Multiply intersecting families of sets

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Abstract Let $[n]$ denote the set $\{1, 2, ..., n\}$, $2^{[n]}$ the collection of all subsets of $[n]$ and $\mathcal{F} \subset 2^{[n]}$ be a family. The maximum of $|\mathcal{F}|$ is studied if any r subsets have an at least s-element intersection and there are no ℓ subsets containing $t+1$ common elements. We show that $|\mathcal{F}| \le \sum_{i=0}^{t-s} \binom{n-s}{i} + \frac{t+\ell-s}{t+2-s}\binom{n-s}{t+1-s} + \ell - 2$ and this bound is asymptotically the best possible as $n \to \infty$ and $t \ge 2s \ge 2$, $r, \ell \ge 2$ are fixed.

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1 Intersecting families: cylinders and Hamming balls

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$ and $2^{[n]}$ the collection of all subsets of $[n]$, and $\binom{[n]}{k}$ is the set of k-element subsets of $[n]$. A simple theorem of Erdős, Ko and Rado [6] says that the maximum of $|\mathcal{F}|$ is 2^{n-1} if every two members of a family $\mathcal{F} \subseteq 2^{[n]}$ have a non-empty intersection. Such an $\mathcal{F}$ is called an **intersecting** family. A maximum intersecting family $\mathcal{M}$ can be obtained by considering all the subsets containing one fixed element. This $\mathcal{M}$ sometimes is called a **cylinder**, or **trivially** intersecting, since $\cap\mathcal{M} \neq \emptyset$. The above theorem was extended by Gy. Katona [13] as follows. If a a set system $\mathcal{F}$ on $[n]$ is s-**intersecting**, i.e., any two sets in the system have intersection of size at least s, then for $n + s$ is even $|\mathcal{F}| \leq |B(n, \geq\frac{1}{2}(n + s))|$ where $B(n, \geq x) := \{A \subset [n] : |A| \geq x\}$, a **Hamming ball**. For the case $n+s$ is odd one has $|\mathcal{F}| \leq 2|B(n-1, \geq\frac{1}{2}(n+s-1))|$, and the optimal family is a combination of a cylinder and a Hamming ball. This is often the case in the theory of intersection families, especially if one considers uniform families (where all sets have the same sizes), see, e.g., Ahlswede and Khachatrian's [1] solution for the Erdős–Frankl conjecture.

In general, we say that $\mathcal{F}$ has the $I(r, \geq s)$ property (also called r-**wise** s-**intersecting**) if

$$|F_1 \cap F_2 \cap \ldots \cap \mathcal{F}_r| \geq s \quad \text{holds for every } F_1, F_2, \ldots, F_r \in \mathcal{F}, \tag{1}$$

and let $f(n; I(r, \geq s))$ denote the size of the largest r-wise s-intersecting family on n elements. Taking all subsets containing a fixed s-element set (i.e., a cylinder) shows that $f(n; I(r, \geq s)) \geq 2^{n-s}$ holds for all $n \geq s \geq 0$. One of the nicest results of the field is due to Frankl [7] that $f(n; I(r, \geq s)) = 2^{n-s}$ holds if and only if $n < r + s$ or $s \leq 2^r - r - 1$ with the possible (but unlikely) exception of the case $(r, s) = (3, 4)$. An excellent survey of these families is due to Frankl [8].

2 Intersecting families: codes and packings

If we have an upper bound on the intersection sizes, then the extremal families are codes, designs, and packings. More precisely, we say that $\mathcal{F}$ has the $I(\ell, \leq t)$ property if

$$|F_1 \cap F_2 \cap \ldots \cap \mathcal{F}_\ell| \leq t \quad \text{holds for every distinct } F_1, F_2, \ldots, F_\ell \in \mathcal{F}, \tag{2}$$

