This appendix material solves the problem faced by liquidity supplying investors in the infinite-horizon version of the simultaneous trade model. It appears in the paper “Inventory Information,” by H. Cao, M. Evans, and R. Lyons (March 2002).

A.1. Proof of Proposition 1: Public Investors

The liquidity-supplying (LS) investors have the following utility defined over intertemporal consumption $c_t$ (per equation 1):

\[
U_i = \sum_{t=0}^{\infty} -\delta^t \exp(-\gamma c_t)
\]

We begin by conjecturing a value function, which we show below is consistent with optimizing behavior on the part of LS investors:

\[
V_t = -\alpha \exp(-\gamma W_t - \psi h_t^2)
\]

where $W_t$ is the LS investors’ nominal wealth at the end of day $t$ and $h_t$ is the total holding of the risky asset at the end of day $t$ (defined in equation 9). We need to determine the conditions under which $h_t$ is willingly held by the LS investors.

Given the proposed price function $P_t = -ah_t$ in text proposition 1, our task is to begin with the Bellman equation corresponding to the maximum of equation (1) and derive explicit expressions for the three coefficient values $\gamma$, $\psi$, and $\alpha$, in this conjectured value function, as well as an expression for the parameter $a$ in the price function. We shall show that we must have:

\[
\hat{\gamma} = \left( \frac{r}{1+r} \right) \gamma
\]

\[
\psi = \left( \frac{1}{2n\sigma_s^2} \right) + (1+r)\alpha \hat{\gamma} - \left( \frac{\alpha \hat{\gamma} + \frac{1}{n\sigma_s^2} (1+r)a - \sigma_\alpha^2 \hat{\gamma}}{2a} \right) + \left( \frac{\sigma_\alpha^2 \hat{\gamma}}{2} \right)
\]

\[
\alpha = \left( \frac{\hat{\gamma}}{\gamma} \alpha \sqrt{1+2n\sigma_s^2 \psi} \right)^{1+r} + \left( \frac{\hat{\gamma}}{\gamma} \alpha \sqrt{1+2n\sigma_s^2 \psi} \right)^{1+r} \left( \frac{\alpha}{1+r} \sqrt{1+2n\sigma_s^2 \psi} \right)
\]
and
\[ a^2 \hat{\gamma} - a \left( \frac{r}{n\sigma_s^2} + 2\psi + 2n\psi \right) + \left( \frac{1}{n\sigma_x^2} + 2\psi \right) \hat{\gamma} \sigma_R^2 = 0 \]

To prove these conditions, write down the Bellman equation:
\[ V_t = \max_{\{c_t, D_t\}} \left( -\gamma c_t \right) - \delta E_t \left[ \alpha \exp\left( -\hat{\gamma} W_{t+1} - \psi h_{t+1}^2 \right) \right] \]
where
\[ W_{t+1} = (1+r) \left( W_t - c_t \right) + D_t \left( P_{t+1} + R_{t+1} - (1+r)P_t \right) \]
and where we have used \( D_t \) to denote the LS investors’ demand for the risky asset. The first order condition with respect to \( c_t \) is:
\[ \gamma \exp\left( -\gamma c_t \right) = \delta \hat{\gamma} \left( 1+r \right) E_t \left[ \alpha \exp\left( -\hat{\gamma} W_{t+1} - \psi h_{t+1}^2 \right) \right] \]

Notice that the consumption decision is unaffected by the investment decision \( D_t \) due to CARA utility. To get an explicit expression for the right-hand side, we calculate the following expectation with respect to the two random variables \( R_{t+1} \) and \( x_{t+1} \), both of which are normally distributed with mean zero and respective variances \( \sigma_R^2 \) and \( \sigma_x^2 \):

(A1) \[ E_t \left[ -\exp\left( -\hat{\gamma} W_{t+1} - \psi h_{t+1}^2 \right) \right] = \]
\[ - \left( \sqrt{1 + 2\hat{\gamma} \psi} \right) \exp\left( -\hat{\gamma} (1+r)(W_t - c_t) \right) + \left( \frac{\alpha \hat{\gamma} D_t + \frac{h_t}{n\sigma_x^2}}{4\psi + 2/(n\sigma_x^2)} \right)^2 + \left( \hat{\gamma}^2 D_t^2 \sigma_x^2 \right) - (1+r)\hat{\gamma} \alpha D_t h_t - \left( \frac{h_t^2}{2n\sigma_x^2} \right) \]

Maximizing with respect to the choice of risky asset demand \( D_t \), we get
\[ D_t = \left( \frac{a(1+r) - \left( \frac{a}{n\sigma_x^2} \right) \left( \frac{1}{n\sigma_x^2} + 2\psi \right)^{-1}}{\hat{\gamma} \sigma_R^2 + a^2 \hat{\gamma} \left( \frac{1}{n\sigma_x^2} + 2\psi \right)^{-1}} \right) h_t \]

For market clearing we must have \( D_t = h_t \), so:
\[
1 = \frac{a(1+r) - \left(\frac{a}{n\sigma^2_x}\right) \left(\frac{1}{n\sigma^2_x} + 2\psi\right)^{-1}}{\hat{\gamma}\sigma^2_x + a^2\hat{\gamma} \left(\frac{1}{n\sigma^2_x} + 2\psi\right)^{-1}}
\]

which equates to the expression above that pins down the pricing parameter “\(a\)”. Now, collecting terms in equation (A1) involving \(h^2\), we get the coefficient \(\psi\) on \(h^2\) in the value function:

\[
\psi = \left(\frac{1}{2n\sigma^2_x}\right) + (1+r)a\hat{\gamma} - \frac{\left(a\hat{\gamma} + \frac{1}{n\sigma^2_x}\right)(a(1+r) - \sigma^2_x\hat{\gamma})}{2a} - \left(\frac{\sigma^2_x\hat{\gamma}}{2}\right)
\]

Substituting the expected value function in the next period back to the Bellman equation, we get the expression for \(\alpha\):

\[
\alpha = \left(\frac{\hat{\gamma}}{\gamma} \sqrt{1 + 2n\sigma^2 \hat{\psi}}\right)^{\frac{1}{1+r}} + \left(\frac{\hat{\gamma}}{\gamma} \alpha \sqrt{1 + 2n\sigma^2 \hat{\psi}}\right)^{\frac{r}{1+r}} \left(\frac{\alpha}{1+r}\right) \sqrt{1 + 2n\sigma^2 \hat{\psi}}
\]

It is easy to show that when \(\gamma\) is sufficiently small, there exists a positive solution for the parameters. Finally, that a value function with this simple exponential form exists ensures that the linear equilibrium pricing rule described in proposition 1 also exists (recall that the mean payoff on the risky asset is zero). Q.E.D.