia, 1999; Ait-Sahalia and Kimmel, 2010). However, we estimate the models using full information maximum likelihood employing an extension of the methods that we develop for pricing options. See Appendix C for a summary of the calculations used in the current context.

We compute the likelihood of the observed time series of data as follows. First, following Pearson and Sun (1994) and Chen and Scott (1993), we suppose that three zero coupon yields and one swaption price are observed exactly for each panel of zero coupon yields and option prices. Given these prices and the underlying parameters, we can then invert the state using Newton's method.[5] The likelihood of the prices of these instruments is then computed by finding the likelihood of the inverted state (as in Appendix C) and applying the Jacobian of the linearized transformation at the observed state. The likelihood of the complete data is then computed with the assumption that all other yields and option prices are observed with independent and identically distributed (i.i.d.) normal measurement errors.

3. Data

The data are obtained from Datastream and consist of London Interbank Offered Rate (LIBOR), swap rates, and at-the-money swaption implied volatilities. We use 3-month LIBOR and the entire term structure of swap rates to bootstrap swap-implied zero rates. The bootstrap procedure assumes that forward rates are constant between observations. From these swap zero rates, we then compute time series of holding period returns. The data period is from June 1997 to January 2016. We split the data into two periods: (a) an in-sample period of June 1997 to June 2006 and (b) an out-of-sample period of July 2006 to January 2016. We estimate the model parameters on the earlier, in-sample period. Using these parameter estimates, we then make inferences about the fit of the model in the later out-of-sample period.

4. Model estimation results

We estimate the model by maximum likelihood as outlined in Section 2. Specifically, we estimate the model using 6-month, 1-, 2-, 3-, 4-, 5-, 7-, and 10-years swap-zero rates and swaptions with expiries of three months, one year, and three years written on swaps with maturities of two years, five years, and eight years. The estimation assumes that the 6-month, 2-year, and 10-year yields are priced without error along with the 1-year into 5-year swaption. The remaining instruments are priced with errors which are assumed to be independent and normally distributed.

The model estimates are in Tables 1–3. Throughout, the superscript *USV* in the model name refers to the estimated model where the unspanned stochastic volatility constraints are imposed. The standard errors are computed using the outer product gradient as in Berndt, Hall, Hall, Hausman, and Berg (1974).

Table 4 presents the in-sample root mean square pricing errors for zeros. For the maturities included in the estimation, pricing errors range from 5 to 10 basis points, with the USV models having slightly higher pricing errors. Also tabulated are pricing errors for maturities over ten years, which were not used in estimation. We discuss these results further in Section 7

Table 5 provides the in-sample pricing errors for swaptions.[6] Data from GovPX indicate that swaption bid-ask spreads range from 1% to 2% implied volatility. With the exception of the 3-months into 2-years swaption, the unconstrained models fit the data quite well with pricing errors under 1.5% and often under 1%. The unconstrained models are less successful fitting the 3-months into 2-years swaption, with root-mean-square pricing errors of 4.1 basis points. The fit for the 3-months into 2-years swaptions does improve a fair amount in low interest rate environments. For example, if the period when the 6-month rate is greater than 2% is dropped, the mean-square-error on the in 3-months-for 2-years swaption drops to 2.8% and 2.9% for the $A_1(4)$ and $A_2(4)$ models, respectively. Thus, generally, the unconstrained models provide a good cross-sectional fit across the options as well

the associated coupon bond. This approach presents a potential problem that Edgeworth expansions do not in general converge. Additionally, to compute the k-th moment of a 10-year coupon bond with semi-annual coupon requires the numerical solution of $\binom{20+k-1}{k}$ differential equations. For $k = 6$ this already reaches 177,100 equations.

[5] The pricing relation was more nearly linear to equate the model-implied Black volatility.

[6] Not reported are pricing errors for interest rates caps; pricing results are similar and the magnitudes are comparable (slightly smaller) than those found in Almeida, Graveline, and Joslin (2011).

Table 1
Drift $\mathbb{Q}$-parameter estimates.
This table provides model estimates drift parameter estimates for the risk-neutral ($\mathbb{Q}$) measure. Standard errors, computed by the gradient of the likelihood function, are given in parentheses. Unreported parameters are set to zero by the normalization constraints. The *USV* superscript denotes an affine model with USV constraints imposed and the † indicates parameters that are constrained by USV. The model is estimated from the in-sample period of June 1997 to June 2006.

	$A_1(4)$	$A_2(4)$	$A_1(4)^{USV}$	$A_2(4)^{USV}$
$K^{\mathbb{Q}}_{0,1}$	0.525 (0.044)	0.5 (0.048)	0.5 (0.041)	1.75 (0.16)
$K^{\mathbb{Q}}_{0,2}$	0	0.5 (0.05)	0	1.05 (0.096)
$K^{\mathbb{Q}}_{1,11}$	−0.317 (0.033)	−8.26 (0.8)	−0.244 (0.018)	−3.83 (0.29)
$K^{\mathbb{Q}}_{1,12}$	0	0.404 (0.033)	0	0†
$K^{\mathbb{Q}}_{1,21}$	0.084 (0.0079)	0.872 (0.083)	0.103†	0.648 (0.058)
$K^{\mathbb{Q}}_{1,22}$	−0.642 (0.071)	−1.09 (0.094)	−0.0338 (0.0036)	−1.63†
$K^{\mathbb{Q}}_{1,23}$	0	0	−0.681 (0.054)	0
$K^{\mathbb{Q}}_{1,24}$	0	0	−4 (0.34)	0
$K^{\mathbb{Q}}_{1,31}$	0.384 (0.045)	1.67 (0.18)	−0.0238†	−1.22 (0.079)
$K^{\mathbb{Q}}_{1,32}$	0	−1.49 (0.12)	0	0.089†
$K^{\mathbb{Q}}_{1,33}$	−0.0503 (0.0051)	−0.624 (0.06)	−0.068†	−0.118 (0.0084)
$K^{\mathbb{Q}}_{1,34}$	0	0	−0.95 (0.076)	−0.402 (0.033)
$K^{\mathbb{Q}}_{1,41}$	0.134 (0.015)	0.758 (0.062)	0†	−1.07 (0.091)
$K^{\mathbb{Q}}_{1,42}$	0	0.0349 (0.0034)	0	−0.356†
$K^{\mathbb{Q}}_{1,43}$	0	0	0	0
$K^{\mathbb{Q}}_{1,44}$	−1.49 (0.13)	−0.527 (0.043)	−1.23 (0.1)	−0.237†

Table 2
Drift $\mathbb{P}$-parameter estimates.
This table provides model estimates drift parameter estimates for the physical ($\mathbb{P}$) measure. Standard errors, computed by the gradient of the likelihood function, are given in parentheses. Unreported parameters are set to zero by the normalization constraints. The *USV* superscript denotes an affine model with USV constraints imposed. The model is estimated from the in-sample period of June 1997 to June 2006.

	$A_1(4)$	$A_2(4)$	$A_1(4)^{USV}$	$A_2(4)^{USV}$
$K^{\mathbb{P}}_{0,1}$	0.839 (0.15)	0.501 (0.14)	0.669 (0.12)	0.764 (0.12)
$K^{\mathbb{P}}_{0,2}$	0.122 (0.028)	0.0915 (0.048)	0.658 (0.56)	0.884 (0.26)
$K^{\mathbb{P}}_{0,3}$	0.143 (0.025)	0.182 (0.045)	0.134 (0.024)	0.273 (0.087)
$K^{\mathbb{P}}_{0,4}$	0.23 (0.045)	0.131 (0.035)	0.241 (0.044)	0.132 (0.05)
$K^{\mathbb{P}}_{1,1,1}$	0.47 (0.33)	0.48 (0.52)	5.29 (0.5)	0.81 (0.91)
$K^{\mathbb{P}}_{1,1,2}$	0	0	−4.39 (0.59)	1.78 (1.6)
$K^{\mathbb{P}}_{1,2,1}$	−2.79 (4.9)	−2.29 (2.9)	0.344 (0.53)	0.9 (0.43)
$K^{\mathbb{P}}_{1,2,2}$	1.22 (0.68)	−5.52 (2.5)	0.044 (0.62)	0.81 (0.75)
$K^{\mathbb{P}}_{1,2,3}$	−0.237 (1.9)	−11.9 (4.3)	0	0
$K^{\mathbb{P}}_{1,2,4}$	−0.517 (1.5)	18.7 (6.8)	0	0
$K^{\mathbb{P}}_{1,3,1}$	−8.06 (3.4)	−1.41 (1.4)	4.62 (1.4)	−0.984 (1.1)
$K^{\mathbb{P}}_{1,3,2}$	0.791 (0.48)	−3.61 (0.96)	−4.72 (0.58)	0.483 (1.4)
$K^{\mathbb{P}}_{1,3,3}$	4.51 (1.3)	−6.65 (1.6)	1.85 (0.51)	1.02 (0.2)
$K^{\mathbb{P}}_{1,3,4}$	3.46 (1.1)	11.3 (2.6)	1.17 (0.42)	0.88 (0.9)
$K^{\mathbb{P}}_{1,4,1}$	6.74 (3.3)	−2.15 (1.5)	8.18 (2)	−0.339 (0.69)
$K^{\mathbb{P}}_{1,4,2}$	−0.701 (0.45)	−4.66 (1.2)	−7.01 (1)	0.214 (0.85)
$K^{\mathbb{P}}_{1,4,3}$	−3.25 (1.2)	−8.95 (2.1)	−1.14 (0.72)	0.0276 (0.14)
$K^{\mathbb{P}}_{1,4,4}$	−2.29 (1)	14.9 (3.4)	−0.498 (0.59)	1.17 (0.61)

as the yields. In contrast, the models with USV imposed have a noticeably worse fit.

To understand the role of risk premia in matching both markets, observe that the likelihood is made up of a component due to the transition dynamics of the economy and a component due to the pricing errors. The pricing component is determined by the risk-neutral drift ($\mu^{\mathbb{Q}}$) and covariance structure (σ) of the risk factors, while the likelihood of the data measured without error is determined by the drift under the physical measure($\mu^{\mathbb{P}}$) and the covariance structure. The drift under the two measures is related by the market prices of risk. Thus, the covariance structure provides a link between the two.

As elaborated in Section 7, convexity plays only a small role in the variation of bond yields. This means that bond prices depend primarily on risk-neutral expectations specified by $\mu^{\mathbb{Q}}$. Provided that the market prices of risk are not restrictive, the likelihood cannot be dominated by the pricing errors and the model will be estimated in a consistent manner. On the other hand, with a constrained market price of risk, there will be a tension between the dynamics and pricing errors. Related points are discussed in

Table 3
Variance parameter estimates.
This table provides model estimates variance parameter estimates. All matrices are symmetric, with the lower diagonals reported. Standard errors, computed by the gradient of the likelihood function, are given in parentheses. Unreported parameters are set to zero by the normalization constraints. The *USV* superscript denotes an affine model with USV constraints imposed. The model is estimated from the in-sample period of June 1997 to June 2006.

	$A_1(4)$	$A_2(4)$	$A_1(4)^{USV}$	$A_2(4)^{USV}$
$\Sigma_{0,\,22}$	8.95 (0.96)	0	1	0
$\Sigma_{0,\,32}$	5.52 (0.81)	0	0	0
$\Sigma_{0,\,33}$	3.41 (0.41)	0.191 (0.019)	1	0.0184 (0.002)
$\Sigma_{0,\,42}$	3.7 (0.47)	0	0	0
$\Sigma_{0,\,43}$	2.28 (0.29)	-0.482 (0.065)	0	0.147 (0.019)
$\Sigma_{0,\,44}$	1.53 (0.16)	1.22 (0.16)	1	2.14 (0.18)
$\Sigma_{1,\,11}$	1	1	1	1
$\Sigma_{1,\,22}$	4.43 (0.49)	0	6.93 (1.1)	0
$\Sigma_{1,\,32}$	1.35 (0.17)	0	0	0
$\Sigma_{1,\,33}$	8.55 (0.97)	0.644 (0.083)	0	0.0265 (0.0029)
$\Sigma_{1,\,42}$	2.47 (0.39)	0	0	0
$\Sigma_{1,\,43}$	−1.4 (0.17)	−2.76 (0.35)	0	0.24 (0.029)
$\Sigma_{1,\,24}$	2.03 (0.25)	11.9 (1.1)	0	5.06 (0.53)
$\Sigma_{2,\,22}$	0	1	0	1
$\Sigma_{2,\,33}$	0	1.42 (0.15)	0	2.15 (0.23)
$\Sigma_{2,\,43}$	0	−4.5 (0.48)	0	0
$\Sigma_{2,\,44}$	0	16.1 (1.7)	0	0

Table 4
Zero coupon pricing errors.
This table computes the root-mean-square zero coupon yield pricing errors (in basis points) for the various models for the in-sample period. Zero coupon yields are computed by bootstrapping the swap curve. The *USV* superscript denotes an affine model with USV constraints imposed. The model is estimated from the in-sample period of June 1997 to June 2006.