and let $f(n; I(\ell, \leq t))$ denote the size of the largest family satisfying $I(\ell, \leq t)$ on n elements. A family of k-subsets of $[n]$ with the $I(\ell, \leq t)$ property is called an $(n, k, t+1, \leq \ell-1)$ **packing** and its maximum size is denoted by $P(n,k,t+1,\leq\ell-1)$. More generally, a family $\mathcal{F}\subseteq 2^{[n]}$ is called an $(n,k^+,j,\leq\lambda)$ **packing** if $|F|\geq k$ holds for every $F\in\mathcal{F}$ and $|\mathcal{F}[X]|\leq\lambda$ for every j-subset $X\subseteq[n]$. (Here, as usual, $\mathcal{F}[X]$ denotes the family $\{F: X\subseteq F\in\mathcal{F}\}$, $\mathcal{F}[x]$ stands for $\mathcal{F}[\{x\}]$, and $\deg_{\mathcal{F}}(x):=|\mathcal{F}[x]|$). The maximum size of such a packing is denoted by $P(n,k^+,j,\leq\lambda)$. Simple double counting gives that

$$P(n,k,j,\leq\lambda)\leq P(n,k^+,j,\leq\lambda)\leq\lambda\frac{\binom{n}{j}}{\binom{k}{j}}. \tag{3}$$

If here equalities hold for $\mathcal{F}\subseteq\binom{[n]}{k}$, then it is called an $S_\lambda(n,k,j)$ **block design**, and in the case $\lambda=1$, a **Steiner system**. For more details about packings see [3, 4].

The existence of designs and the determination of the packing number is a very difficult question, here we only recall a folklore result about the case $j=k-1$.

$$\begin{aligned}\frac{\lambda}{k}\frac{n-k}{n}\binom{n}{k-1}\leq\quad & P(n,k,k-1,\leq\lambda)\\ \leq\quad & P(n,k^+,k-1,\leq\lambda)\quad\leq\frac{\lambda}{k}\binom{n}{k-1}.\end{aligned} \tag{4}$$

PROOF: The upper bound follows from (3). To show the lower bound we give a construction $\mathcal{F}$. For every $x\in[n]$ consider $\mathcal{F}_x:=\{F\in\binom{[n]}{k}:\sum_{f\in F}f\equiv x \pmod{n}\}$. It satisfies the property that every $(k-1)$-element set is contained in at most one member of $\mathcal{F}_x$, so $\mathcal{F}_1\cup\mathcal{F}_2\cup\cdots\cup\mathcal{F}_n$ is a decomposition of the complete hypergraph $\binom{[n]}{k}$ into $(n,k,k-1,\leq 1)$ packings. Define $\mathcal{F}$ as the union of the λ largest ones among these $\mathcal{F}_x$'s. Then $|\mathcal{F}|\geq\frac{\lambda}{n}\binom{n}{k}=\frac{\lambda}{k}\frac{n-k}{n}\binom{n}{k-1}$. □

One can see that the determination of $f(n;I(\ell,\leq t))$ is equivalent to the determination of the packing function estimated in (4).

Proposition 1 *Suppose that $\ell\geq 2$, $t\geq 1$ are integers. Then*

$$f(n;I(\ell,\leq t))=\sum_{0\leq i\leq t+1}\binom{n}{i}+P(n,(t+2)^+,t+1,\leq\ell-2).$$

PROOF: Easy. Let $\mathcal{F}\subseteq 2^{[n]}$ be a set system with property $I(\ell,\leq t)$. We may suppose that all subsets of $[n]$ with at most t elements belong to $\mathcal{F}$ because if $|X|\leq t$, then $\mathcal{F}\cup\{X\}$ has the $I(\ell,\leq t)$ property. We also show that we may suppose that all $(t+1)$-element subsets belong to $\mathcal{F}$. Indeed, if $X\in\binom{[n]}{t+1}$, $X\notin\mathcal{F}$ and $|\mathcal{F}[X]|\leq\ell-2$, then again $\mathcal{F}\cup\{X\}$ also has the $I(\ell,\leq t)$ property. Finally, if $|\mathcal{F}[X]| = \ell - 1 > 0$, then take a member $F \in \mathcal{F}[X]$ and replace it by X. The obtained new family $\mathcal{F} \setminus \{F\} \cup \{X\}$ has the $I(\ell, \leq t)$ property, too. One can repeat this process until all members of $\binom{[n]}{t+1}$ belong to $\mathcal{F}$.

