Intersecting families of sets, no l containing two common elements

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Abstract

Let H denote the set $\{f_1, f_2, ..., f_n\}$, $2^{[n]}$ the collection of all subsets of H and $\mathcal{F} \subseteq 2^{[n]}$ be a family. The maximum of $|\mathcal{F}|$ is studied if any k subsets have a non-empty intersection and the intersection of any l distinct subsets $(1 \leq k < l)$ is empty. This problem is reduced to a covering problem.

If we have the conditions that any two subsets have a non-empty intersection and the intersection of any l distinct subsets contains no two different elements we show that the maximum of $|\mathcal{F}|$ is $(l-1)n + o(n)$.

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1 Introduction

Let H denote the set $\{f_1, f_2, ..., f_n\}$ and $2^{[n]}$ the collection of all subsets of H. A simple theorem of Erdős, Ko and Rado [4] says that the maximum of $|\mathcal{F}|$ is 2^{n-1}, if every two members of a family $\mathcal{F} \subseteq 2^{[n]}$ have a non-empty intersection. An optimal construction consists of all the subsets containing one fixed element. Andrew Szilard [5] has asked what the maximum of $|\mathcal{F}|$ is if every two subsets have a non-empty intersection and no three have a common element. We solve this problem with an easy dual approach.

Section 2 is about a more general problem: what is the maximum of $|\mathcal{F}|$ if any k members of $\mathcal{F}$ have a non-empty intersection and the intersection of any l members is empty. With the dual approach mentioned above we can reduce this problem to a covering problem.

The main result of the present paper is in Section 3. We prove that $|\mathcal{F}|$ is at most $(l-1)n+o(n)$ if any two members of $\mathcal{F}$ have a non-empty intersection but the intersection of any l distinct members contains at most one element. This generalizes (in an asymptotic sense) the well-known theorem [3] stating that the number of subsets (of an n-element set) with pairwise intersection of size exactly one cannot exceed n.

Thorough the paper we will consider families $\mathcal{F}$ consisting of different subsets.

2 k-intersecting, l-non-intersecting families

Given the integers $1 < k < l$ suppose that the following two properties hold:

$$F_1 \cap \ldots \cap F_k \neq \emptyset, F_1, \ldots, F_k \in \mathcal{F} \tag{2.1}$$

$$F_1 \cap \ldots \cap F_l = \emptyset, F_1, \ldots, F_l \in \mathcal{F}, \tag{2.2}$$

where $F_1, ..., F_l$ are distinct in (2.2). Let $f(n, k, l)$ denote the maximal size of $\mathcal{F}$ satisfying the above conditions.

Definition 2.1 *Call the $n \times |\mathcal{F}|$ matrix $A = (a_{ij})$ the characteristic matrix of the family $\mathcal{F}$ if $a_{ij} = 1$ when $f_i \in F_j$ otherwise $a_{ij} = 0$ ($1 \leq i \leq n$, $1 \leq j \leq |\mathcal{F}|$).*

Conversely call $\mathcal{F}$ the family defined by the ($n \times m$) matrix A (consisting of 0s and 1s) if A is the characteristic matrix of $\mathcal{F}$.

2.1 k=2, l=3

First we discuss this special case. The following properties are supposed:

$$F_1 \cap F_2 \neq \emptyset, F_1, F_2 \in \mathcal{F} \tag{2.3}$$

$$F_1 \cap F_2 \cap F_3 = \emptyset, F_1, F_2, F_3 \in \mathcal{F}, \tag{2.4}$$

where F_1, F_2, F_3 are distinct in the latter case. Let A be the characteristic matrix of $\mathcal{F}$. There are at most two 1s in every row because at most two subsets can contain a fixed element by (2.4). But there is a row for any two columns where we have a 1 in that two columns (2.3). One row is associated with at most one pair of columns, so $n \geq \binom{|\mathcal{F}|}{2}$.

