1. Observe that \( \Delta W = \Delta v_1 + \Delta v_2 - \Delta C \), where, with the exception of \( \Delta C \), the notation is the same as §5.1 of “A Primer on Pricing.” Define \( \Delta C = C(x_1^* + x_2^*) - C(x_1^U + x_2^U) \). Because \( C(\cdot) \) is strictly convex, it lies everywhere above any line tangent to it; that is,

\[
C(x_1^* + x_2^*) > C(x_1^U + x_2^U) + C''(x_1^U + x_2^U)(\Delta x_1 + \Delta x_2) \quad \text{and} \quad C(x_1^U + x_2^U) > C(x_1^* + x_2^*) - C''(x_1^* + x_2^*)(\Delta x_1 + \Delta x_2).
\]

Hence,

\[
-C''(x_1^* + x_2^*)(\Delta x_1 + \Delta x_2) < -\Delta C < -C''(x_1^U + x_2^U)(\Delta x_1 + \Delta x_2). \tag{1}
\]

Define \( MC^U = C'(x_1^U + x_2^U) \) and \( MC^* = C'(x_1^* + x_2^*) \). Recalling expression (36) of “A Primer on Pricing,” we can conclude that

\[
(p^U - MC^U)(\Delta x_1 + \Delta x_2) > \Delta W > (p_1^* - MC^*)\Delta x_1 + (p_2^* - MC^*)\Delta x_2. \tag{2}
\]

Observe that expression (2) has the same interpretation as expression (37) of “A Primer on Pricing.”

2. Recall that for linear inverse demand of the form \( p = A - Bx \), with constant marginal cost, \( c \), the profit-maximizing quantity under linear pricing is \( \frac{A-c}{2B} \) (see, e.g., footnote 6 of “A Primer on Pricing”). Hence, under an unconstrained third-degree price discrimination scheme (i.e., linear pricing to the different populations or segments), total sales are \( a_2 + a_1 \) (recall, here, \( c = 0 \)). Given the condition

\[
\frac{a_2 - a_1}{2b} < K < \frac{a_2 + a_1}{2b}, \tag{3}
\]

this proves the seat constraint is binding. Given the constraint is binding, let \( x_2 \) be the seats sold to population 2 and \( K - x_2 \) be, therefore, the seats sold to population 1. The optimal \( x_2 \) is an interior solution (i.e., between 0 and \( K \)) if equating the marginal revenues has an interior solution:

\[
a_1 - 2b(K - x_2) = a_2 - 2bx_2. \]

Obviously \( x_2 > 0 \) because \( a_2 > a_1 \). So the question is whether \( x_2 < K \). Solving the last equation for \( x_2 \):

\[
x_2 = \frac{1}{2} K + \frac{1}{2} \left( \frac{a_2 - a_1}{2b} \right), \tag{4}
\]
which is less than $K$ from the first inequality of (3).

If $K = 100$, $a_2 = 200$, $a_1 = 80$, and $b = 1$, then (4) yields $x_2 = 80$; that is, 20 seats are sold to population-1 consumers. Note that $p_1(0) = 80 < p_2(100) = 100$; this means that it is inefficient to sell any seats to population-1 consumers. The welfare loss from doing so is, thus, the forgone benefit of selling the last 20 seats to population-2 consumers less the benefit of selling them to population-1 consumers:

$$\int_{80}^{100} p_2(t)dt - \int_{0}^{20} p_1(t)dt = \int_{0}^{20} \left( [200 - (t + 80)] - [80 - t] \right) dt$$

$$= \int_{0}^{20} 40dt$$

$$= 800.$$

3. Observe that $\partial v(x_2, 2)/\partial x = 10 - x$. Equate to marginal cost (recall, no distortion at the top), so $x_2^* = 9$. Now consider, $x_1$. $I(x) = v(x, 2) - v(x, 1) = 5x$. The left-hand side of expression (60) from “A Primer on Pricing” is, thus,

$$5 - x_1 - 5 - 1 = -x_1 - 1.$$

This cannot equal zero for a positive $x_1$; hence, we can conclude that population-1 is excluded from buying. In other words, the seller sells only to population 2. The solution is, therefore, $x_1^* = T_1 = 0$ and $x_2^* = 9$ and $T_2 = v(9, 2) = 49.5$. 