Then, obviously, the family $\mathcal{F} \setminus \bigcup_{0 \leq i \leq t+1} \binom{[n]}{i}$ is an $(n; (t+2)^+, t+1, \leq \ell - 2)$ packing. □

Note that the above proof implies the following slightly stronger result. If $\mathcal{F}$ has the $I(\ell, \leq t)$ property on $[n]$ and each $F \in \mathcal{F}$ has size at least $t+1$, then

$$|\mathcal{F}| \leq \binom{n}{t+1} + P(n, (t+2)^+, t+1, \leq \ell - 2). \tag{5}$$

Conjecture 2 *Suppose that k, j and λ are positive integers. Then for sufficiently large n, $n > n_0 := n_0(k, j, \lambda)$ the packing functions in* (4) *have the same values*

$$P(n, k, j, \leq \lambda) = P(n, k^+, j, \leq \lambda).$$

3 Simultaneous restrictions

The Problem. Determine $f(n; I(r, \geq s), I(\ell, \leq t))$, the maximal size of $\mathcal{F} \subseteq 2^{[n]}$ satisfying both the conditions (1) and (2).

Although there were several partial results (and we will cite some of those later) this problem was proposed in this generality only in [14]. We suppose that $r, \ell \geq 2$, although the cases $r = 1$ and $\ell = 1$ are also interesting. It was also proved in [14] that the following asymptotic holds for fixed ℓ as $n \to \infty$

$$f(n; I(2, \geq 1), I(\ell, \leq 1)) = (\ell - 1)n + o(n).$$

Our first result is a simple proof for a more precise version of this.

Theorem 3 *If any $r \geq 2$ members of $\mathcal{F} \subseteq 2^{[n]}$ have a non-empty intersection but the intersection of any ℓ distinct members contains at most one element, then $|\mathcal{F}| \leq (\ell - 1)n$, i.e.,*

$$f(n; I(r, \geq 1), I(\ell, \leq 1)) \leq (\ell - 1)n$$

holds for every n and $r, \ell \geq 2$.

The proof is postponed to Section 4. There we also discuss related results, linear (and almost linear) hypergraphs, a topic started by de Bruijn and Erdős [5], who proved the case $\ell = 2$ of Theorem 3.

In Section 5 we give an asymptotic solution of the general problem for the case $t \geq 2s$. More exactly, we reduce it to a packing function discussed in (4). We show that

Theorem 4 *Suppose that $t \geq 2s \geq 2$, $r \geq 2$, $\ell \geq 2$ and $n > n_0 := n_0(r, s, \ell, t)$. Then for $r \geq 3$*

$$f(n; I(r, \geq s), I(\ell, \leq t)) = f(n - s; I(\ell, \leq t - s)).$$

If $r = 2$ then

$$f(n; I(2, \geq s), I(\ell, \leq t)) \leq f(n - s; I(\ell, \leq t - s)) + \ell - 2,$$

and here equality holds if Conjecture 2 holds for $(n - s, t + 2 - s, t + 1 - s, \leq \ell - 2)$-packings and $s \geq \ell - 1$.

Then Proposition 1 and (4) imply (for $r \geq 3$) that

$$\begin{aligned} f &= \sum_{i=0}^{t+1-s} \binom{n-s}{i} + P(n - s; (t + 2 - s)^+, t + 1 - s, \leq \ell - 2) \\ &= (1 + o(1)) \left(1 + \frac{\ell - 2}{t + 2 - s}\right) \binom{n}{t + 1 - s}, \end{aligned}$$

and the same asymptotic result holds for $r = 2$.

4 The Case s = t = 1

Here we prove Theorem 3. The following is a slight generalization of a result of Motzkin [15] who proved the case $c = 1$. In fact, the case $c = 1$ implies the lemma for all real $c > 0$.

Lemma 1 *Let $G = G(A, B; E)$ be a bipartite graph with $E \neq \emptyset$, and let c be a positive real number. Suppose that no vertex in A is adjacent to all vertices in B and that for every pair of non-joined vertices $a \in A, b \in B$, $(a, b) \notin E$ one has $\deg(a) \leq c \deg(b)$. Then $|A|c \geq |B|$.*

PROOF: Let p and q be positive integers satisfying $p/q \geq c$. We will show $(p/q)|A| \geq |B|$, and since $(p/q) - c$ could be arbitrarily small and since G is finite this will suffice to show $c|A| \geq |B|$.