We can show a construction for every $m = |\mathcal{F}|$ which satisfies $n \geq \binom{m}{2}$. Pick the subsets with 2 elements of $\{1, 2, \ldots, m\}$ and list them: $C_1, \ldots C_{\binom{m}{2}}$. Let the two elements of C_i be $c_{i,1}$ and $c_{i,2}$. Put two 1s into the ith row of the $n \times m$ matrix A in the $c_{i,1}$th and $c_{i,2}$th column. Finally let $\mathcal{F}$ be the family defined by A. We have proved the following theorem.

Proposition 2.2 *f(n,2,3) is the largest integer m satisfying $\binom{m}{2} \leq n$.* □

2.2 The general case

In the present subsection we suppose that the properties (2.1), (2.2) hold for some given integers $2 \leq k < l$. Let A be the characteristic matrix of $\mathcal{F}$. There are at most $l-1$ 1s in every row because at most $l-1$ subsets can contain a fixed element by (2.2). But there is a row for any k columns where we have a 1 in that k columns (2.1). One row is associated with at most $\binom{l-1}{k}$ k-element set of columns, so

$$n \geq \frac{\binom{|\mathcal{F}|}{k}}{\binom{l-1}{k}}. \tag{2.5}$$

Definition 2.3 *$C(t,h,v)$ =min$\{|\mathcal{B}| : \mathcal{B} \subset \binom{[v]}{h}$ and each $T \in \binom{[v]}{t}$ is contained in some $B \in \mathcal{B}\}$ $(t \leq h \leq v)$.*

The problem of determining $C(t,h,v)$ is the widely investigated covering problem (see e.g. [1]. We need, however a somewhat modified version.

Definition 2.4 *A family* $\mathcal{B} = \{B_1, B_2, \ldots, B_{|\mathcal{B}|}\} \subset 2^{[n]}$ *is a separating system if there is a* k *for every pair* $1 \le i \ne j \le n$ *such that* $b_i \in B_k$,$b_j \notin B_k$ *or* $b_i \notin B_k$,$b_j \in B_k$.

Definition 2.5 $C_{\mathrm{sep}}(t, \le h, v)$ $=$*min*$\{|\mathcal{B}| : \mathcal{B} \subset \binom{[v]}{1} \cup \binom{[v]}{2} \cup \ldots \cup \binom{[v]}{h}$ *is separating and each* $T \in \binom{[v]}{t}$ *is contained in some* $B \in \mathcal{B}\}$ $(t \le h \le v)$.

Remark 2.6 $C_{\mathrm{sep}}(t, \le h, v) \ge C(t, h, v) \ge \frac{\binom{v}{t}}{\binom{h}{t}}$. □

We show that determining $f(n, k, l)$ is the dual form of this modified covering problem. Transposing A we get $\mathcal{F}^T$:= the family defined by A^T. Properties (2.1) and (2.2) mean that every k-element subset in $2^{[|\mathcal{F}|]}$ is contained in some $F \in \mathcal{F}^T$ and $|G| \le l - 1$ holds for any $G \in \mathcal{F}^T$. $\mathcal{F}^T$ is a separating system because we have distinct sets in $\mathcal{F}$. This gives $n = |\mathcal{F}^T| \ge C_{\mathrm{sep}}(k, \le l - 1, |\mathcal{F}|)$.

Conversely if $n \ge C_{\mathrm{sep}}(k, \le l - 1, m)$ let us construct an $\mathcal{F}$ satisfying (2.1), (2.2) and $|\mathcal{F}| = m$. Let $\mathcal{C}$ be a separating covering of minimum size and C its characteristic matrix. Let A be the transposed of C and $\mathcal{F}$ the family defined by A (the columns of C are different since $\mathcal{C}$ is separating so $\mathcal{F}$ is a family of different members). $\mathcal{F}$ is satisfyng properties (2.1) and (2.2) because of the matrix tranposition. We have proved the following theorem.