	$A_1(4)$	$A_1(4)^{USV}$	$A_2(4)$	$A_2(4)^{USV}$
6 Month	0	0	0	0
1 Year	7.4	12.7	7.4	10.3
2 Year	0	0	0	0
3 Year	4.1	10.3	4.1	6.3
4 Year	5.2	15.1	5.2	8.2
5 Year	5.3	16.4	5.3	8.3
7 Year	3.8	13.0	3.8	6.1
10 Year	0	0	0	0
12 Year	3.9	12.9	3.9	6.9
15 Year	8.9	38.8	9.1	19.7
20 Year	12.9	96.4	13.2	39.0
25 Year	13.3	176.0	13.6	53.0
30 Year	17.3	279.1	17.4	66.2

Table 5
Swaption implied volatility errors.
This table computes the root-mean-square errors for swaption implied volatilities (in percentage points) for the various models for the in-sample period. Swaptions are considered to be at-the-money in the model. The *USV* superscript denotes an affine model with USV constraints imposed. The model is estimated from the in-sample period of June 1997 to June 2006.

	$A_1(4)$	$A_1(4)^{USV}$	$A_2(4)$	$A_2(4)^{USV}$
3 Months into 2 Years	4.1	7.2	4.1	21.5
3 Months into 5 Years	1.5	2.1	1.4	5.4
3 Months into 8 Years	1.4	1.5	1.4	3.1
1 Year into 2 Years	1.0	3.6	0.9	5.7
1 Year into 5 Years	0	0	0	0
1 Year into 8 Years	0.5	0.5	0.5	0.6
3 Years into 2 Years	1.0	1.5	0.8	2.0
3 Years into 5 Years	0.7	0.8	0.7	1.8
3 Years into 8 Years	0.9	0.8	0.8	1.9

Dai and Singleton (2003). The completely affine market price of risk allows for risk premia to depend on the state in two important ways. First, it allows for risk premia to depend on the slope of the yield curve and change sign over time. Second, it also allows the risk premium demanded for holding volatility risk to not shrink to zero as volatility drops to zero—that is, investors may still be averse to volatility risk, even when volatility is low.

The risk premium for volatility risk is particularly important in matching the cross-section of option prices. Agents are exposed to interest rate risks directly through holding bonds and also indirectly through asset prices linked to interest rates, such as home values. When interest rate volatility is low, these assets become less risky. This means that when volatility is low, an increase in volatility turns a portion of the investor's portfolio from a riskless asset to a risky asset. If the price of volatility risk is proportional to the level of volatility, the agent is effectively close to risk-neutral to changes in the risk-level of large portions of their portfolio.

Overall, the model fits the prices in the data fairly well during the in-sample period (with the possible exception noted above). In general, fitting only prices for a reduced-form model is not the most stringent test, but given the cross-section of both bond and bond options and some prior difficulties fitting both of these simultaneously with a low-dimensional model, this does build some confidence in the model. In Section 8, we build on this by showing the models' abilities to explain conditional moments of bonds and bond options and the models' ability to match prices out-of-sample.

5. Volatility and the cross section of yields

In general, volatility factors may affect the shape of the yield curve as in, for example, Longstaff and Schwartz (1992). This may be through a correlation with the short rate (a nonzero entry in ρ_1), an effect on risk-neutral expectations (a nonzero entry in $K_1^{\mathbb{Q}}$), or through a stochastic convexity effect. This section assesses the magnitude of the incremental effect of volatility on the yield curve (relative to the level, slope, and curvature) through the lens of the model.

To study this effect, we consider the component of variance risk which is locally uncorrelated with the level, slope, and curvature of the yield curve. For this purpose, define the residual variance

$$V_t^R = V_t - \alpha_1 \ell_t - \alpha_2 s_t - \alpha_3 c_t, \tag{9}$$

where V_t is the (instantaneous) variance of the 5-year zero rate and $(\ell_t,\ s_t,\ c_t)$ are measures of the level, slope, and curvature of the yield curve: $\ell_t = \frac{1}{3}(y_{6m} + y_{2y} + y_{10y})$, $s_t = y_{10y} - y_{6m}$, $c_t = -y_{6m} + 2y_{2y} - y_{10y}$. α is chosen so that V_t^R is locally uncorrelated with $(\ell_t,\ s_t,\ c_t)$: $\alpha = \Sigma_y^{-1}\Sigma_{Vy}$.[7] Because the covariance of the factors is time-varying, the weights (α) are time-varying as well. Here, it is convenient to work with variance instead of volatility since variance is an affine function of the state. Implicit in this definition is the idea that the residual variance primarily drives volatility of the yield curve rather than the shape of the yield curve. In the case of the models with USV, the residual variance factor has exactly no incremental effect on the yield curve. This turns out to be approximately true also in the case of the unconstrained model. Fig. 1 plots the effect of changes in the local residual variance, fixing the 6-month, 2-year, and 10-year yields, on the cross-section of yields for the estimated $A_1(4)$ model.[8] The effect of the residual variance on the yield curve is nonzero, but quite small with a one standard deviation monthly shock resulting in a shift of less than half a basis point in all but very short maturity yields. This indicates that, although the model does not precisely have unspanned volatility, the residual variance, and thus volatility itself, is only very poorly identified from the cross-section of bond yields. Thus, while in principle one could extract volatility from the cross-section, in practice this may be difficult to do in the presence of measurement errors much larger than the effect of volatility. This can also be seen by computing condition numbers for the required matrix inversion, which are quite large.

These results agree well with Litterman and Scheinkman (1991), who show that three principal components explain nearly all of the variation in the yield curve. Thus, one would anticipate that a fourth factor likely would have only a small effect on yields. Duffee (2011) finds support, within the context of a Gaussian model, for a factor which has an effect only on returns. In the next section, we show that the residual variance drives both volatility *and* expected excess returns. We elaborate further on the mechanism that generates this effect in comparison to the restrictions in the USV model in Section 7.

6. Pricing yield risk

Fama and Bliss (1987), Campbell and Shiller (1991), and others have suggested that the shape of the yield curve drives risk premia that investors demand for holding long maturity bonds over short maturity bonds. A number of results (see, e.g., Cochrane and Piazzesi, 2008; Duffee, 2011; Joslin, Priebsch, and Singleton, 2014; Ludvigson and Ng, 2000; Rudebusch, Swanson, and Wu, 2006; Wright and Zhou, 2009) suggest that factors which have little incremental impact on the shape of the yield curve may be important for predicting excess returns for holding long maturity bonds. Indeed, Ludvigson and Ng (2000) find evidence that inflation and real macroeconomic activity risk factors have the ability to predict variation in bond excess returns above and beyond the level of interest rates. This raises the question of whether fixed income derivatives may be useful for identifying time-series variation in expected excess returns for holding long maturity bonds.

The theoretical possibility that volatility may incrementally forecast bond returns can be seen as follows. The (local) risk premium for exposure to a risk factor F_t is given by

$$\text{risk premium} = \mu_F^{\mathbb{P}}(X_t) - \mu_F^{\mathbb{Q}}(X_t).$$

That is, the expected excess returns for exposure to a risk is determined by the difference between the $\mathbb{P}$ and $\mathbb{Q}$ expected changes in the risk factor. We can reparameterize the model so that rather than the latent state variable X_t, we have the state variable $Y_t = (\ell_t, s_t, c_t, V_t^R)$ as in Section 5. In these terms, we can transform the model to be given in terms of Y_t which will have

$$\mu_Y^{\mathbb{Q}} = K_{0Y}^{\mathbb{Q}} + K_{1Y}^{\mathbb{Q}} Y_t, \tag{10}$$

$$\mu_Y^{\mathbb{P}} = K_{0Y}^{\mathbb{P}} + K_{1Y}^{\mathbb{P}} Y_t. \tag{11}$$

As seen in Fig. 1, volatility has little incremental impact on bond yields relative to the level, slope, and curvature. Moreover, we show in Section 7, volatility induces only minor variation in the convexity effect across maturities. Together, these observations imply that volatility has little incremental impact on $\mathbb{Q}$-forecasts of the level, slope, and curvature. In this case (where volatility does not affect the $\mathbb{Q}$-forecasts of the yield factors), volatility will be useful for forecasting bond returns whenever volatility is incrementally informative for predicting future yields through $\mu_Y^{\mathbb{P}}$.

We decompose the risk premia into a component associated with the yield curve and a component due to the residual variance. A one-year standard deviation increase

[7] Alternatively, one could use different maturities or use principal components (as in Joslin, Singleton, and Zhu, 2011) with the volatility of the level factor instead of individual maturities.

[8] These loadings are found by transforming the original risk factors X_t from the drift-normalized model to the risk factors $Y_t = (\ell_t, s_t, c_t, V_t^R) = C + DX_t$. The new loadings for maturity τ are transformed by $B(\tau) \mapsto (D^{-1})' B(\tau)$.

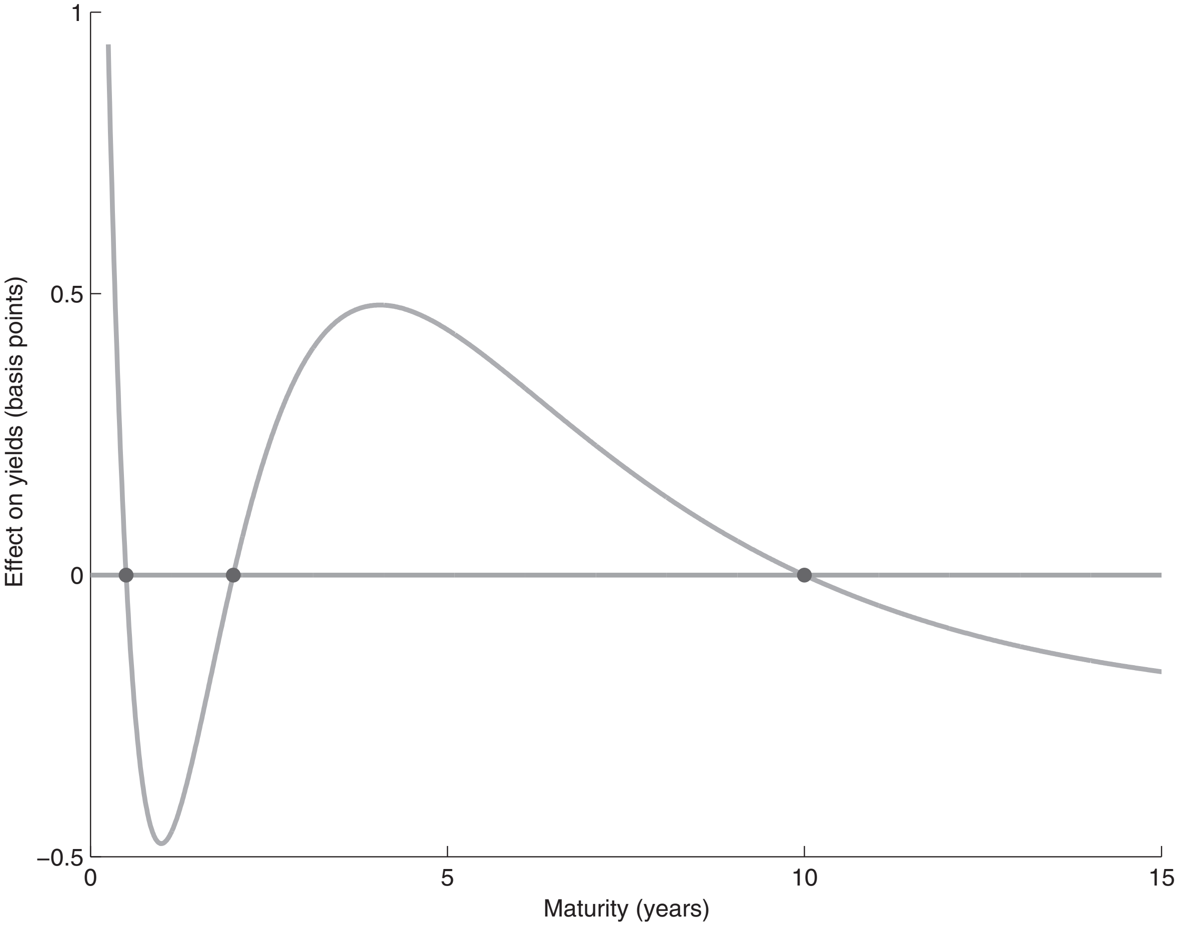

Fig. 1. Effect of residual variance on the yield curve. This figure plots the effect of the residual variance on the yield curve. The residual variance is defined as the risk which is locally uncorrelated with the 6-month, 2-year, and 10-year yields. The figure plots the effect of a monthly one standard deviation shock in the residual variance on the yield curve, fixing the 6-month, 2-year, and 10-year yields for the $A_1(4)$ model on June 21, 2006.

