3-wise exactly 1-intersecting families of sets

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Abstract

Let f(l,t,n) be the maximal size of a family $\mathcal{F} \subset 2^{[n]}$ such that any $l \geq 2$ sets of $\mathcal{F}$ have an exactly $t \geq 1$-element intersection. If $l \geq 3$, it trivially comes from [8] that the optimal families are trivially intersecting (there is a t-element core contained by all the members of the family). Hence it is easy to determine $f(l,t,n) = \left\lfloor \frac{l}{2}(n-1) \right\rfloor + 1$. Let $g(l,t,n)$ be the maximal size of an l-wise exaclty t-intersecting family that is not trivially t-intersecting. We give upper and lower bounds which only meet in the following case: $g(3,1,n) = n^{2/3}(1+o(1))$.

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1 Introduction

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$, $2^{[n]}$ the collection of all subsets of $[n]$ and $\binom{[n]}{k}$ the set of k-element subsets of $[n]$. A simple theorem of Erdős, Ko and Rado [6] says that the maximum of $|\mathcal{F}|$ is 2^{n-1}, if every two members of a family $\mathcal{F} \subseteq 2^{[n]}$ have a non-empty intersection. Such an $\mathcal{F}$ is called an intersecting family. A maximum intersecting family $\mathcal{F}$ can be obtained by considering all the subsets containing one fixed element. This $\mathcal{F}$ sometimes is called a cylinder, or trivially intersecting, since $\cap\mathcal{F} := \cap_{F\in\mathcal{F}} F \neq \emptyset$.

A family $\mathcal{F} \subset 2^{[n]}$ (consisting of distinct sets) is called l-wise exactly t-intersecting if $|F_1 \cap \ldots \cap F_l| = t$ holds for any distinct members $F_1, \ldots, F_l$ of $\mathcal{F}$ and is called trivially l-wise exactly t-intersecting if $|\cap \mathcal{F}| = t$. Let $f(l,t,n)$ denote the maximal size of an l-wise exactly t-intersecting family $\mathcal{F} \subset 2^{[n]}$. The well-known theorem of Erdős and De Brujin [5] states that $f(2,1,n) = n$. There are three types of optimal families here. One is trivially intersecting: $\{\{1\}\} \cup \{\{1,i\} | n \geq i \geq 2\}$, another one is almost the same: $\{[n] - \{1\}\} \cup \{\{1,i\} | n \geq i \geq 2\}$, but the third one is $PG(2,q)$ (if $q^2 + q + 1 = n$). The generalization for $t \geq 2$ is known as Fisher's inequality. R.C. Bose was the first to apply the linear argument to solve a combinatorial problem [2], but the following theorem was first proved by Majumdar .

Theorem. [15] *If* $t \geq 1$ *then* $f(2,t,n) \leq n$.

We are interested in the case $l \geq 3$. The general problem of r-wise at least s-intersecting and l-wise ($l \geq r \geq 2$) at most t-intersecting families is solved in [9] only for $t \geq 2s \geq 2$. To determine $f(l,t,n)$ is a special case of the general problem ($t = s$,$r = l$) not included in [9]. It easily comes from a theorem of Füredi [8] that the optimal families here are trivially intersecting.

Another approach is the linear algebraic method. The well known, far reaching generalizations of the Erdős-de Bruijn theorem are the results of Ray-Chaudhuri-Wilson [16] and Frankl-Wilson [7] about 2-intersections. Continuing this way Grolmusz and Sudakov [11] gives upper bounds for families with given l-intersections. The referee called my attention to two recent papers in the field. Füredi and Sudakov in [10] and Szabó and Vu in [18] also study families with given l-intersections and they prove more general theorems than the following. For the sake of completeness, in Section 2, we determine the exact value of $f(l,t,n)$, proving this theorem.

Theorem 1. *Suppose* $l \geq 3$ *and* $n \geq \frac{l(l+t)}{l-2}$, *then* $f(l,t,n) = \left\lfloor \frac{l}{2}(n-t) \right\rfloor + 1$.