Let G' be a bipartite graph with vertices $A' \cup B'$ such that A' is p copies of A, and B' is q copies of B and if $a' \in A'$ is a copy of $a \in A$ and $b' \in B'$ is a copy of $b \in B$ then a' is joined to b' in G' if and only if a is joined to b in G. Then in case of $(a', b') \notin E'$,

$$\deg_{G'}(a') = q \deg_G(a) \leq q \left(\frac{p}{q} \deg_G(b) \right) = p \deg_G(b) = \deg_{G'}(b').$$

Thus the conditions of the Motzkin's lemma (the case $c = 1$) hold for G'. This implies $|A'| \geq |B'|$. Hence $\frac{p}{q}|A| \geq |B|$ holds, as desired. □

PROOF OF THEOREM 3. Let $\mathcal{F}$ be an intersecting family, with no ℓ sets containing two common elements. Define a bipartite graph $G(A, B; E)$ as follows. $A := [n]$, $B := \mathcal{F}$ and $x \in [n]$ is adjacent to $F \in \mathcal{F}$ if $x \in F$. If this graph has no edge then $F = \emptyset$ for every $F \in \mathcal{F}$, implying $|\mathcal{F}| \leq 1$.

If there is an $a \in A$ which is adjacent to all vertices of B, then the element a is contained in every member of $\mathcal{F}$. However there are at most $\ell - 1$ members of $\mathcal{F}$ containing a pair of $[n]$, i.e.,

$$|\mathcal{F}[x, y]| \leq \ell - 1, \tag{6}$$

where $\mathcal{F}[x, y] := \{X \in F : x, y \in X\}$. Using this inequality for the pairs (a, y), we get that every element $y \neq a$ is contained in at most $\ell - 1$ sets of $\mathcal{F}$. Thus $|\mathcal{F}| \leq 1 + (\ell - 1)(n - 1) \leq (\ell - 1)n$. We will show that in any other case the main constraint of Lemma 1 holds with $c = \ell - 1$ (which is positive, since $\ell \geq 2$).

Let $x \in [n] = A$ and $F \in \mathcal{F} = B$ be a non adjacent pair in G, i.e., $x \notin F$. Consider $\mathcal{F}[x] := \{X \in \mathcal{F} : x \in X\}$. We have $|\mathcal{F}[x]| = \deg_G(x)$. Since $\mathcal{F}$ is intersecting, all members of $\mathcal{F}[x]$ intersect F, thus $\mathcal{F}[x] = \cup_{y \in F} \mathcal{F}[x, y]$. Then (6) implies $|\mathcal{F}[x]| \leq (\ell - 1)|F|$. This means $\deg_G(x) \leq (\ell - 1) \deg_G(F)$. Hence Lemma 1 can be applied to G giving $(\ell - 1)n = (\ell - 1)|A| \geq |B| = |\mathcal{F}|$. □

5 Proof of Theorem 4

Let

$$\Sigma := \sum_{i=0}^{t+1-s} \binom{n-s}{i} + P(n - s, (t + 2 - s)^+, t + 1 - s, \leq \ell - 2).$$

One can give a family of Σ sets which have s common elements and satisfies the intersection properties. Let $\mathcal{P}'$ be a packing with $P(n - s, (t + 2 - s)^+, t + 1 - s, \leq \ell - 2)$ members on the underlying set $[n]\setminus[s]$. Let $\mathcal{P}=\{P\cup[s] : P\in\mathcal{P}'\}$. Let $\mathcal{F}_0=\{F\subset[n] : |F|\le t+1; [s]\subset F\}\cup\mathcal{P}$. Hence

$$f(n; I(r,\ge s), I(\ell,\le t)) \ge |\mathcal{F}_0| = \Sigma. \tag{7}$$

For $r=2$ one can construct a larger family. Suppose that there is an optimal $(n-s,(t+2-s)^+,t+1-s,\le\ell-2)$ packing consisting of only $(t+2-s)$-element sets, i.e. Conjecture 2 holds for these values. Then remove $[s]$ from $\mathcal{F}_0$ and add $\min\{s,\ell-1\}$ sets of the form $[n]\setminus\{i\}$, $1\le i\le s$. One can easily see that the obtained family $\mathcal{F}_1$ is pairwise s-intersecting and satisfies the $I(\ell,\le t)$ property.