Theorem 2.7 *f(n,k,l) is not larger than the largest* m *integer satisfying* $n \ge C_{\mathrm{sep}}(k, \le l - 1, m)$ *and this is best possible.* □

By this theorem our problem is reduced to a covering problem.

Definition 2.8 *A Steiner system* $\mathcal{S}(t,h,v)$ $(2 \le t < h < v)$ *is a collection of some* h*-element subsets of an* v*-element set, such that every* t*-element subset of* $2^{[v]}$ *is contained in exactly one* h*-element subset.*

Lemma 2.9 *Every Steiner system* $\mathcal{S}(t,h,v)$ *is separating.*

Proof: Without loss of generality, it is sufficient to see that $f_1 \in [v]$ and $f_2 \in [v]$ are separated. Let $A := \{f_1, \ldots, f_h\} \in \mathcal{S}(t,h,v)$. There is a set $B \in \mathcal{S}(t,h,v)$ which contains $\{f_2, f_3, \ldots, f_t, f_{h+1}\}$ $(h < v)$. $f_1 \notin B$ because only one set contains $\{f_1, \ldots, f_t\}$ and $A \neq B$ since $f_{h+1} \notin A$. We have separated f_1 and f_2 with B. $\square$

It is easy to see that $|\mathcal{S}(t,h,v)| = \frac{\binom{v}{t}}{\binom{h}{t}}$, hence this lemmma implies $C_{\text{sep}}(t, \le h, v) = |\mathcal{S}(t,h,v)| = \frac{\binom{v}{t}}{\binom{h}{t}}$. By this (2.5) is sharp if an appropriate Steiner system exists.

If $k = 2$, it is known [6] that there are many Steiner systems, therefore Theorem 2.5 yields a very good approximation. We believe that the same is true for $k > 2$, however in this case very little is known about the Steiner systems therefore our reduction does not give a final solution.

Of course if $\mathcal{F}$ is not supposed to have different members, then the same statements hold without the separating condition.

3 Intersecting families, no l containing more than one common element

Suppose that the following conditions hold for $\mathcal{F}$ (a family of subsets of H). Any two members of $\mathcal{F}$ have a non-empty intersection, the intersection of any $2 \leq l$ distinct members has at most one common element:

$$F_1 \cap F_2 \neq \emptyset, F_1, F_2 \in \mathcal{F}$$

$$|F_1 \cap \ldots \cap F_l| \leq 1, F_1, \ldots, F_l \in \mathcal{F}$$

where $F_1, \ldots, F_l$ are distinct in the latter condition.

3.1 Upper bound

Let $k(> 0)$ be the size of the smallest subset. First we discuss the case of $k = 1$. Let K be one of the one-element subsets. Let $f_1, \ldots, f_{n-1}$ be the elements of $H' = H \backslash K$. Every subset contains K because any two subsets have a non-empty intersection. Delete K and the element contained in it. Let $\mathcal{F}' = \{F'_1, \ldots, F'_{|\mathcal{F}|-1}\}$ be the rest of $\mathcal{F}$. This is a family on H'. The size of every subset is decreased by one, and any l subsets have an empty intersection, so f_j $(1 \leq j \leq n-1)$ is contained in at most $l-1$ subsets. For every $1 \leq g \leq |\mathcal{F}|-1$ we put a weight $\frac{1}{|F'_g|}$ on every element of F'_g. Since every element is conatined in at most $l-1$ subsets we have at most $l-1$ weights on every element. The two largest weights cannot be both 1s because all subsets are different and two 1s mean two different 1-element subsets. The second largest possible weight is $\frac{1}{2}$, so the sum of the weigths is at most $1 + \frac{l-2}{2} = \frac{l}{2}$ on every element. The sum of the weigths on all elements is at most $(n-1)\frac{l}{2}$, and is equal to $|\mathcal{F}'|$ because the sum of the weigths is 1 for each subset. So $|\mathcal{F}'| \le \frac{l}{2}(n-1)$, which means