There are several examples for optimal families that are not trivially intersecting. The finite projective plane is one of these which is non-trivially 2-wise exactly 1-intersecting. Also in [14] where intersecting but l-wise at most 1-intersecting families are studied, the only asymptotically optimal family is originated from the finite projective plane.

This leads us to examine the maximal non-trivially l-wise exactly t-intersecting families in Sections 3 and 4. Let $g(l,t,n)$ =max$|\mathcal{F}|$ where $\mathcal{F}$ is l-wise exactly t-intersecting and $|\cap\mathcal{F}| < t$. Of course $g(l,t,n) \leq f(l,t,n)$. By [5] $g(2,1,n) = n$. In Section 3 we prove the upper bounds, in Section 4 the lower bounds of the following theorems.

Theorem 2. $g(3,1,n) = n^{2/3}(1+o(1))$.

In general, our lower and upper estimates are, unfortunately, rather different.

Theorem 3. *If* $l \geq 2$, *then*

$$\frac{1}{t}n^{1/l}(1+o(1)) \leq g(l,t,n) \leq n^{(t+1)/(l+t-1)}(1+o(1)).$$

These theorem yield too, that the optimal families in Theorem 1 are trivially intersecting for large n's.

2 Trival intersection allowed

Proof of Theorem 1. A special case of a theorem of Füredi says

Theorem. [8] *If* $l \geq 3, t \geq 1$ *and* $\mathcal{F}$ *is* l*-wise non-trivially exactly* t*-intersecting then* $|\mathcal{F}| \leq n+l-2$.

A much stronger theorem (Theorem 3) will be proved in Section 3 but we wanted to keep the linear ordering of the paper. We will construct a family with size $\left\lfloor \frac{l}{2}(n-t)\right\rfloor +$ $1 > n+l-2$ (for $n \geq 2+\frac{l}{l-2}t$) if $|\cap\mathcal{F}| = t$ so we only have to deal with the trivially intersecting case. (Suppose that $\cap\mathcal{F} = \{n, n-1, ..., n-t+1\}$. Define the family

$\mathcal{F}' = \{F \backslash \{n, n-1, ..., n-t+1\} | F \in \mathcal{F}\}$. Trivially $|\mathcal{F}| = |\mathcal{F}'|$. For every $F' \in \mathcal{F}'$ let us associate a weight $\frac{1}{|F'|}$ with every element of F' except the empty set. Since every element of $\{1, 2, \ldots n-t\}$ is contained in at most $l-1$ subsets we have at most $l-1$ weights on every element. The two largest weights cannot be both 1s because all subsets are different and two 1s would mean two different 1-element subsets. The second largest possible weight is $\frac{1}{2}$, so the sum of the weigths is at most $1 + \frac{l-2}{2} = \frac{l}{2}$ on every element. The sum of the weigths on all elements is at most $(n-t)\frac{l}{2}$, and is equal to $|\mathcal{F}'|$ or $|\mathcal{F}'| - 1$ (depending on the empty set) because the sum of the weigths is 1 for each subset. Thus

$$|\mathcal{F}| = |\mathcal{F}'| \leq \frac{l}{2}(n-t) + 1. \tag{1}$$

If $n \geq l + t$ we can construct an $\mathcal{F}$ with $|\mathcal{F}| = \left\lfloor \frac{l}{2}(n-t) \right\rfloor + 1$. It is enough to exhibit an $\mathcal{F}'$ with $|\mathcal{F}'| = \left\lfloor \frac{l}{2}(n-t) \right\rfloor + 1$ on the underlying set $\{1, 2, \ldots, n-t\}$. In the case of even l let $\mathcal{F}'$ be $\emptyset \cup \{1\} \cup \{2\} \cup \ldots \cup \{n-t\} \cup \mathcal{F}_1' \cup \mathcal{F}_2' \cup \ldots \cup \mathcal{F}_{\frac{l-2}{2}}'$ where $\mathcal{F}_i' = \{1, i+1\} \cup \{2, i+2\} \cup \ldots \cup \{n-t, n-t+i\}$ (every element is understood (mod $n-t$)). In the case of odd l let $\mathcal{F}'$ be $\emptyset \cup \{1\} \cup \{2\} \cup \ldots \cup \{n-t\} \cup \mathcal{F}_1' \cup \mathcal{F}_2' \cup \ldots \cup \mathcal{F}_{\left\lfloor \frac{l-2}{2} \right\rfloor}' \cup \{1, \left\lfloor \frac{n-t}{2} \right\rfloor + 1\} \cup \{2, \left\lfloor \frac{n-t}{2} \right\rfloor + 2\} \cup \ldots \cup \{\left\lfloor \frac{n-t}{2} \right\rfloor, 2\left\lfloor \frac{n-t}{2} \right\rfloor\}$. □