in the level of residual variance results in a decrease in expected return of approximately 1% for the five-year zero coupon bond. The variation in risk premia due to residual variance accounts for approximately 40% of the total variation in risk premia. Although not the dominant term, the residual variance drives an economically meaningful portion of the risk premium.

The period from June 2004 to June 2006 is suggestive of this relationship. During this period, the Federal Open Market Committee raised the target Fed funds rate 25 basis points for 17 consecutive meetings. This period has been referred to as a conundrum by then Fed Chairman Alan Greenspan because during this period the long rate remained relatively constant despite the increasing short rate. This conundrum is resolved either through changing expectations of future short rates (i.e., long rates could remain unchanged if investors anticipated the future Fed policy actions) or through declining term premiums. The model estimates indicate that this flattening of the yield curve was largely associated with declining risk premia. Thus, we can associate the flattening of the yield curve with a decline in risk premia for holding long maturity bonds (Section 7 further discusses the decomposition of the yield curve into expectations, term premia, and convexity effects). Fig. 2 plots the slope of the yield curve (the 10-year rate minus the 6-month rate) on the left axis and the implied volatility of an in 1-year-for-5-year swaption.[9] Here it is evident that the period is also associated with a decline in yield volatility (perhaps due to increased transparency of monetary policy, as some suggest). These observations, albeit over a short time period, support the empirical results that volatility is an important determinant of the risk premium demanded for holding long maturity bonds.[10]

7. Role of convexity in bond pricing

Long maturity bond yields reflect expectations of future interest rates, risk premia, and convexity effects.

[9] These results partially reflect the fact that swaption implied volatility is quoted on a log of yields scale, rather than on a level of yields scale.

[10] As a robustness check, we also estimate the non-USV models over the shorter period ending in April 2004, which excludes this episode. The results were generally similar with the exception of declining statistical significance due to larger standard errors.

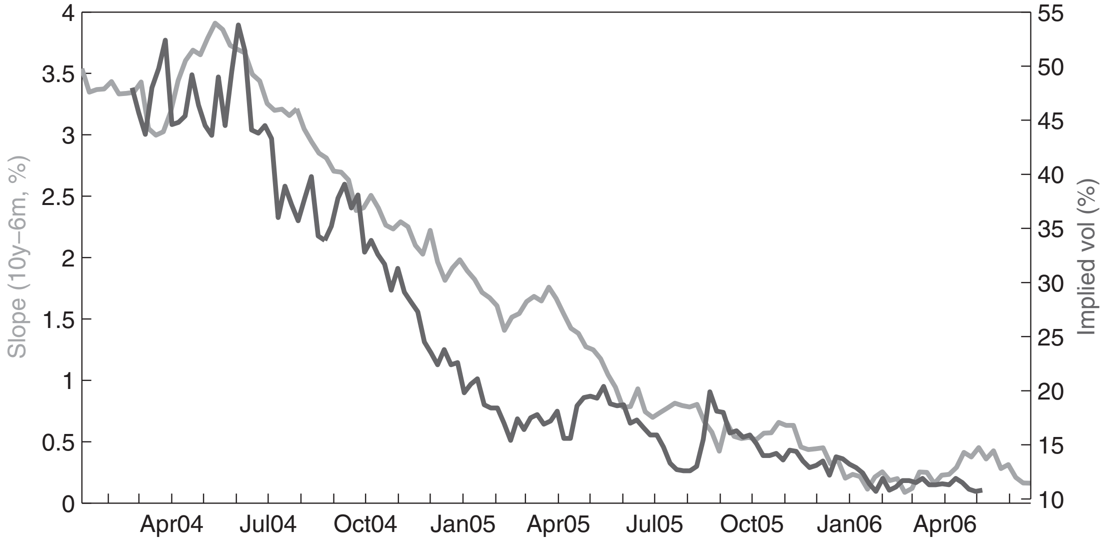

Fig. 2. Long-yield conundrum. The figure plots the slope of the yield curve (10-year swap-implied zero rate minus the 6-month LIBOR rate) on the left axis and the implied volatility of the in 6-month-for-2-year swaption on the right axis. During this period, the yield curve became flat as the Fed continually raised interest rates. Also, implied volatilities declined similarly.

Fixing (risk-neutral) forecasts of future interest rates, convexity affects long maturity bond yields through Jensen's inequality as

$$\exp\left(E^{\mathbb{Q}}\left[-\int_0^T r_\tau d\tau\right]\right) < E^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_\tau d\tau\right)\right].$$

The size of the convexity effect will be determined by the volatility of interest rates. We now turn to analyze the relative importance of this channel and its impact on bond prices and the modeling of the yield curve.

The T-year zero coupon yield can be decomposed into an expectations effect ($y_{t,\,E}$), a risk premium ($y_{t,\,RP}$), and a convexity effect ($y_{t,\,C}$) as

$$y_t^T = y_{t,E}^T + y_{t,RP}^T + y_{t,C}^T\,, \tag{12}$$

where

$$\begin{aligned} y_{t,E}^T &\equiv \frac{1}{T}\int_t^{t+T} E_t^{\mathbb{P}}[r_\tau]d\tau, \\ y_{t,RP}^T &\equiv \frac{1}{T}\int_t^{t+T} (E_t^{\mathbb{Q}}[r_\tau] - E_t^{\mathbb{P}}[r_\tau])d\tau\,, \\ y_{t,C}^T &\equiv -\frac{1}{T}\left(\log E_t^{\mathbb{Q}}\left[e^{-\int_t^{t+T} r_\tau d\tau}\right] + \int_t^{t+T} E_t^{\mathbb{Q}}[r_\tau]d\tau\right). \end{aligned} \tag{13}$$

The expectations term represents the bond price discounted with a yield to maturity equal to the average expected future short rate.

Fig. 3 plots the decomposition of the 10-year zero coupon yield in terms of expectations, risk premia, and convexity effects as defined above. Each of the terms are an affine function of the state variable whose loading can be computed by solving a Riccati differential equation or a linear constant coefficient ordinary differential equation. The figure shows that the variation in yields is dominated by expectations and risk premium effects and that the convexity effects are quite small (see also Gupta and Subrahmanyam, 2000).

Table 6
Convexity effects in zero coupon bond yields.
The top panel provides the average model-implied convexity effect in basis points for different bond maturities. The bottom panel provides the sample standard deviation, in basis points, of monthly changes in the model-implied convexity effects. The convexity effect of a T-year yield:

$$C_t(T) \equiv \frac{1}{T}\log E\left[e^{-\int_t^{t+T} r_\tau d\tau}\right] + \frac{1}{T}E^{\mathbb{Q}}\left[\int_t^{t+T} r_\tau d\tau\right].$$

The USV superscript denotes an affine model with USV constraints imposed. The model is estimated from the in-sample period of June 1997 to June 2006.

Panel A: *Average convexity effects*					
	$A_1(3)$	$A_1(4)$	$A_1(4)^{USV}$	$A_2(4)$	$A_2(4)^{USV}$
2 year	0.48	0.46	0.79	0.46	1.80
5 year	4.06	4.01	4.78	4.02	5.20
10 Year	16.19	15.65	17.79	15.83	15.00
30 Year	85.48	79.15	168.29	80.06	90.95
Panel B: *Time variation of convexity effects*					
	$A_1(3)$	$A_1(4)$	$A_1(4)^{USV}$	$A_2(4)$	$A_2(4)^{USV}$
2 year	0.02	0.03	0.07	0.04	0.11
5 year	0.13	0.28	0.36	0.27	0.28
10 Year	0.39	0.84	1.01	0.83	0.38
30 Year	0.80	1.84	2.69	1.82	0.27

Table 6 provides the magnitude of the convexity effect across maturities for the various models' specifications. For comparison, an $A_1(3)$ model, estimated on the same data but not inverting the swaption, is added to both tables. Panel A shows that the average convexity effects are small for the 2-year zero coupon bond, around one basis point. Extending the maturity to ten years, the average size of the convexity effects becomes more economically meaningful, reaching around 15 basis points. However, Panel B shows that, although the 10-year convexity effect is larger, the variation is still quite small with a monthly standard

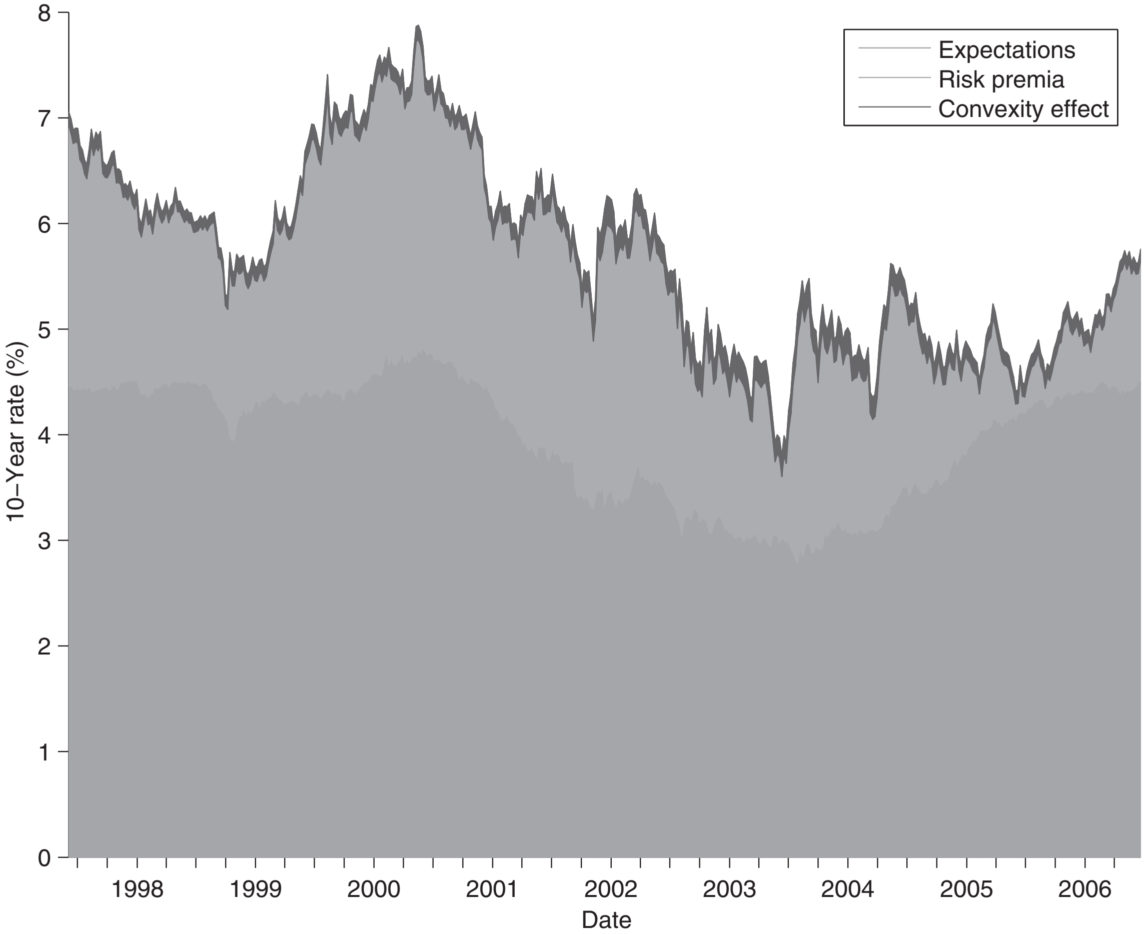

Fig. 3. Yield decomposition. This figure plots model-implied decomposition of the 10-year yield into expectations, risk premiums, and convexity components for the estimated $A_1(4)$ model. Expectations, risk premiums, and convexity terms are defined by