Let $\mathcal{F}$ be a family with the properties (1) and (2). We have to show the other inequality $|\mathcal{F}|\le\Sigma$ for $r\ge 3$, and $|\mathcal{F}|\le\Sigma+\ell-2$ for $r=2$.

First, we discuss the case $|\cap\mathcal{F}|\ge s$. This means that all the members of $\mathcal{F}$ have s common elements, say, $[s]\subset F$ for every $F\in\mathcal{F}$. Consider $\mathcal{F}'=\{F\setminus[s] : F\in\mathcal{F}\}$. In $\mathcal{F}'$ the intersection of any ℓ members has at most $(t-s)$ elements, it satisfies the property $I(\ell,\le t-s)$. So every $t+1-s$ elements are contained in at most $\ell-1$ members of $\mathcal{F}'$. Hence $|\mathcal{F}|=|\mathcal{F}'|\le\Sigma$ by Proposition 1.

From now on, we suppose that $|\cap\mathcal{F}|<s$. Define $\mathcal{F}(i):=\{F\in\mathcal{F} : |F|=i\}$ and $\mathcal{F}(\ge j)=\mathcal{F}(j)\cup\mathcal{F}(j+1)\cup\ldots$ We have $\mathcal{F}=\mathcal{F}(1)\cup\mathcal{F}(2)\cup\ldots$. Trivially $\mathcal{F}(i)=\emptyset$ if $i<s$. The family $\mathcal{F}(i)$ is i-uniform on $[n]$ vertices and satisfies $I(2,\ge s)$. So Erdős-Ko-Rado theorem [6] says that

$$|\mathcal{F}(i)|\le\binom{n-s}{i-s} \quad \text{for} \quad n>n(i,s). \tag{8}$$

We use this estimate if $s\le i\le t$.

By (7) and (4) we can suppose that $|\mathcal{F}|\ge\sum_{i=0}^{t-s}\binom{n-s}{i}+\left(1+\frac{\ell-2}{t+2-s}\frac{n-t-2}{n-s}\right)\binom{n-s}{t+1-s}$. Hence (8) implies

$$|\mathcal{F}(\ge t+1)|\ge\left(1+\frac{\ell-2}{t+2-s}\,\frac{n-t-2}{n-s}\right)\binom{n-s}{t+1-s} \tag{9}$$

We need an upper bound on the number of large sets.

Lemma 2 *If $|A\cap F|\ge s$ for every $F\in\mathcal{F}$ then*

$$|\mathcal{F}(\ge k)|\le\frac{\binom{|A|}{s}}{\binom{k-s}{t+1-s}}\binom{n-s}{t+1-s}(\ell-1).$$

PROOF: Let $\mathcal{A}=\{X\in\binom{[n]}{t+1} : |A\cap X|\ge s\}$. Clearly $|\mathcal{A}|\le\binom{|A|}{s}\binom{n-s}{t+1-s}$. Every $t+1$ elements are contained in at most $\ell-1$ members of $\mathcal{F}$, and every $F\in\mathcal{F}(\ge k)$ contains at least $\binom{k-s}{t+1-s}$ members of $\mathcal{A}$. Hence $|\mathcal{F}(\ge k)|\binom{k-s}{t+1-s}\le|\mathcal{A}|(\ell-1)$. □

Since $t \geq 2s \geq 2$ the fraction $\binom{k}{s}/\binom{k-s}{t+1-s}$ is arbitrarily small for all sufficiently large $k \geq k_0$. For example, for $k > k_0(t,s) := 4s^2 + 6t^2$ we get

$$\frac{\binom{k}{s}}{\binom{k-s}{t+1-s}} < \frac{2t}{k} \quad < \frac{1}{3t}.$$

Split $\mathcal{F}(\geq t+1)$ into two parts, $\mathcal{G} \cup \mathcal{F}(\geq k_0)$, i.e., $\mathcal{G} := \mathcal{F}(t+1) \cup \mathcal{F}(t+2) \cup \ldots \cup \mathcal{F}(k_0 - 1)$. We are going to give an upper bound on the size of the family $\mathcal{F}(\geq k_0)$. If it is non-empty, then choose $F_0 \in \mathcal{F}(\geq k_0)$ with the minimum size among its members, i.e., $|F| \geq |F_0|$ for every $F \in \mathcal{F}(\geq k_0)$. Denote the size of F_0 by f_0. We can use F_0 as A in Lemma 2. We obtain

$$|\mathcal{F}(\geq k_0)| = |\mathcal{F}(\geq f_0)| \leq \frac{\binom{f_0}{s}}{\binom{f_0-s}{t+1-s}} \binom{n-s}{t+1-s} (\ell - 1) < \frac{\ell - 1}{3t} \binom{n-s}{t+1-s}.$$