$$|\mathcal{F}| \le \frac{l}{2}(n-1) + 1 \le \frac{l}{2}n. \tag{3.1}$$

From now on we will suppose that $k \ge 2$. Choose one of the k-element subsets and denote it by K. Let $a_1, a_2, \dots, a_k$ be the elements of K. Define the following subfamilies:

$$\mathcal{A}_{i,j} = \{F \in \mathcal{F} : a_i \in (F \cap K), |F \cap K| = j\} \ (1 \le i \le k, 1 \le j < k)$$

$$\mathcal{A}_{i,k} = \{F \in \mathcal{F} : a_i \in (F \cap K), |F \cap K| = k, F \ne K\} \ (1 \le i \le k).$$

Let $\mathcal{B}_i$ be equal to $\mathcal{A}_{i,1} \cup \mathcal{A}_{i,2} \cup \dots \cup \mathcal{A}_{i,k}$ and $\mathcal{C}_j = \mathcal{A}_{1,j} \cup \mathcal{A}_{2,j} \cup \dots \cup \mathcal{A}_{k,j}$. In the next subsubsections we will give two upper bounds for $|\mathcal{F}|$ (3.1.1 and 3.1.2), finally we combine these in Subsubsection 3.1.3.

3.1.1 Case of large sets

Fix an integer $1 \le i \le k$ and choose a $y \in (H \backslash K)$. Let $\mathcal{Y} = \{Y : y \in Y \in B_i\}$.

Lemma 3.1 $|\mathcal{Y}| \le l - 1$.

Proof: Suppose on the contrary that there are l distinct subsets $(E_1, \dots, E_l)$ in $\mathcal{F}$, which intersect K in a_i and contain y. Then $|E_1 \cap \dots \cap E_l| \ge 2$. This is a contradiction. □

Since this lemma is true for any $y \in H\backslash K$,

$$\sum_{I\in\mathcal{B}_i} |I \cap (H\backslash K)| \le (l-1)(n-k), \tag{3.2}$$

because of $|H\backslash K| = n-k$. Summing (3.2) over i

$$\sum_{i=1}^{k}\sum_{I\in\mathcal{B}_i} |I \cap (H\backslash K)| \le (l-1)k(n-k)$$

is obtained. By the definiton of $\mathcal{B}_i$ we have

$$\sum_{i=1}^{k}\sum_{j=1}^{k}\sum_{J\in\mathcal{A}_{i,j}} |J \cap (H\backslash K)| \le (l-1)k(n-k),$$

and changing the order of the first two sums we obtain

$$\sum_{j=1}^{k}\sum_{i=1}^{k}\sum_{J\in\mathcal{A}_{i,j}} |J \cap (H\backslash K)| \le (l-1)k(n-k). \tag{3.3}$$

Consider the last two sums:

$$\sum_{i=1}^{k}\sum_{J\in\mathcal{A}_{i,j}} |J \cap (H\backslash K)| = j\sum_{G\in\mathcal{C}_j} |G \cap (H\backslash K)|, \tag{3.4}$$

because $\sum_{i=1}^{k}\sum_{J\in\mathcal{A}_{i,j}}$ means all subsets in $\mathcal{C}_j$ exactly j times. On the other hand

$$j\sum_{G\in\mathcal{C}_j} |G \cap (H\backslash K)| \ge j\sum_{G\in\mathcal{C}_j} (k-j),$$

because $|G| \ge k$, $|G \cap K| = j$, so $|G \cap (H\backslash K)| \ge k-j$. For the right hand side we have

$$j\sum_{G\in\mathcal{C}_j} (k-j) = j(k-j)|\mathcal{C}_j|,$$

so by (3.3)

$$\sum_{i=1}^{k}\sum_{J\in\mathcal{A}_{i,j}}|J\cap(H\backslash K)|\geq j(k-j)|\mathcal{C}_j| \tag{3.5}$$

holds.