$$y_{\text{expectation}} = \frac{1}{10}\int_t^{t+10} E_t^P[r_\tau]d\tau,$$

$$y_{\text{risk premia}} = \frac{1}{10}\int_t^{t+10} (E_t^{\mathbb{Q}}[r_\tau] - E_t^P[r_\tau])d\tau, \text{ and}$$

$$y_{t,\text{convexity}} = -\frac{1}{10}\left(\log E_t[e^{-\int_t^{t+10} r_\tau d\tau}] + \int_t^{t+10} E_t^{\mathbb{Q}}[r_\tau]d\tau\right).$$

deviation of less than one basis point. Extending to 30-year bonds, the average convexity effect becomes important, but the variation still remains quite small. The reason behind this is intuitive: over short horizons, convexity effects are generally unimportant; over long horizons, mean reversion in the level of volatility implies that the current level of volatility has only a small impact on long maturity yields.

These results show that convexity plays only a small role in bond prices. The fact that convexity effects are small implies that a dynamic term structure model may exhibit arbitrary correlation between the first few principal components and volatility under very parsimonious conditions. For example, consider the $A_1(4)$ model and approximate (6) by eliminating the quadratic convexity term,

$$\dot{B} \approx -\rho_1 + (K_1^{\mathbb{Q}})' B. \tag{14}$$

If there is a risk factor that affects the conditional volatility of yields but does not affect risk-neutral expectations of future rates, volatility will only be related to the yield curve through the small convexity effect and correlation between the risk factors.[11]

As Collin-Dufresne, Goldstein, and Jones (2009) argue, the fact that convexity effects are small suggests that volatility may be poorly identified from the cross-section of bond prices.[12] Indeed, even in a model where the constraints for USV are strongly violated, volatility may only be identified through the convexity effect in a very

[11] More formally, the precise condition is the existence of an eigenvector of $K_1^{\mathbb{Q}}$ which is orthogonal to ρ_1 and loads on the volatility factors.

[12] Andersen and Benzoni (2010) stress the theoretical deficiency of general affine models to produce low correlation between volatility changes and yield changes.

sensitive manner. More precisely, although volatility may be directly inferred from bond prices, it is only through solving the numerically unstable equation $Ax = b$ where A is nearly singular. The near singularity of A means that small errors, for example, measurement errors in the yields or estimation errors, may result in large errors in the inferred volatility. For example, if 6-month, 2-year, 5-year, and 10-year zero coupon yields are used to infer volatility, the unconstrained $A_1(4)$ estimates indicate that the matrix A will have a very high condition number (6,299—condition numbers over 30 suggest multicollinearity) and thus will be nearly non-singular.

To highlight the mechanism, define a model with a factor that has volatility *unspanned by expectations* by

Defintion 7.1. A diffusive $\mathbb{Q}$-short rate model,

$$dr_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t,$$

has volatility unspanned by expectations if there exists a change of variable $Y_t = f(X_t)$ such that Y_t^1 drives volatility but does not affect $\mathbb{Q}$-expectations of future interest rates $(\partial\mu/\partial y^1 \equiv 0)$.

A non-degenerate affine term structure model has volatility unspanned by expectations if there exists an eigenvector of K_1 which is orthogonal to ρ_1. In an affine model, we can obtain this representation as follows. First, assuming K_1 is diagonalizable, let $\{u_1, u_2, \ldots, u_n\}$ be a basis of eigenvectors of K_1 so that u_1 is an eigenvector corresponding to a zero eigenvalue which is also orthogonal to ρ_1. Letting $U = [u_1, \ldots, u_n]$, define a change of variables $Y_t = UX_t$. We will then have (1) $\rho_{Y1,1} = 0$ and (2) $Y_{t,\,1}$ will not have any feedback on the conditional mean of the other variables (i.e., $K_{Y1,j,1} = 0$ for $j > 1$); thus, $E_t^{\mathbb{Q}}[r_{t+\tau}]$ will not depend on $Y_{t,\,1}$ for any horizon. In case K_1 is not diagonalizable, we can use a real Schur form as in Joslin, Singleton, and Zhu (2011).

Not all classes of term structure models admit volatility risk factors unspanned by expectations. Gaussian affine term structure models clearly cannot admit expectations unspanned volatility risk factors (the eigenvector condition implies the model will be degenerate). Additionally, it is easy to show that quadratic affine term structure models, as in Longstaff (1989), Ahn, Dittmar, and Gallant (2002), Li and Zhao (2006), and others, do not admit volatility unspanned by expectations.

The mechanism at work in models with unspanned volatility and expectations unspanned volatility are very different. In USV models, there exist small convexity effects whose differential effects across maturities must be exactly canceled by a corresponding expectations effect. This requires restrictions on the number of stochastic convexity effects generated and the rates of mean reversions of the risk factors. For example, in the $A_1(3)$ specification, there are ten parameter constraints required. In contrast, when volatility is unspanned by expectations, it will by spanned by yields. However, this is only through the convexity effects which we have seen empirically show very small time-series variation. Furthermore, this requires only a single parameter constraint.

The simple condition in Defintion 7.1 is very different from the multiple conditions required for unspanned stochastic volatility, which explicitly cancel the convexity effect. For example, Joslin (2017) shows that in order for the convexity effect to cancel, there must be some factors whose mean reversions are related in a 2:1 ratio in order to possibly cancel a quadratic convexity effect.[13] We can examine this particular condition further by looking at the rates of mean reversion (determined by the eigenvalues of $\kappa^{\mathbb{Q}} = -K_1$) for each of the models. The eigenvalues under the risk-neutral measure are given in Table 7. We see that both USV models have *two* persistent risk factors with long half-lives. This is because the convexity effect generated by a persistent risk factor with stochastic volatility can only be canceled by a risk factor with twice the rate of mean reversion. This stands in contrast to the unconstrained models that have only a single persistent risk factor.

Table 7
Eigenvalues under the risk-neutral measure.
Eigenvalues of the mean-reversion matrix, $\kappa^{\mathbb{Q}}$, under the risk-neutral measure. Eigenvalues of the CIR factors are denoted with a $\circ$. The *USV* superscript denotes an affine model with USV constraints imposed. The model is estimated from the in-sample period of June 1997 to June 2006.

	$A_1(4)$	$A_1(4)^{USV}$	$A_2(4)$	$A_2(4)^{USV}$
1st Eigenvalue	1.492	1.233	1.563°	3.833°
2nd Eigenvalue	0.642	0.244°	0.624	1.628°
3rd Eigenvalue	0.317°	0.067	0.348°	0.237
4th Eigenvalue	0.050	0.034	0.053	0.118

We can examine the severity of this constraint on the eigenvalues both through statistical tests and through the out-of-sample pricing performance of the models. Table 8 shows that both a Lagrange-multiplier test using the restricted estimates and a Wald test using the unrestricted estimates reject the restriction on the rates of mean reversion. Additionally, a likelihood ratio test of the constrained USV model against the unconstrained model strongly rejects the USV restrictions for both the $A_1(4)$ and $A_2(4)$ models. The economic effect of the mean-reversion restrictions can also be understood by comparing the ability of the models to price 30-year zero coupon bonds in Table 4. These bonds were not used in estimation and thus represent an out-of-sample comparison of the models. For the longer maturities, the non-USV models price the yields reasonably well with root-mean-square errors ranging from 10 to 17 basis points. The errors for the USV models are much larger. The cause for the larger error can be attributed to the restrictions on the rates of factor mean reversion imposed by USV.

8. Out-of-sample analysis

This section provides additional analysis that focuses on out-of-sample tests. Specifically, it first focuses on out-of-sample assessment of the ability of the models to accu-

[13] Joslin (2017) shows that in order for the convexity effect to cancel, three types of restrictions must hold: (1) some factor mean reversions must be related in a 2:1 ratio in order to possibly cancel a quadratic convexity effect, (2) some factors must have constant volatility in order to not generate convexity effects, and (3) volatility must affect expectations of future rates in exactly the right way to cancel the convexity effect.

Table 8
Statistical tests of USV constraints.
This table gives the test statistics for the constraints required for the $A_1(4)$ and $A_2(4)$ models to exhibit unspanned stochastic volatility. The USV constraint imposes nine parameter restrictions on the unconstrained model. The mean-reversion constraint is tested by a Lagrange-multiplier test. In addition, the complete set of restrictions is rejected by a likelihood ratio test and by a Wald test.

	$A_1(4)$	$A_2(4)$
Lagrange multiplier (mean reversion)	6.31 (0.006)	14.73 ($\ll$ 0.001)
Wald test (full model)	987 ($\ll$ 0.001)	1,320 ($\ll$ 0.001)
Likelihood ratio (full model)	2,690 ($\ll$ 0.001)	4,800 ($\ll$ 0.001)

Table 9
Out-of-sample zero coupon pricing errors.
This table computes the out-of-sample root-mean-square zero coupon yield pricing errors (in basis points) for the various models. Zero coupon yields are computed by bootstrapping the swap curve. The *USV* superscript denotes an affine model with USV constraints imposed. The model is estimated from the in-sample period of June 1997 to June 2006. The out-of-sample period is from July 2006 to January 2016.

	$A_1(4)$	$A_1(4)^{USV}$	$A_2(4)$	$A_2(4)^{USV}$
6 Month	0	0	0	0
1 Year	7.2	14.5	7.2	12.4
2 Year	0	0	0	0
3 Year	4.4	12.4	4.7	7.5
4 Year	5.7	18.2	5.7	9.9
5 Year	5.8	19.6	5.8	10.0
7 Year	4.0	14.7	4.3	7.0
10 Year	0	0	0	0
12 Year	4.2	15.0	4.4	7.8
15 Year	10.9	47.8	10.2	25.3
20 Year	14.7	125.0	14.8	49.6
25 Year	15.3	221.5	15.5	66.1
30 Year	18.5	362.5	18.6	80.2

Table 10
Out of sample swaption implied volatility errors.
This table computes the out-of-sample root mean square errors for for swaption implied volatilities (in percentage points) for the various models. Swaptions are considered to be at-the-money in the model. The *USV* superscript denotes an affine model with USV constraints imposed. The model is estimated from the in-sample period of June 1997 to June 2006. The out-of-sample period is from July 2006 to January 2016.