3 Upper bounds for the non-trivial case

Here we prove the upper bounds of Theorem 2 and 3 by induction on l. We will show $|\mathcal{F}| \leq n^{(t+1)/(l+t-1)}(1 + o(1))$ if $\mathcal{F}$ is l-wise exactly t-intersecting ($l \geq 2$) and $|\cap \mathcal{F}| < t$. The first case, $l = 2$ is Fisher's inequailty. Before proving the induction step we have to remark that the upper bound is true and the proof is valid if we allow multisets in $\mathcal{F}$ (if it is a collection of not necessary different subsets of $2^{[n]}$) and we will use this fact in the induction step. Suppose for $l \geq 3$ that the bound is true for $l-1$ and consider a non-trivially l-wise exactly t-intersecting family $\mathcal{F}$. Denote the smallest member of $\mathcal{F}$ by X and suppose $|X| = k$. Take the intersections $X \cap F$ for all $F \in \mathcal{F}$. Consider them as a collection of not necessary different subsets of X as mentioned above. One can see that these intersections are non-trivially $l-1$-wise exactly t-intersecting ($|\cap_{F\in\mathcal{F}} X \cap F| < t$). By induction this implies

$$|\mathcal{F}| \le k^{(t+1)/(l+t-2)}(1+o(1)). \tag{2}$$

Hence we are done with the proof if $k \le n^{(l+t-2)/(l+t-1)}$ so we can suppose $k > n^{(l+t-2)/(l+t-1)}$. Before continuing the proof extend the notation $\binom{x}{t}$ for non-integer x's. Let $\binom{x}{l} = f(x) = \frac{x(x-1)\cdots(x-l+1)}{l!}$ if $x \ge l-1$ and $f(x) = 0$ otherwise. One can see that this function is monotone and convex so we can apply Jensen's inequality. For an $A \in \binom{[n]}{t}$ let d_A denote $|\{F \in \mathcal{F} | A \subset F\}|$, the number of sets of $\mathcal{F}$ containing A. The l-wise exactly t-intersecting condition means that for every l sets in $\mathcal{F}$ there is a set $|A| = t$ which is their intersection. A can be the intersection of $\binom{d_A}{l}$ choices of l members of $\mathcal{F}$ if $d_A \ge l$. Hence

$$\frac{\binom{|\mathcal{F}|}{l}}{\binom{n}{t}} = \frac{\sum_{A\in\binom{[n]}{t}} \binom{d_A}{l}}{\binom{n}{t}} \ge \binom{\frac{\sum_A d_A}{\binom{n}{t}}}{l} \tag{3}$$

by the Jensen-inequality. For a fixed A there are d_A F's ($\in \mathcal{F}$) containing A and for a fixed $F \in \mathcal{F}$ there are $\binom{|F|}{t}$ A's in it, so we can continue (3):

$$\binom{\frac{\sum_A d_A}{\binom{n}{t}}}{l} = \binom{\frac{\sum_{F\in\mathcal{F}} \binom{|F|}{t}}{\binom{n}{t}}}{l} \ge \binom{|\mathcal{F}|\frac{\binom{k}{t}}{\binom{n}{t}}}{l}. \tag{4}$$