We discuss the case $j=k$ separately. Start with (3.4). $\mathcal{C}_k$ does not contain K, so the right hand side of (3.4) is greater or equal to $k|\mathcal{C}_k|$, so (3.4) implies

$$(k-1)|\mathcal{C}_k|\leq\sum_{i=1}^{k}\sum_{J\in\mathcal{A}_{i,k}}|J\cap(H\backslash K)|. \tag{3.6}$$

Substituting (3.5) and (3.6) to (3.3) we obtain

$$(k-1)|\mathcal{C}_k|+\sum_{j=1}^{k-1}j(k-j)|\mathcal{C}_j|\leq\sum_{j=1}^{k}\sum_{i=1}^{k}\sum_{J\in\mathcal{A}_{i,j}}|J\cap(H\backslash K)|\leq(l-1)k(n-k).$$

For $1\leq j\leq k-1$ the inequality $k-1\leq j(k-j)$ holds, so

$$(k-1)\sum_{j=1}^{k}|\mathcal{C}_j|=\sum_{j=1}^{k}(k-1)|\mathcal{C}_j|\leq(l-1)k(n-k).$$

By definition $\dot\cup_{j=1}^{k}\mathcal{C}_j=\mathcal{F}\backslash\{K\}$ holds, so $\sum_{j=1}^{k}|\mathcal{C}_j|=|\mathcal{F}|-1$, which yields $(k-1)(|\mathcal{F}|-1)\leq(l-1)k(n-k)$,

$$|\mathcal{F}|\leq(l-1)(n-k)\frac{k}{k-1}+1\leq(l-1)n\frac{k}{k-1}+1 \tag{3.7}$$

is obtained.

3.1.2 Case of small k

We can suppose without loss of generality that $|\mathcal{A}_{1,1}|\geq|\mathcal{A}_{2,1}|\geq\ldots\geq|\mathcal{A}_{k,1}|$. Let $x=|\mathcal{A}_{1,1}|$.

If $x < \frac{(l-1)n}{k}$ then $|\mathcal{C}_1| = \sum_{i=1}^{k} |\mathcal{A}_{i,1} \leq (l-1)n$ and $\sum_{j=2}^{k} |\mathcal{C}_j| \leq (l-2)\binom{k}{2}$ because there are at most $l-2$ subsets Z for any two distinct elements a_x, a_y of K such that $a_x, a_y \subseteq (K \cap Z)$. We obtain the following upper estimate

$$|\mathcal{F}| = |\mathcal{C}_1| + \sum_{j=2}^{k} |\mathcal{C}_j| + 1 \leq (l-1)n + (l-2)\frac{k^2}{2} + 1 \leq \frac{3l-4}{2}k^2 + (l-1)n + 1. \quad (3.8)$$

From now on we can suppose $x \geq \frac{2n}{k}$.

Lemma 3.2 *$|F \backslash \{a_i\}| \geq \frac{x}{l-1}$ holds for any $F \in \mathcal{A}_{i,1}$ $(2 \leq i \leq k)$.*

Proof: Suppose on the contrary that $|F \backslash \{a_i\}| < \frac{x}{l-1}$ for some $F \in \mathcal{A}_{i,1}$. Let $a_i, f_1, \ldots, f_m$ be the elements of F. Every two members of $\mathcal{F}$ have a non-empty intersection, so $F \cap G \neq \emptyset$ for any $G \in \mathcal{A}_{1,1}$. For every such G there is an element of F which is not a_i and is contained in $F \cap G$. $|F \backslash \{a_i\}| < \frac{x}{l-1}$, so $F \backslash \{a_i\}$ has an element f_f, which is contained in at least l distinct subsets (say $G_1, \ldots, G_l$) from $\mathcal{A}_{1,1}$. So $f_f \in G_1 \cap \ldots \cap G_l$ and $a_1 \in G_1 \cap \ldots \cap G_l$, which is a contradiction because the intersection of any l distinct subsets contain at most one element and $f_f \neq a_1$. □