Comparing this to (9) we get (for $n > k_0$) that

$$\begin{aligned} |\mathcal{G}| &= |\mathcal{F}(\geq t+1)| - |\mathcal{F}(\geq f_0)| \geq \left(1 + \frac{\ell-2}{t+2-s}\frac{n-t-2}{n-s} - \frac{\ell-1}{3t}\right)\binom{n-s}{t+1-s} \\ &\geq \frac{\ell-1}{3t}\binom{n-s}{t+1-s}. \end{aligned} \tag{10}$$

Proposition 5 $\quad |\cap \mathcal{G}| = s.$

PROOF: If $|\cap \mathcal{G}| \leq s-1$, then there is a set $A \subset [n]$, $|A| < 3k_0$ such that $|A \cap G| \geq s+1$ for every $G \in \mathcal{G}$. Indeed, either there is an $A \in \mathcal{G}$ meeting all other members of $\mathcal{G}$ in at least $s+1$ elements, or we can find $G_1, G_2 \in \mathcal{G}$ with $|G_1 \cap G_2| = s$. Then there exists a $G_3 \in \mathcal{G}$ not containing $G_1 \cap G_2$. Thus $|G_1 \cap G_2 \cap G_3| \leq s-1$. Then $G_1 \cup G_2 \cup G_3$ is suitable for A. The existence of such an A in the case $|\cap \mathcal{G}| \geq s+1$ is obvious.

The $I(\ell, \leq t)$ property implies that

$$|\mathcal{G}| \leq \binom{|A|}{s+1}\binom{n-s-1}{t-s}(\ell-1) \leq \binom{3k_0}{s+1}\frac{t+1-s}{n-s}\binom{n-s}{t+1-s}(\ell-1).$$

This contradicts (10) for $n > n_0(k,s)$. □

Thus, we may assume that $[s] \subset \cap\mathcal{G}$. Let $\mathcal{S} := \{F \in \mathcal{F}(\geq t+1) : [s] \subset F\}$ and let $\mathcal{H} = \{F \in \mathcal{F}(\geq t+1) : [s] \not\subset F\}$. We have $\mathcal{F}(\geq t+1) = \mathcal{S} \cup \mathcal{H}$. The family $\mathcal{S}' := \{F \setminus [s] : F \in \mathcal{S}\}$ has the

$I(\ell, {\le}t-s)$ property on $n-s$ elements. Moreover each member of $\mathcal{S}'$ has size at least $t+1-s$. So Proposition 1, more exactly (5), implies that

$$|\mathcal{S}| = |\mathcal{S}'| \le \binom{n-s}{t+1-s} + P(n-s, (t+2-s)^+, (t+1-s), {\le}\ell-2). \tag{11}$$

If $\mathcal{H} = \emptyset$ then $\mathcal{S} = \mathcal{F}({\ge}t+1)$ and (11) and (8) imply $|\mathcal{F}| \le \Sigma$, and we are done.

From now on we suppose that $\mathcal{H} \ne \emptyset$. Let H_1 be a minimal size member in $\mathcal{H}$, $|H_1| = h$.