We may suppose $|\mathcal{F}| > n^{(t+1)/(l+t-1)}$, otherwise we would be done with the proof. Since $k > n^{(l+t-2)/(l+t-1)}$ the quantity $|\mathcal{F}|\frac{\binom{k}{t}}{\binom{n}{t}}$ tends to infinity ($n \to \infty$). The combination of (3) and (4) leads to

$$\frac{1}{\binom{n}{t}l!}|\mathcal{F}|^l \ge \binom{|\mathcal{F}|\frac{\binom{k}{t}}{\binom{n}{t}}}{l} \ge \frac{1}{l!}\left(|\mathcal{F}|\frac{\binom{k}{t}}{\binom{n}{t}} - l + 1\right)^l \ge \frac{1-\varepsilon}{l!}\left(|\mathcal{F}|\frac{\binom{k}{t}}{\binom{n}{t}}\right)^l \tag{5}$$

for any $\varepsilon > 0$ if n is large enough. Hence we obtain

$$(1+o(1))\binom{n}{t}^{l-1} \ge \binom{k}{t}^l \tag{6}$$

and this yields $k \le (1+o(1))n^{(l-1)/l} \le (1+o(1))n^{(l+t-2)/(l+t-1)}$, so by (2) $|\mathcal{F}| \le (1+o(1))n^{t+1/(l+t-1)}$. □

4 Non-trivial constructions

Constructions will prove the lower bounds of Theorem 2 and 3. We will use projective spaces, namely $PG(N,q)$, the N-dimensional finite projective space of order q. It consists of $q^N + q^{N-1} + ... + 1$ hyperplanes on the same number of points. Considering the hyperplanes as sets on the underlying points, any N of them have a non-empty intersection, which can be a K-dimensional subspace $K < N-1$.

First we show $g(3,1,n) \geq n^{2/3}(1+o(1))$ to complete the proof of Theorem 2 giving a family $\mathcal{F}$ consisting of $n^{2/3}(1+o(1))$ subsets of $[n]$ which are 3-wise 1-intersecting, and $\cap\mathcal{F} = \emptyset$. Take $PG(3,q)$, such that $q^3+q^2+q+1 = n$ (if there is such prime power q). By [2] and [17] the maximal size of an *ovoid* is q^2+1 here, which means that the maximal number of points, such that there at most two on any line, is q^2+1. Take such q^2+1 points. Obviously, these points can not lie in one plane. Their dual planes (there are also q^2+1 of them) will be the members of $\mathcal{F}$. They have the property that no three of them contain the same line. This implies that the intersection of 3 planes is a point, so $\mathcal{F}$ is 3-wise 1-intersecting, and of course, non-trivially. Hence $g(3,1,n) \geq n^{2/3}(1+o(1))$ because the underlying set has q^3+q^2+q+1 points. If there is no q such that $q^3+q^2+q+1 = n$ and q is prime power, then we take the largest $m \leq n$ such that $q^3+q^2+q+1 = m$ with a prime power q. We know that there is a prime between n and $(1-\varepsilon)n$ for any ε if n is large enough.

To prove $g(l,1,n) \geq n^{1/l}(1+o(1))$ for $l \geq 4$ (Theorem 3) we generalize the above construction. We would like to find as many points as we can in $PG(l,q)$ ($q^l + ... + q + 1 = n$) such that there are at most $l-1$ of them in any $l-2$-dimensional (2-codimensional) subspaces (these points are called a *k-track*, if there are k of them). Unfortunately the exact number of these points is not known for $l \geq 4$, but we have a lower bound $q + 2\sqrt{q}$ ([4] and [1]). Taking these $q+2\sqrt{q}$ points, their dual hyperplanes will form $\mathcal{F}$, that will be l-wise 1-intersecting as above and so $g(l,1,n) \geq n^{1/l}(1+o(1))$.

Finally $g(l,t,n) \geq (1/t)n^{1/l}(1+o(1))$ comes with replacing every element of the underlying set with t other ones.

Remark 1. *If the exact value of the size of a track was* $q^2(1 + o(1))$ *(this is the existing upper bound) this would imply* $g(l, 1, n) = n^{2/l}(1 + o(1))$ *with the idea above.*

The reader may be interested in arcs, caps, ovoids and (n,r)-sets. All recent results on them can be found in the survey [12] and its update [13].

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