By definition $\mathcal{A}_{b,1} \subseteq \mathcal{B}_b$ holds, so (3.2) implies

$$\sum_{I \in \mathcal{A}_{b,1}} |I \cap (H \backslash K)| \leq (l-1)(n-k), \quad (3.9)$$

Consider the case $2 \leq b$. For any $I \in \mathcal{A}_{b,1}$ we have $|I \cap (H \backslash K)| = |I \backslash \{a_b\}| \geq \frac{x}{l-1}$ by the lemma, therefore

$$|\mathcal{A}_{b,1}| \frac{x}{l-1} \leq \sum_{I \in \mathcal{A}_{b,1}} |I \cap (H \backslash K)| \leq (l-1)(n-k),$$

hence $|\mathcal{A}_{b,1}| \leq \frac{(l-1)^2(n-k)}{x} \leq \frac{(l-1)^2 n}{x}$. We supposed that $x \geq \frac{(l-1)n}{k}$, so $|\mathcal{A}_{b,1}| \leq (l-1)\frac{kn}{n} = (l-1)k$.

For any $I \in \mathcal{A}_{1,1}$ the inequaltity $|I \cap (H \setminus K)| \geq k-1$ holds since K is the smallest subset in $\mathcal{F}$. By (3.9) we obtain $|\mathcal{A}_{1,1}| \leq (l-1)\frac{n-k}{k-1}$.

Summation gives $|\mathcal{C}_1| = \sum_{i=1}^{k} |\mathcal{A}_{i,1}| \leq (l-1)k(k-1) + (l-1)\frac{n-k}{k-1} \leq (l-1)k^2 + (l-1)\frac{n}{k-1}$. On the other hand $\sum_{j=2}^{k} |\mathcal{C}_j| \leq (l-2)\binom{k}{2}$ because there are at most $l-2$ subsets Z for any two distinct elements a_x, a_y of K such that $a_x, a_y \subseteq (K \cap Z)$. We obtain

$$|\mathcal{F}| = |\mathcal{C}_1| + \sum_{j=2}^{k} |\mathcal{C}_j| + 1 \leq$$

$$\leq \frac{3l-4}{2}k^2 + (l-1)\frac{n}{k-1} + 1 \leq \frac{3l-4}{2}k^2 + (l-1)n + 1 \qquad (3.10)$$

since $k \geq 2$. This is true in the case of $x < \frac{(l-1)n}{k}$ too (see (3.8)).

3.1.3 Asymptotic results

By (3.7) and (3.10), $|\mathcal{F}| \leq \min\{(l-1)n\frac{k}{k-1} + 1, \frac{3l-4}{2}k^2 + (l-1)n + 1\}$ for $k \geq 2$. If $k = 1$, then $|\mathcal{F}| \leq \frac{l}{2}n$ holds by (3.1). In the case $1 < k \leq \sqrt[3]{n}$, we have $|\mathcal{F}| \leq \frac{3l-4}{2}k^2 + (l-1)n + 1 \leq (l-1)n + 1 + \frac{3l-4}{2}n^{\frac{2}{3}}$. On the other hand, if $\sqrt[3]{n} < k \leq n$, then $|\mathcal{F}| \leq (l-1)n\frac{k}{k-1} \leq (l-1)n\frac{n^{\frac{1}{3}}}{n^{\frac{1}{3}}-1} + 1$, so $|\mathcal{F}| \leq \max\{\frac{l}{2}n, (l-1)n + 1 + \frac{3l-4}{2}n^{\frac{2}{3}}, (l-1)n\frac{n^{\frac{1}{3}}}{n^{\frac{1}{3}}-1} + 1\}$. For every ϵ we have an N, such that $\max\{\frac{l}{2}n, (l-1)n+1+\frac{3l-4}{2}n^{\frac{2}{3}}, (l-1)n\frac{n^{\frac{1}{3}}}{n^{\frac{1}{3}}-1}+1\} \leq (l-1)n(1+\epsilon)$ for $n \geq N$, so $|\mathcal{F}| \leq (l-1)n + o(n)$ holds.