	$A_1(4)$	$A_1(4)^{USV}$	$A_2(4)$	$A_2(4)^{USV}$
3 Months into 2 Years	3.3	9.3	3.4	26.9
3 Months into 5 Years	1.7	2.6	1.5	6.8
3 Months into 8 Years	1.5	1.9	1.5	4.0
1 Year into 2 Years	1.1	4.4	1.0	7.0
1 Year into 5 Years	0	0	0	0
1 Year into 8 Years	0.5	0.6	0.6	0.7
3 Years into 2 Years	1.1	1.9	0.8	2.4
3 Years into 5 Years	0.8	1.0	0.8	2.2
3 Years into 8 Years	1.0	1.0	0.9	2.4

rately price the cross-section of yields and swaption implied volatilities. The section then evaluates the ability of the model to match conditional first and second moments of yields in- and out-of-sample. In addition, we extend the evidence in the model to examine the role of volatility in bond risk premia through model-free regression tests. To briefly summarize the findings described in the rest of the section below: (a) the model is able to price quite well the cross-section of yields and swaption implied volatilities out-of-sample, supporting the in-sample evidence previously provided; (b) the model can match conditional first moments of yields and swaption implied volatilities in- and out-of-sample; (c) the model can match conditional second moments of yields in- and out-of-sample; and (d) there is additional model-free regression evidence on the role of volatility in determining bond risk premia. Taken together, the additional findings in this section indicate that the model describes interest rate dynamics and they provide further confirmatory evidence on the impact of volatility on interest rates and their dynamics.

Table 9 provides the out-of-sample fit to the swap-implied zero curve. The results show that the out-of-sample fit is generally similar to the in-sample fit. In particular, compared to the in-sample fit in Table 4, the fit is generally only slightly worse with root-mean-square errors within one basis point. A few notable exceptions are that: (a) the fit for the one-year zero is slightly better out-of-sample for the unconstrained models; and (b) the fit of the longer maturities beyond ten years (which were not used in estimation) is slightly worse for the unconstrained models (increases in root-mean-square errors of around one to two basis points) and considerably worse relative to an already poor fit for the USV-constrained models.

Table 10 provides the out-of-sample fit to the swaption implied volatilities. As with the zero curve, the out-of-sample fit is similar to the in-sample fit. In particular, compared to the in-sample fit in Table 5, most of the root-mean-square errors (RMSEs) are within around 10 to 20 basis points (0.10–0.20%) in implied volatility. An exception is the in 3-months-for-2-years swaption that has the shortest expiration and shortest underlying maturity. Here, the out-of-sample fit actually improves somewhat for the unconstrained models. This finding is consistent with the fact that in-sample the model fits this swaption better in lower interest rate environments and the fact that the out-of-sample period is dominated by low interest rates. In general, the same conclusion for the in-sample period holds for the out-of-sample period. That is, the unconstrained models match well the cross-section of swaptions with the exception of a poorer fit to the short expiration-short maturity swaption.

To further evaluate the fit of the model to first conditional moments, we compare the ability of the models to predict changes in yields and changes in implied volatilities relative to standard baseline models. Table 11 presents the root-mean-square errors for predicting monthly and annual changes for the ten-year swap-implied zero rate

Table 11
Predicting changes in yields.
This table presents the ability of the in-sample estimated term structure to forecast in-sample and out-of-sample changes in 10-year swap-implied zero rate. Panel A (Panel B) presents the results for forecasting changes one (12) month(s) ahead. The first (third) data column presents the root-mean-square prediction error for the models for the in-sample (out-of-sample) period. The second (fourth) data column presents the *t*-statistic from Diebold-Mariano-West tests of equal predictive accuracy compared to a random walk. Double (single) asterisk indicates statistical significance at the 5% (10%) level using two-tailed tests. The 3 PC row uses a regression with the level, slope, and curvature to predict future changes. The *USV* superscript denotes an affine model with USV constraints imposed. The in-sample period is June 1997 to June 2006 and the out-of-sample period is from July 2006 to January 2016.

Panel A: *One-month-ahead yield changes*

	In		Out	
	RMSE	t_{DMW}	RMSE	t_{DMW}
$A_1(4)$	27.0	2.50**	30.8	0.76
$A_2(4)$	27.2	1.81	31.0	0.44
$A_1(4)^{USV}$	28.0	0.51	31.3	−0.02
$A_2(4)^{USV}$	29.0	−0.80	31.6	−0.52
3 PC	27.6	1.38	35.3	−1.03
Random walk	28.5		31.3	

Panel B: *12-Month-ahead yield changes*

	In		Out	
	RMSE	t_{DMW}	RMSE	t_{DMW}
$A_1(4)$	76.0	2.21**	76.4	1.95*
$A_2(4)$	77.0	1.99**	77.9	1.77*
$A_1(4)^{USV}$	86.1	0.57	90.5	−0.06
$A_2(4)^{USV}$	89.1	−0.22	92.5	−0.42
3 PC	59.8	2.03**	97.5	−1.04
Random walk	88.2		90.3	

for both the in- and out-of-sample periods. For the out-of-sample period, we use the model estimated on the in-sample period and computed the implied forecasted changes in rates based on the states extracted in the out-of-sample period. For comparison, we also compute the root-mean-square forecast errors for a random walk. Additionally, for predicting changes in swap-implied zeros, we consider the predictions of a standard three-factor Gaussian model ($A_0(3)$) estimated using yield data only. We estimate the model using the in-sample period as in the main estimation except that we do not fit the swaption implied volatilities.

The results show, unsurprisingly, that all the models are able to beat a random walk for the in-sample period. Out-of-sample, the fit of all of the models declines. The unconstrained models maintain lower root-mean-square errors than the random walk out-of-sample for nearly all zero rates and implied volatilities. The models with USV constraints imposed have generally worse forecasting ability out-of-sample. We additionally conduct a Diebold-Mariano-West (DMW) test (Diebold and Mariano, 1995; West, 1996) based on the root-mean-square errors (that is, a test of whether the models provide superior forecasting ability versus a random walk). We compute the spectral density at frequency zero using a Newey and West (1987) estimator with 12 lags.

Table 11 shows that the unconstrained models generally provide superior forecasting ability for predicting changes in yields in- and out-of-sample. Unsurprisingly, the DMW test (uncorrected for the in-sample overfit), rejects the null of equal forecasting ability for the uncontrained models in-sample. As is commonly observed (see, e.g., Diebold and Li, 2006), at a one-month horizon, we cannot reject the null of equal forecasting ability with a random walk for any of the models. At a one-year horizon, we see that both unconstrained models reject the null of equal forecasting power at the 10% level.

As additional analysis, we also conduct several additional tests to examine results using a wide array of maturities and horizons. The results generally extend if we consider other yield maturities and horizons between one month and one year. For almost all maturities and horizons, the unconstrained models have lower RMSEs than a random walk. Generally, for longer (shorter) horizons, there is evidence (less evidence) that we can reject the null of equal forecasting ability between the models and a random walk. This holds for most maturities except less so for the six-month zero.

The results are also broadly similar if we consider predicting changes in the implied volatilities (for various underlying maturities and expirations) at different horizons. One key difference between the yields and the implied volatilities is that there was a marked increase in the volatilities of implied volatilities between the in-sample and out-of-sample periods. One measure of this is the increase in the RMSE of a random walk going from the in-sample to the out-of-sample period. For example, for the one-year into five-year swaption, the RMSE increases from 2.20% in-sample to 3.78% out-of-sample. A result of this is that simple time-series models such as an AR(1) actually perform quite a bit worse out-of-sample than a random walk due to inaccurate forecasts of large changes in implied volatilities. The estimated models (with or without the USV constraints) perform substantially better than s simple AR(1) model at all horizons. However, even at the one-year horizon there is not enough evidence to reject a null of equal forecasting ability with a random walk. However, using a loss function in the DMW test based on mean absolute error (which is less sensitive to outliers) allows a significant rejection of the null of equal forecasting ability with a random walk for the one-year horizon for several specifications.

Next, we turn to consider the ability of the models to match conditional second moments in- and out-of-sample. The fact that the (forward-looking) volatility of the zero rates is not observed complicates inference on the ability of the models to match conditional second moments. To assess the ability of the model to fit conditional moments, we follow Patton (2010), Meddahi (2002), and others and assess the ability of the models to predict a conditionally unbiased volatility proxy. As a baseline, $h_{1,\,t}$, the (scaled) past-month realized daily variance, is used as a predictor of the future conditional variance, and we then test whether the models provide a superior forecast relative to this baseline. Specifically, we use $\hat{\sigma}_t^2$, the subsequent squared monthly change in the yield, as a proxy for the unobserved conditional variance. From the model,

Table 12
Model fit to conditional second moments.
This table presents the t-statistics from Diebold-Mariano-West tests of equal predictive accuracy for predicting future variance by a 30-day rolling window forecast and by forecasts from estimated affine term structure models for the 10-year swap-implied zero rate. Future variance is proxied by the square return and the loss function is given by QLIKE. Double (single) asterisk indicates statistical significance at the 5% (10%) level using two-tailed tests. The *USV* superscript denotes an affine model with USV constraints imposed. The in-sample period is June 1997 to June 2006 and the out-of-sample period is from July 2006 to January 2016.

	$A_1(4)$	$A_2(4)$	$A_1(4)^{USV}$	$A_2(4)^{USV}$
In -sample	1.78*	1.82*	1.42	1.32
Out-of-ample	2.03**	2.10**	1.79*	1.68*
Full sample	2.50**	2.61**	2.16**	2.02**

we compute the conditional variance of the yields: $h_{2,t} \equiv E_t[(y^m_{t+1} - \bar{y}^m_{t+1})^2]$, which is readily found as the solution to an ordinary differential equation. We then test using a DMW test whether $E[L(\sigma_t^2, h_{1,t})] = E[L(\sigma_t^2, h_{2,t})]$. Here we use the QLIKE loss function of $L(\hat{\sigma}^2, h) = \log(h) + \hat{\sigma}^2/h$. Patton (2010) shows that this loss function has several appealing theoretical properties and is robust to extreme observations.

The results of this analysis for the ten-year swap-implied zero is in Table 12. The reported DMW statistic is positive when the term structure model provides a better fit than the prior month daily realized variance (i.e., whether the loss function is lower for the term structure model). Over all the sample periods, the evidence is generally consistent with the fact that the models are able to match the conditional second moments better than the realized variance, although it is not always possible to reject the null hypothesis. Results are similar for other maturities. These findings, together with the results on the conditional first moments, show that the models are able to capture a number of features of the dynamics of interest rates and implied volatilities.

We now further examine the validity of the model implication shown in Section 6 relating volatility and bond returns. This section reconsiders this examination in a model-free way through a regression framework. In particular, here we consider whether a variance factor has incremental ability to predict bond returns in- and out-of-sample for various holding periods. For this, we construct h-month realized log excess returns for the m-month zero coupon bond as

$$r_{t,t+h} = -(m-h)y^{m-h}_{t+h} + my^m_t - y^h_t. \tag{15}$$

We then consider whether a variance factor incrementally predicts bond returns relative to the level, slope, and curvature of the yield curve. To do this, we convert the three-months into five-years swaption implied volatility to variance factor rather than using volatilities themselves, following the implication of the model.[14] In particular, we conduct the variance factor as follows. FIrst, we convert the implied volatility of the in 3-months-for 5 years swaption (which is implicitly measured in logs) to a realized variance by first multiplying by the level of the 5-year rate and then squaring. Second, for each month we compute the variance of the daily changes in the 5-year rate over the prior month. The variance factor is then the difference between these implied and realized variance measures. Alternatively, one could construct a measure based on model-free estimates of implied volatility (and variance) as in Britten-Jones and Neuberger (2000). This would provide a cleaner implied volatility measure that is free from modeling assumption implicit in the implied volatility.

The results are shown in Table 13 for the excess return of holding a ten-year zero coupon bond for a 12 month holding period.[15] Standard errors are computed as in Newey and West (1987) using 12 lags. With the exception of the univariate regression in the in-sample period, the variance factor predicts excess returns with at least a 10% significance level and typically, a 5% significance level. This applies as well even when controlling for the level, slope, and curvature factors. Similar results obtain for excess returns for other maturities and holding periods (though the statistical significance and magnitude attenuate for shorter holding periods). These results validate the implications of the estimated model in Section 6 and show that the variance factor is incrementally informative about future bond returns.