To estimate $|\mathcal{S}|$ consider the family $\mathcal{C} := \{C \subset [n] : C \supset [s], |C| = t+1\}$. Since $\mathcal{F}(t+1) \subset \mathcal{G} \subset \mathcal{S}$ we have that $\mathcal{F}(t+1) \subset \mathcal{C}$. Every member of $\mathcal{S}({\ge}t+2)$ contains at least $t+2-s$ members of $\mathcal{C}$. On the other hand, every member of $\mathcal{C}$ is contained in at most $\ell-1$ members of $\mathcal{F}$. We obtain, that

$$(t+2-s)\left(|\mathcal{S}| - |\mathcal{F}(t+1)|\right) + |\mathcal{F}(t+1)| \le (\ell-1)|\mathcal{C}| = (\ell-1)\binom{n-s}{t+1-s}.$$

Rearranging we get

$$|\mathcal{S}| \le \left(1 + \frac{\ell-2}{t+2-s}\right)\binom{n-s}{t+1-s} - \frac{t+1-s}{t+2-s}\left(|\mathcal{C}| - |\mathcal{F}(t+1)|\right).$$

For $F \in \mathcal{F}(t+1)$, $(F \setminus [s])$ can not be contained in $[n] \setminus H_1$. Hence $|\mathcal{C}| - |\mathcal{F}(t+1)| \ge \binom{n-s-h}{t+1-s}$. Also the fraction $(t-s+1)/(t-s+2)$ is at least $2/3$. We obtain

$$|\mathcal{S}| \le \left(1 + \frac{\ell-2}{t+2-s}\right)\binom{n-s}{t+1-s} - \frac{2}{3}\binom{n-h-s}{t+1-s}. \tag{12}$$

To give upper bound to $|\mathcal{H}|$ we use Lemma 2 with an arbitrary $A \in \mathcal{G}$. Since $|A| \le k_0$ and $|H| \ge h$ for every $H \in \mathcal{H}$, we have

$$|\mathcal{H}| \le \frac{\binom{k_0}{s}}{\binom{h-s}{t+1-s}}\binom{n-s}{t+1-s}(\ell-1) < \frac{2t}{h}\binom{n-s}{t+1-s}(\ell-1). \tag{13}$$

Adding up the upper bounds (12) and (13) and comparing to the lower bound (9) we get the following.

$$\left(1 + \frac{\ell-2}{t+2-s}\,\frac{n-t-2}{n-s}\right)\binom{n-s}{t+1-s} \le |\mathcal{F}({\ge}t+1)| = |\mathcal{S}| + |\mathcal{H}|$$
$$\le \left(1 + \frac{\ell-2}{t+2-s}\right)\binom{n-s}{t+1-s} - \frac{2}{3}\binom{n-h-s}{t+1-s} + \frac{2t}{h}\binom{n-s}{t+1-s}(\ell-1).$$

Rearranging we obtain

$$\frac{2}{3(\ell-1)}\binom{n-h-s}{t+1-s} \le \frac{2t}{h}\binom{n-s}{t+1-s} + \frac{1}{n-s}\binom{n-s}{t+1-s}. \tag{14}$$

We can redefine $k_0(t,s)$ as $k_0(t,s,\ell)$, sufficiently large depending only on t, s, and ℓ, and suppose that n is sufficiently large compared to this new k_0, i.e., $n > n_0(t,s,\ell)$. Then (14) implies $h > \frac{\ell-1}{\ell}(n+t)$. Hence the $I(\ell,\leq t)$ property implies that $|\mathcal{H}| \leq \ell - 1$.

Again (11) and (8) imply $|\mathcal{F}| \leq \Sigma + |\mathcal{H}| \leq \Sigma + (\ell-1)$. Since $|\mathcal{F}| \geq \Sigma$, we get from (8) that $|\mathcal{F}(s+1)| \geq \binom{n-s}{1} - |\mathcal{H}| \geq (n-s) - (\ell-1) > k_0 > s+2$. The members of $\mathcal{F}(s+1)$ pairwise meet in s elements so either $|\cup \mathcal{F}(s+1)| \leq s+2$ and then $|\mathcal{F}(s+1)| \leq \binom{s+2}{s+1}$, or $|\cap \mathcal{F}(s+1)| = s$. Let S_0 be the s-set contained in every $F \in \mathcal{F}(s+1)$. If a set F meets every member of $\mathcal{F}(s+1)$ in at least s elements and $|F| < |\mathcal{F}(s+1)|$, then $S_0 \subset F$. This implies that $\cap\mathcal{G} \supset S_0$ and also that S_0 is contained in every member of $\mathcal{F}(i)$ for $s \leq i \leq t$, too. Then Proposition 5 implies that $S_0 = [s]$. We also get that

$$|H \cap [s]| = s-1 \text{ and } H \supset (\cup\mathcal{F}(s+1) \setminus [s])$$

holds for every $H \in \mathcal{H}$.