3.2 Construction

We give a construction for $\mathcal{F}$ using finite projective planes, such that $|\mathcal{F}| = (l-1)n + o(n)$. Csima and Füredi has proved [2] that we can color the points of a finite desarguesian projective plane of order q, PG(q) with $q+1$ colors, that there are no three points of the same color on a line. Using the duality of points and lines in PG(q) it follows that the lines can also be colored in such a way that no three lines of the same color go through one point.

Take PG(q), where q is the largest prime such that $q^2 + l(q+1) \leq n$. Let $a_1, \dots, a_{q^2+q+1}$ be the points and $L_1, \dots, L_{q^2+q+1}$ the lines. Color the lines as described above, the color of the line L_i will be $c(i)$. Take $(l-2)(q+1)$ extra points with the colors of the lines ($l-2$ of each color). Call them $e_{1,1}, \dots, e_{q+1,1}, e_{2,2}, \dots, e_{q+1,l-2}$ ($e_{i,j}$ is the jth extra point with the ith color). Let $F_{i,j} = L_i \cup \{e_{c(i),j}\}$ $(1 \leq i \leq q^2+q+1, 1 \leq j \leq l-2)$, $F_{i,l-1} = L_i$.

The underlying set will be $\{a_1, \dots, a_{q^2+q+1}, e_{1,1}, \dots, e_{q+1,l-2}\}$, and the family is defined by $\mathcal{F} = \{F_{i,j} : 1 \leq i \leq q^2+q+1, 1 \leq j \leq l-1\}$. It is easy to see that the so defined members of $\mathcal{F}$ are different, therefore $|\mathcal{F}| = (l-1)(q^2+q+1)$.

Any two subsets have a common element because $L_i \subseteq F_{i,j}$, $L_a \subseteq F_{a,b}$, and $L_i \cap L_a \neq \emptyset$.

Let us see that if we take l distinct subsets, the intersection has at most one element. Two of them must have different first indexes ($F_{a,b}, F_{e,f}, a \neq e$). If $b \neq f$ then $F_{a,b} \cap F_{e,f} = L_a \cap L_e$ and its size is exactly one. If $b = f$

take a third one: $F_{g,h}$. If $h \neq b$ then $g \neq a$ or $g \neq e$ (suppose $g \neq a$) so $F_{a,b} \cap F_{g,h} = L_a \cap L_g$ and its size is exactly one. If $b = f = h$ then in the case of $c(a) = c(e) = c(g)$ $F_{a,b} \cap F_{e,f} \cap F_{g,h} = \{e_{c(a),b}\}$ since $L_a \cap L_e \cap L_g = \emptyset$. In the case when $c(a), c(e), c(g)$ are not all the same colors we have $F_{a,b} \cap F_{e,f} \cap F_{g,h} = L_a \cap L_e \cap L_g$ and its size is at most one.

In this construction $n \geq q^2 + l(q+1)$ and $|\mathcal{F}| = (l-1)(q^2+q+1)$. So $\frac{|\mathcal{F}|}{n} \leq l - 1$. On the other hand

$$\frac{|\mathcal{F}|}{n} \geq (l-1)\frac{q^2+q+1}{((1+\epsilon)q)^2 + l((1+\epsilon)q+1)}$$

for any ϵ if n is large enough, because there is prime between q and $(1+\epsilon)q$ if q is enough large. So $\frac{|\mathcal{F}|}{n}$ converges to $l-1$ if n tends to infinity. Combining this with subsection 3.1 we have proved the following theorem:

Theorem 3.3 *There are at most $(l-1)n + o(n)$ distinct subsets of an n-element set, such that any two have a common element, but there are no $(2 \leq)l$ distinct ones with intersection of size greater than* 1.

The theorem is still true if $\mathcal{F}$ is not supposed to have different members but we need some modifications in the proof, e.g. (3.1) is not true.

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