We now turn to investigate some structural causes of the reduced form relationships that we have documented. In order to understand some of the sources of variation in interest rate volatility, we consider the relationship between interest rate volatility and the macroeconomy. More specifically, our model so far studies the properties of interest rate volatility in a reduced form manner. We add to this by examining a source of time series variation in interest rate volatility. Because of the major role of the corporate sector in the macroeconomy including in affecting monetary policy, we analyze how the corporate sector is linked to interest rate volatility. In standard economics models, both the risk free rate and the overall discount rate are endogenously determined in equilibrium. Higher uncertainty can lead to lower interest rates through the precautionary savings channel (decreased inflationary pressures stemming from reduced consumption and increased saving). On the other hand, higher uncertainty may lead to higher interest rates and higher cost of capital through changes in investment opportunity sets. In a real options model of investment, higher uncertainty can make real options more valuable and increase asset values with a concurrent increase in volatility. Given that higher uncertainty can be driven by higher cross-sectional dispersion in corporate unexpected performance, we operationalize the real options investment channel by examining how interest rate volatility is linked to cross-sectional dispersion in firms earnings information. To conduct this analysis, we obtain quarterly accounting variables from Compustat and use all available data with consecutive observations

[14] This also follows the equity literature as in, for example, Bollerslev, Tauchen, and Zhou (2009).

[15] Generally, results are similar, though statistical significance is attenuated for a one-month holding period.

Table 13
Volatility and interest rate risk.
This table presents the ability of term structure factors (PCs1–3: level, slope, and curvature) and a variance factor to predict 12-month holding period excess returns for holding a ten-year zero coupon bond. The variance factor is computed by converting the implied volatility of the 3-months into 5-year swaption. Standard errors are based on Newey and West (1987) with 12 lags. Double (single) asterisk indicates statistical significance at the 5% (10%) level using two-tailed tests. The in-sample period is June 1997 to June 2006 and the out-of-sample period is from July 2006 to January 2016.

Panel A: *In-sample period*					
Variance factor	−116*	−131**	−114**		
	(62)	(45)	(37)		
PC1			0.095		0.118
			(0.078)		(0.087)
PC2		0.242	0.435	0.117	0.381
		(0.38)	(0.31)	(0.46)	(0.34)
PC3			0.131		−0.050
			(1.31)		(1.3)
Panel B: *Out-of-sample period*					
Variance factor	−96.7	−87.2*	−110**		
	(62)	(51)	(43)		
PC1			−0.182**		−0.17**
			(0.070)		(0.069)
PC2		−0.100	−0.253	−0.303	−0.498
		(0.40)	(0.28)	(0.42)	(0.32)
PC3			0.81		0.81
			(1.0)		(1.1)
Panel C: *Full sample*					
Variance factor	−105**	−103**	−123**		
	(43)	(37)	(31)		
PC1			−0.066		−0.047
			(0.058)		(0.061)
PC2		−0.027	−0.004	−0.149	−0.149
		(0.31)	(0.29)	(0.34)	(0.33)
PC3			−0.055		−0.225
			(0.79)		(0.85)

when considering the intersection of available Treasury rates and Compustats earnings. First, tracking the time series of interest rate volatilities (untabulated for brevity) reveals that (a) volatilities across different horizons are highly correlated, with slightly higher volatilities for short-term maturity horizons, and (b) volatilities spike around recessions, demonstrating the sensitivity of yields to economic uncertainty such as recessions. Second, we calculate cross-sectional dispersion in firms earnings information by measuring each firms scaled earnings information (scaled quarterly income before extraordinary items) in each quarter and then computing the cross sectional dispersion as the standard deviation of the cross section of earnings changes (year-over-year changes in scaled earnings) across firms in each period. To avoid negative denominator problems, we scale earnings by sales. Time series of dispersion in earnings information are calculated every month throughout the year based on the cross-sectional standard deviation across all firms with fiscal quarter ends falling in that month. We then estimate regression models of future one-month and two-months interest rate volatility for each horizon in Federal Reserve H.15 reports on current period dispersion in earnings information.

The main takeaway from these regression results, untabulated for brevity, is that there is a strong link between corporate accounting information and future interest rate volatilities, especially over the short to middle terms. In particular, the cross-sectional earnings information dispersion is significantly tied to subsequent interest rate volatility, which is consistent with the real options investment channel.

9. Conclusion

This paper provides theory and evidence showing that when the covariance structure of risk factors and market prices of risk are not restricted, low-dimensional dynamic term structure models are able to simultaneously capture the price dynamics in bond and bond option markets. We show that under parsimonious conditions there can exist a residual component of volatility risk largely uncorrelated with yield changes. This residual volatility risk is an important determinant of the risk premium that investors demand for holding long maturity bonds. We find empirical evidence rejecting conditions for unspanned volatility. We find evidence for a component of volatility that has very little effect on the cross-section of bond yields due to having a small effect on risk-neutral expectations of future short rates and inducing little variation in convexity effects. We show that our model is able to capture conditional first and second moments in the data and provide model-free regression analysis demonstrating the role of volatility in bond risk premia. This paper also shows an important link between corporate performance and interest rate volatility. Finally, we develop computational methods for pricing options and extend the technique to provide maximum likelihood estimation of general affine diffusions which can be used in a number of contexts.

Appendix A. Model specification

For the affine term structure model

$$r_t = \rho_0 + \rho_1 \cdot X_t,$$
$$dX_t = \mu_t dt + \sigma_t dB_t,$$
$$\mu_t = K_0 + K_1 X_t,$$
$$\sigma_t \sigma_t' = H_0 + H_1 \cdot X_t,$$

where

$$K_1 = \begin{bmatrix} K_V & 0 \\ K_{VG} & K_G \end{bmatrix}, \quad H_0 = \begin{bmatrix} 0 & 0 \\ 0 & \Sigma_0^G \end{bmatrix}, \quad H_1^i = \begin{bmatrix} \Sigma_i^V & 0 \\ 0 & \Sigma_i^G \end{bmatrix},$$

with K_V an $M \times M$ matrix, K_G and Σ_i^G $(N-M)\times(N-M)$ matrices, and Σ_i^V $M \times M$ matrices. The *drift-normalized* canonical representation is as follows. For the parameters $\Theta = (\rho_0, \rho_1, K_0^P, K_1^P, K_0^Q, K_1^Q, C_{M+1}^G, C_{M+2}^G, \ldots, C_N^G)$, impose the constraints:

1. K_G^Q is diagonal with entries increasing on the diagonal.
2. C_i^G is lower triangular and gives the Cholesky factorization of Σ_i^G: $\Sigma_i^G = (C_i^G)^T (C_i^G)$.
3. $K_{0,n}^Q = 0, \ n > M$.
4. $\Sigma_{i,jk}^V = 1$, if $j = k = i$, or 0 otherwise.
5. $\rho_{1,n} = 1, \ n > M$.
6. $\rho_{1,n} < \rho_{1,n+1}, \ n < M$.
7. $K_{V,ij}^P > 0, \ K_{V,ij}^Q > 0$ if $i \neq j$.
8. $K_{0,n}^P \geq \frac{1}{2}$, $K_{0,n}^Q \geq \frac{1}{2}$, $n \leq M$.

Appendix B. Pricing

This appendix presents a computationally efficient method for computing the transform given in Duffie, Pan, and Singleton (2000):

$$\begin{aligned} G(y) &= E^Q[e^{-\int_0^P r_\tau d\tau + d \cdot X_T}\{\delta \cdot X_T \leq y\}], \\ \hat{G}(y) &= E^Q[e^{-\int_0^P r_\tau d\tau + d \cdot X_T + i\delta \cdot X_T}], \\ G(y) &= \frac{\hat{G}(0)}{2} - \frac{1}{\pi}\int_0^\infty \frac{1}{t}\mathrm{Im}(\hat{G}(t)e^{ity})dt. \end{aligned} \tag{A1}$$

In computing the transform, we can use the fact that $Y = \delta \cdot X_P$ is roughly normally distributed under the forward measure, F, where

$$\frac{dF}{dQ} = \frac{e^{-\int_0^P r_\tau d\tau + d \cdot X_T}}{E^Q[e^{-\int_0^P r_\tau d\tau + d \cdot X_T}]}. \tag{A2}$$

More precisely, $\hat{G}(t) \approx c e^{-\sigma_Y^2 t^2/2 + it\mu_Y}$.[16] In the case of an $A_0(N)$ Gaussian model, this equation is exact. Considering this case for now, the Levy integral then becomes:

$$\begin{aligned} I &= \int_0^\infty \frac{1}{t}\mathrm{Im}(\hat{f}(t)e^{-ity})dt \\ &= \int_0^\infty \frac{1}{t}\mathrm{Im}(e^{-\frac{\sigma^2 t^2}{2}} e^{it\mu} e^{-ity})dt \\ &= \int_0^\infty \frac{\sin(t(\mu - y))}{t} e^{-\frac{\sigma^2 t^2}{2}} dt \\ &= \int_0^\infty g(t)w(t)dt, \end{aligned}$$

where $w(t) = e^{-\sigma^2 t^2/2}, g(t) = \sin((\mu - y)t)/t$. $w(x)$ is a scaling of the weighting function e^{-t^2} used in Gauss-Hermite quadrature. By using flexibility in both the choice of nodes and weights, Gauss-Hermite quadrature allows very accurate computations for integrals of the form $\int g(t)e^{-t^2}dt$ with very few nodes. This suggests that, after appropriate scaling, Gauss-Hermite quadrature will be an accurate way to compute the inversion integral.

In general, we can write the Levy integral as:

$$\begin{aligned} I &= \int_0^\infty \underbrace{\frac{1}{t}\mathrm{Im}(\hat{f}(t)e^{-ity}e^{\sigma^2 t^2})}_{g(t)} e^{-\sigma^2 t^2} dt \\ &\approx \sum_i g(t_{i,\sigma}) w_{i,\sigma}. \end{aligned}$$

Two points also become clear:

1. Scale matters. If we are computing the transform integral, we must integrate on approximately $t \in [-\frac{3}{\sigma}, \frac{3}{\sigma}]$ before rescaling. This means if we are directly computing this integral and we are using options with various maturities (so that σ will vary), *any* quadrature scheme must take this into account.
2. Out-of-moneyness increases oscillation of integrand. By rescaling to change the integral to:

$$I = \int_0^\infty \sigma \frac{\sin(\frac{\mu - y}{\sigma}u)}{u} e^{-u^2/2} du$$

we see that the integral will have a weighting function times a decaying oscillatory terms. The frequency of oscillation increases as we move more standard deviation for μ.

B.1. Example

We now turn to an example of computing forward probabilities. Consider the risk-free $A_1(2)$ term structure model in Duffie, Pedersen, and Singleton (2003). To emphasize the generality of the approach, we augment the model with jumps occurring with intensity $\lambda = 1$ of size $\pm 1.5\%$ in the short rate.

We then compute a term involved in pricing a zero coupon bond option:

$$E_0\left[e^{-\int_0^T r_s ds} e^{B(\tau) \cdot X_T}\left\{-\frac{A(\tau) + B(\tau) \cdot X_T}{\tau} \geq f_0 + m\right\}\right].$$

Here, τ is the maturity of the underlying zero coupon bond (which has log price $A(\tau) + B(\tau) \cdot X_0$ when the state is X_0), T is the expiry of the option, and f_0 is the corresponding forward rate with m a moneyness adjustment. We compute this term for an option on a 5-year zero coupon bond with expiry of six months ($T = 0.5, \tau = 5$.) The strikes are adjusted from the corresponding forward rate of 7.27%. The initial state was taken to be the long run mean, $X_0 = \theta_P$.

[16] It is important to note that a smooth density function implies fast decay of the Fourier transform.