Consider $F_1 \neq F_2 \in \mathcal{F}(s+1)$, and $H_1 \in \mathcal{H}$. Then $|F_1 \cap F_2 \cap H_1| \leq s-1$. This is a contradiction for $r \geq 3$ hence $|\mathcal{F}| \leq \Sigma$ in this case.

Finally, suppose that $r = 2$. Since $\mathcal{H} \neq \emptyset$ the set system $\mathcal{F}$ cannot contain $[s]$, so $|\mathcal{F}(s)| = 0$ and thus $|\mathcal{F} \setminus \mathcal{H}| \leq f(n-s, I(\ell, \leq t-s)) - 1$. Thus $|\mathcal{F}| \leq \Sigma - 1 + (\ell-1)$. □

6 Multihypergraphs

In the previous theorems we did not allow multiplied sets. Consider a sequence of sets $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ of subsets of $[n]$ with properties $I(r,\geq s)$ and $I(\ell,\leq t)$ where now repetition is allowed. If we can have multi-sets with size $s \leq |F| \leq t$ then this sequence can be arbitrarily long. Define $f'(n; I(r,\geq s), I(\ell,\leq t))$ as $\max m$ where $\mathcal{F}$ is a family of multi-subsets of $[n]$ satisfying the intersection conditions (1), (2) with all members having at least $t+1$ elements.

Theorem 6 *Suppose that $t \geq 2s \geq 2$, $r \geq 2$, $\ell \geq 2$ and $n > n_0 := n_0(r,s,\ell,t)$. Then for $r \geq 3$*

$$f'(n; I(r,\geq s), I(\ell,\leq t)) = f'(n-s; I(\ell,\leq t-s)) = (\ell-1)\binom{n-s}{t+1-s}$$

If $r = 2$ *then*

$$\begin{aligned} f'(n; I(2, \geq s), I(\ell, \leq t)) &= f'(n-s; I(\ell, \leq t-s)) + \ell - 1 \\ &= (\ell-1)\binom{n-s}{t+1-s} + \ell - 1. \end{aligned}$$

The proof of the upper bound is nearly the same as of Theorem 4. The packing problem giving the lower bound is trivial here, the extremal family consists of only $(t+1)$-element sets ($\ell - 1$ copies each) for $r \geq 3$ and some $(n-1)$-element sets in the case $r = 2$.

The analog of Theorem 3 holds.

Theorem 7

$f'(n; I(2, \geq 1), I(\ell, \leq 1)) = (\ell - 1)n$

holds for every n *and* $\ell \geq 2$.

One can take multiple copies of the very same extremal configurations as in [5] and [15], namely $\ell - 1$ copies of the lines of a finite projective plane (if such exists, so in this case $n = q^2 + q + 1$) or $\ell - 1$ copies of $n - 1$ pairs through an element x and the set $[n] \setminus \{x\}$ (for all n).

7 Conclusion

It was not unknown in the literature to investigate intersecting families of sets with upper and lower bounds on the intersection sizes. For example it was conjectured in [9] that (using our notation)

$$f(n; I(2, \geq 1), I(2, \leq k)) = \sum_{0 \leq i \leq k} \binom{n-1}{i}.$$

Taking all the at most $(k+1)$-element sets containing a given element shows that this is, indeed, the best possible. This was proved for $n > 100k^2/\log(k+1)$ by Frankl and Füredi [9] using the so-called Δ-system method, for $n \leq 2k+2$ and for $6(k+1) \leq n \leq (1/5)(k+1)^2$ by Pyber [16] using the permutation method in an ingenious way. Finally, Ramanan [17] proved the conjecture for all n (without characterizing the extremal families) using the method of multilinear polynomials, building on earlier successes by (among others) Alon, Babai and Suzuki [2]. A second proof was given based on the same technique by Sankar and Vishwanathan [18].

In the present paper we extended those results for multiple intersections whenever n is large. There is a renewed interest to multiple intersection problems, see, e.g., Grolmusz [10] and Grolmusz and Sudakov [11] for recent developments.

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