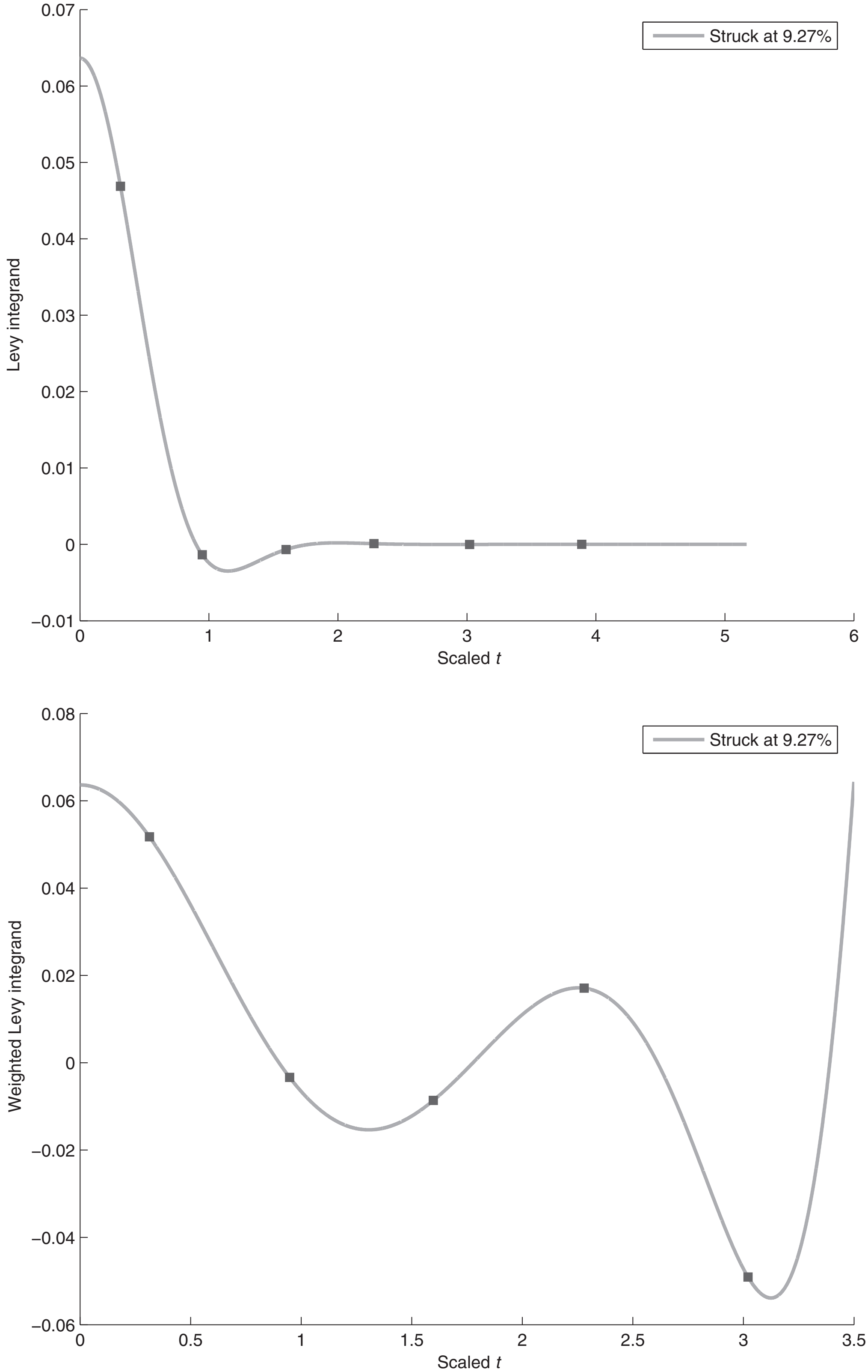

Fig. A1. Levy integrand. The top panel plots the Levy-integrand used in computing the value of an out-of-the-money bond option. The integrand is approximately a scaled normal density times an oscillatory function. The bottom panel plots the oscillatory multiplier by weighting the integrand. The squares indicate nodes used in Gauss-Hermite quadrature.

Table A1
Accuracy of quadrature.
This table presents the accuracy of the adaptive Gauss-Hermite quadrature scheme for computing option prices. The first column gives the strike of the 6-month option on 5-year zero coupon bond. The exact price of the option is given in the second column and the number of standard deviations (under the risk-neutral measure) the option is out-of-the-money is given in the third column. The remaining columns give the accuracy of the quadrature scheme.

Strike	Price	n_σ	3 Nodes	5 Nodes	8 Nodes
7.27%	30.55	0.03	0.00029	1.02e−006	8.47e−009
8.27%	6.60	1.24	0.00409	2.18e−005	3e−008
9.27%	0.55	2.46	0.178	0.00247	7.49e−006

Fig. A1 shows the integrand (scaled by σ) in the Levy inversion integral for the various strikes. In each case the integrand can be seen to be $e^{-x^2}g(t)$ where $g(t)$ is a decaying oscillatory function which is more oscillatory the more the option is out-of-the money. The left panels plot the integrand itself, where the right panel plots $g(t)$. The figure also plots in the nodes used for the quadrature with $n = 5$ nodes.

Another method of computing the forward probability $P(b \cdot X_T \leq y)$ would be to use a cumulant expansion for the random variable $b \cdot X_T$.[17] This amounts to doing a Taylor series expansion of the right-hand panel. As can be seen, when the option is near the money, the Taylor series will be accurate, except for large values of t which are given little weight in the integral. However, as the options become more out-of-the money (the lower panel), the Taylor series approximation will become inaccurate. In contrast, the quadrature scheme is able to both pick up the oscillatory nature of the integrand and focus on the region which is important for the integral.

Table A1 reports the accuracy of the quadrature for various numbers of nodes. The reference value was computed using Simpson's rule with 10,000 nodes spaced on the interval [0, $6/\sigma$]. The variable n_σ measures how many standard deviations (under the forward measure) the option is out-of-the money. For a fixed number of nodes, the accuracy decays as the option goes out-of-the money since the Levy integrand becomes more oscillatory. For options which are within a standard deviation of being at-the-money, the quadrature scheme is quite accurate with even just three nodes.

Appendix C. Computing exact likelihood

Because the conditional characteristic function is known in terms of the solution to an ordinary differential equation, the transition likelihood for an affine diffusion can be recovered from the characteristic function:

$$\hat{f}(s|x_t) = E[e^{is \cdot X_{t+1}}|X_t = x_t]$$
$$f(x_{t+1}|x_t) = \frac{1}{(2\pi)^n}\int \hat{f}(s|x_t)\, e^{-ix_{t+1} \cdot s} ds. \tag{A3}$$

[17] The cumulants will be affine in the state and can be obtained by repeatedly differentiating the original Riccati equation. In the case of a forward measure, the cumulants can again be computed by differentiating a different Riccati equation.

However, direct computation of this integral is often intractable. Two ideas are used in order to simplify the computations involved. First, the integrand in the inverse Fourier transform of a transition becomes more oscillatory as the transition varies from the expected transition. In order to remove the oscillations, a transition measure is defined where the observed transition becomes a likely transition. That is,

$$\hat{f}(s) \approx e^{i\mu_t \cdot s - \frac{1}{2}s^\top \Sigma_t s}$$
$$\mathrm{Real}\left(\hat{f}(s)e^{-is \cdot x_{t+1}}\right) \approx \mathrm{Real}\left(e^{i(\mu_t - x_{t+1})\cdot s - \frac{1}{2}s^\top \Sigma_t s}\right)$$
$$= \cos(s \cdot (\mu_t - x_{t+1}))e^{-\frac{1}{2}s^\top \Sigma_t s}.$$

If we define $dT/dP = e^{a \cdot X_{t+1}}/E_t[e^{a \cdot X_{t+1}}]$, under T, $E_t[X_{t+1}] \approx \mu + \Sigma a$.[18] So, by choosing $a = \Sigma^{-1}(x_{t+1} - \mu)$, $E_t^T[X_{t+1}] \approx x_{t+1}$ and

$$f(x_{t+1}|x_t) = f^T(x_{t+1}|x_t) \times \frac{dP}{dT}(x_{t+1}).$$

After this change of measure, $\hat{f}^T \approx e^{ix_{t+1}\cdot s - \frac{1}{2}s^\top \Sigma_t s}$ and so the integrand in the inverse Fourier transform to compute $f^T(x_{t+1})$ is approximately $e^{-\frac{1}{2}s^\top \Sigma s}$. Thus, the integral

$$f^T(x_{t+1}) = \frac{1}{(2\pi)^n}\int \hat{f}^T(s)e^{-is\cdot x_{t+1}} ds$$
$$= \frac{1}{(2\pi)^n}\int w(s)e^{-\frac{1}{2}s^\top \Sigma s} ds,$$

where $w(s) = e^{-\frac{1}{s}^\top \Sigma s}/\hat{f}^T(s)e^{-is\cdot x_{t+1}} ds \approx 1$. This multidimensional integral is suitable to be evaluated by Gauss-Hermite quadrature with very few nodes. Also, since the integrand is odd, only half of the evaluations need actually be done. Though the curse of dimensionality is still present, the computation now become tractable since even four nodes give reasonable accuracy. Evaluating the inverse transform for rare transition with highly oscillatory inverse Fourier transform integrands would require solving hundreds of millions of differential equations (4^4 versus 100^4, for example).

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[18] Note that when there are Cox-Ingersoll-Ross factors we must consider that $E_t[e^{a \cdot X_\tau}]$ is finite for all τ only when a is in the domain of attraction of the fixed point of the affine differential equation. However, even when this is not the case the expectation will be finite for τ small and calculations show this range is reasonably large when boundary non-attainment is enforced.

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Online Appendix to

"Interest Rate Volatility, the Yield Curve, and the Macroeconomy"

Scott Joslin and Yaniv Konchitchki*

These notes supplement Joslin and Konchitchki (2017). Joslin and Konchitchki (2017) considers the residual variance – the portion of the variance of the five-year zero rate that is uncorrelated with the level, slope, and curvature of the yield curve. In the macro-finance model described in the paper, there are stochastic correlations among all of the risk factors.

Figure 1 plots the time series of the local correlation of the residual variance with the total variance of the 5-year yield for the $A_1(4)$ model. Note that a high correlation between the variance and residual variance indicates that the yield curve, as summarized by its level, slope, and curvature, explains a small part of the variation in the yield volatility. The correlation typically ranges from 40% to 60%, thus indicating that about half of the variation in the variance of the 5-year yield is accounted for by factors unrelated to the level slope and curvature of the yield curve. A notable exception it that in late 1998, around the time of the Russian default and the LTCM bailout, the model indicates that the variance was nearly perfectly correlated with the residual variance. This indicates that in this period movements in the volatility of the yield curve were almost completely uncorrelated to movements in the level, slope, and curvature of the yield curve.

In our model and its estimation, we study the properties of interest rate volatility as if interest rate volatility is a given construct. We also examine a source of time series variation in interest rate volatility. Because of the major role of the corporate sector in the macroeconomy including in affecting monetary policy (Fischer and Merton, 1984). As we describe, we employ a real options model of investment. In a real options model of investment, higher uncertainty can make real options more valuable and increase asset values with a concurrent increase in volatility. Given that higher uncertainty can be driven by higher cross-sectional dispersion in corporate unexpected performance, we operationalize the real options investment

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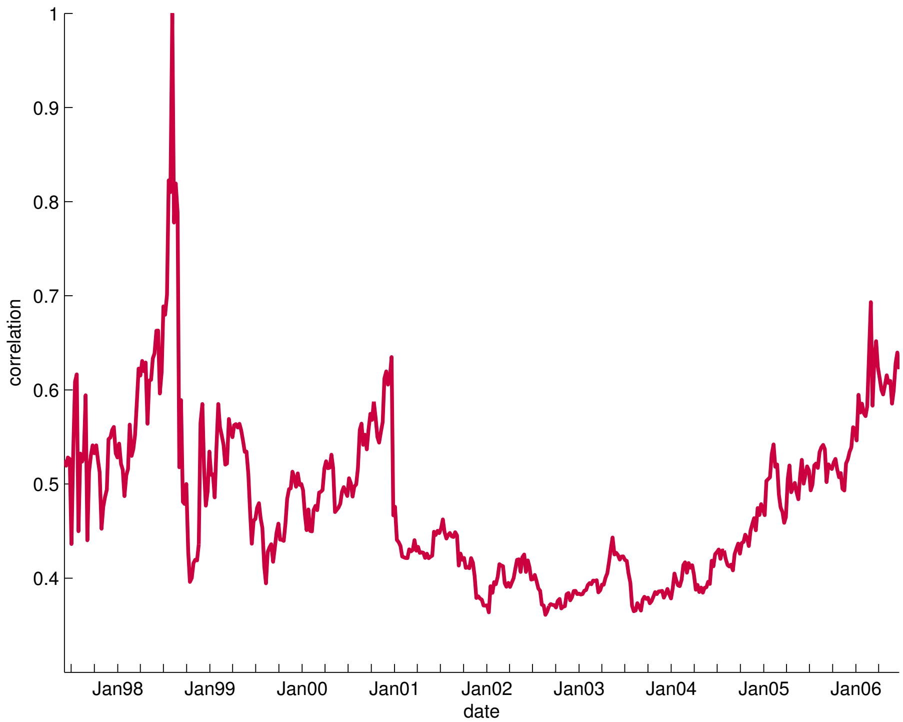

Figure 1: Correlation of variance with residual variance

This figure plots the correlation of the variance of the 5-year yield with the residual variance for the $A_1(4)$ model. The residual variance is defined as the risk which is locally uncorrelated with the 6-month, 2-year, and 10-year yields. A correlation of 1 between the variance and residual variance indicates that 6-month, 2-year, and 10-year yields are uncorrelated with volatility.

channel by examining how interest rate volatility is linked to cross-sectional dispersion in firms earnings information. Indeed, in standard economics models, both the risk free rate and the overall discount rate (or cost of capital) are endogenously determined in equilibrium. Higher uncertainty can lead to lower interest rates through the precautionary savings channel (decreased inflationary pressures stemming from reduced consumption and increased saving). On the other hand, higher uncertainty may lead to higher interest rates and higher cost of capital through changes in investment opportunity sets. In sum the empirical analysis focuses on how the corporate sector is linked to interest rate volatility.

First, Figure 2 plots time series of interest rate volatilities calculated as the monthly standard deviation of daily interest rate yields on Treasury bonds and bills for all horizons available from the Federal Reserve Board of Governors H.15 Reports. The figure shows that the volatilities across different horizons are highly correlated, with slightly higher volatilities for short-term maturity horizons. The figure also shows that volatilities spike around recessions, demonstrating the sensitivity of yields to economic uncertainty such as recessions.

The next analysis focuses on the accounting-volatility link, where again we operationalize the real options investment channel by examining how interest rate volatility is linked to cross- sectional dispersion in firms' earnings information. Cross-sectional dispersion in firms' earnings information is the standard deviation of each quarterly cross section of all quarterly earnings changes in the period. We measure accounting earnings for firm i in quarter q ($Earn_{i,q}$) as scaled quarterly income before extraordinary items, and earnings change ($\Delta Earn_{i,q}$) as the year-over-year change in $Earn_{i,q}$. To mitigate possible outlier effects, we trim each firm's $Earn_{i,q}$ and $\Delta Earn_{i,q}$ based on the top and bottom one percentile of each periodic cross-section. To avoid negative denominator problems, we scale earnings by sales.

Time series of dispersion in earnings information are calculated every month throughout the year based on the cross-sectional standard deviation across all firms with fiscal quarter ends falling in that month. We then estimate regression models of future one-month and two-months interest rate volatility for each horizon in the Fed H.15 reports on current period dispersion in earnings information.

Table 1 reports these regressions results. The major finding is that there is a strong link between corporate accounting information and future interest rate volatilities, especially over the short to middle terms. In particular, the cross-sectional earnings information dispersion is significantly tied to subsequent interest rate volatility,which is consistent with the real options investment channel.

We also conduct two additional analyses. First, in the regressions we employ additional subsequent horizons of up to four periods ahead. Second, we use IBES data to measure the dispersion in earnings information. In this case we obtain analyst earnings forecasts from the IBES Summary History Summary Statistics with Actuals (EPS for U.S. Region) dataset.

We use a firm's analyst median earnings-per-share consensus forecast as the analyst forecast of the firm, which is then multiplied by the number of shares outstanding, adjusted for stock splits and stock dividends, on the last day of the period for which the forecast is calculated. We then subtract the actual earnings minus its expectation for each firm. The inferences are unchanged as a result of these additional analyses.

The results of Joslin and Konchitchki (2017) (and related results of Cochrane and Piazzesi (2008), Ludvigson and Ng (2000), Rudebusch et al. (2006), Wright and Zhou (2009) and others) suggest that the cross-section of bond yields may poorly identify expected excess returns for holding long maturity bonds. This raises the question of whether expected excess returns may be accurately identified using all fixed income security prices – including fixed income derivatives. To see why it may be the case that expected excess returns are not identified from prices, consider the simple two-factor short rate model where the short rate, r_t, is simply the sum of observable inflation, i_t, and a nominal-real spread, s_t. Suppose that the state variable $X_t = (i_t, s_t)$ follows the diffusion

$$dX_t = \begin{bmatrix} .2 & 0 \\ 0 & .1 \end{bmatrix} \Big(\theta - X_t\Big)dt + \begin{bmatrix} \sigma_i & 0 \\ 0 & \sigma_s \end{bmatrix} dB_t^P \tag{1}$$

Suppose also that the degree to which agents are risk averse to inflation risk depends on the current level of inflation ($\lambda_t = \lambda_0 + \lambda_1 i_t$) so that under Q:

$$dX_t = \begin{bmatrix} .1 & 0 \\ 0 & .1 \end{bmatrix} \Big(\theta - X_t\Big)dt + \begin{bmatrix} \sigma_i & 0 \\ 0 & \sigma_s \end{bmatrix} dB_t^Q \tag{2}$$

In this case, risk-neutrally, i_t and s_t affect fixed income security prices only through their sum and as far as prices are concerned this is a one-factor model. However, knowing both i_t and s_t are informative for learning about expected excess returns for holding long maturity bonds. That is, fixed income security prices (both bonds and derivatives) *do not* determine the state of the economy and *do not* identify expected excess returns.

Indeed, Ludvigson and Ng (2000) find evidence that real and inflation risk factors have the ability to predict variation in bond excess returns above and beyond the level of interest rates. Excess returns will be weakly identified from bond prices when there is a risk factor (after a change of variables) which does not affect Q-expectation of future interest rates but does affect P-expectations of future interest rates. In the example, we change the variable to $Y_t = (r_t, i_t)$. Then fixing r_t and changing i_t do not affect future Q-expectations of interest rates but does

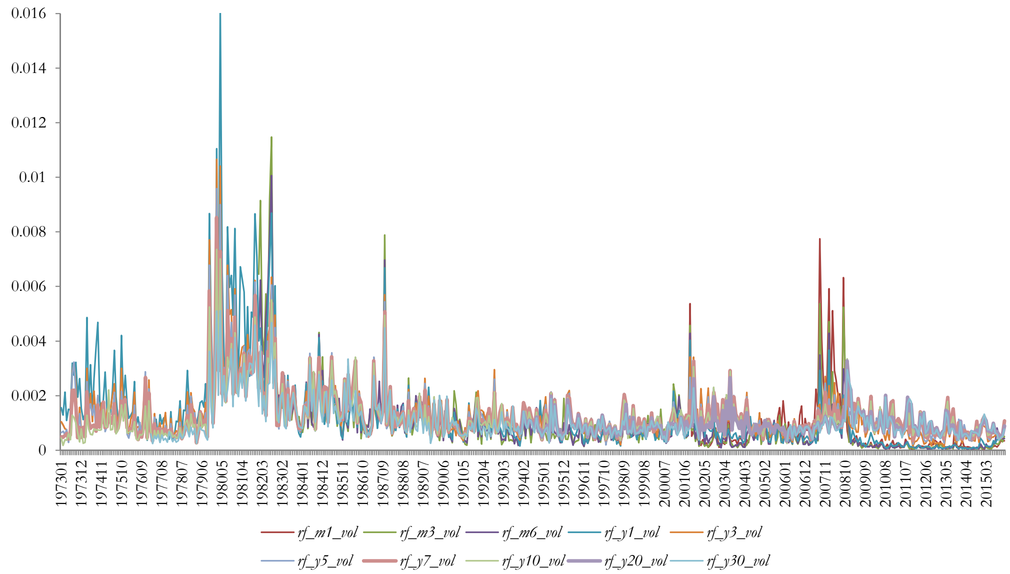

Figure 2: Time series of interest rate volatilities

This figure plots time series of yield volatilities of constant maturity Treasury bonds and bills for all horizons available from the Federal Reserve Board of Governors H15 Reports. We obtain data from the Federal Reserve Board of Governors H15 Reports available on Wharton Research Data Services (WRDS). We use all data with consecutive observations when considering the intersection of Fed's yields and Compustat's accounting earnings (January 1973 to January 2016). The available horizons are for the 1-month (rf_m1), 3-month (rf_m3), 6-month (rf_m6), 1-year (rf_y1), 3-year (rf_y3), 5-year (rf_y5), 7-year (rf_y7), 10-year (rf_y10), 20-year (rf_y20), and 30-year (rf_y30) yields. The suffix "_vol" refers to volatility, calculated as the monthly standard deviation of the daily interest rate yields during that month.

(a) Corporate earnings effect on one month ahead volatilities for different horizons of constant maturity Treasury bills and bonds

	Estimate	StdErr	t-statistic	Adj.R^2
1 Month	0.00052**	0.00025	2.11253	0.08378
3 Month	0.00044**	0.00020	2.19679	0.10889
6 Month	0.00027**	0.00011	2.45221	0.06559
3 Year	0.00018**	0.00005	3.78813	0.04870
5 Year	0.00015**	0.00005	3.01000	0.04082
7 Year	0.00013**	0.00005	2.55691	0.02877
10 Year	0.00011**	0.00004	2.53991	0.02220
20 Year	0.00007*	0.00004	1.93235	0.00712
30 Year	0.00005	0.00004	1.44384	0.00107

(b) Corporate earnings effect on two months ahead volatilities for different horizons of constant maturity Treasury bills and bonds

	Estimate	StdErr	t-statistic	Adj.R^2
1 Month	0.00040**	0.00014	2.76060	0.04413
3 Month	0.00036**	0.00013	2.82923	0.07172
6 Month	0.00032**	0.00011	2.86632	0.09705
1 Year	0.00030**	0.00009	3.36788	0.11335
3 Year	0.00018**	0.00008	2.42936	0.04794
5 Year	0.00015**	0.00006	2.58699	0.03804
7 Year	0.00013**	0.00005	2.59619	0.02833
10 Year	0.00013**	0.00005	2.59764	0.03052
20 Year	0.00009**	0.00004	1.95786	0.01323
30 Year	0.00004	0.00004	1.10180	-0.00311

Table 1: Accounting information and interest rate volatilities. This table provides results from regression models of future interest rate volatilities on cross sectional dispersion in earnings information. Accounting data items are from the Compustat North America Fundamentals Quarterly dataset and yield volatilities are from the Federal Reserve Board of Governors H15 Reports dataset, both available on Wharton Research Data Services (WRDS). Yield volatilities are the constant maturity Treasury bonds and bills for all horizons available from the Federal Reserve Board of Governors H15 Reports. We use all data with consecutive observations when considering the intersection of Fed's yields and Compustat's accounting earnings (January 1973 to January 2016). The available horizons are for the 1-, 3-, and 6-month as well as for the 1-, 3-, 5-, 7-, 10-, 20-, and 30-year yields. We calculate yield volatilities as the monthly standard deviation of the daily interest rate yields during that month, and earnings dispersion information as the periodic cross sectional dispersion in earnings changes.

affect future P-expectations. This is because the drift of r_t is $\kappa\theta - .2i_t - .1s_t = \kappa\theta - .1i_t - .1r_t$.[1] Additionally, there is the possibility that excess returns may be identified accurately not through bond prices but through derivative prices. This will be precluded when there another orthogonality condition on the volatility is obtained.

The econometric method used in Joslin and Konchitchki (2017) precludes such identification of risk premia since it is assumed that the state of the economy can be inferred from fixed income security prices. To allow for such flexibility would require the use of a filtering technique which is complicated both by computational burden and by the non-linearity in the option prices.

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[1]Precisely, the condition in an affine model is the existence of an eigenvector of κ^Q which is orthogonal to ρ_1 but which is not an eigenvector of κ^P