1 Introduction

These notes sketch a 1 hour and 50 minute lecture on auctions.

The main points are:
1. Set up the basic auction problem.
2. Derive the revenue equivalence theorem.
3. Derive an optimal auction.
4. Risk aversion
5. Talk about experiments and applications in this area.

2 Preliminaries

- Perhaps the most fruitful area for the application of optimal screening contracts is in auction theory
- 1 principle, many bidders.
- The goal might be to allocate efficiently (and maximize revenues while doing so)
- Or it might just be to maximize revenues.

The main result we will establish is:

Revenue Equivalence Theorem (RET): Assume each of a given number of risk-neutral potential buyers has a privately-known valuation independently drawn from a strictly-increasing atomless distribution, and that no buyer wants more than one of the $k$ identical indivisible prizes.

Then any mechanism in which

- (i) the prizes always go to the $k$ buyers with the highest valuations and
- (ii) any bidder with the lowest feasible valuation expects zero surplus,

yields the same expected revenue (and results in each bidder making the same expected payment as a function of her valuation).
2.1 Model
Throughout, we study the archetypal auction model:

- \( n \) ex ante identical potential bidders
- independent private values, \( v_i \), drawn from some atomless distribution \( F \) on \([0, 1]\).
- Single object to be auctioned.
- Seller has commonly known valuation \( v_0 \).

Game form
Seller chooses the “contract” (i.e. auction form).
Bidders bid.
Payoffs are realized.

3 Second Price Auction
Consider an auction where the winning bidder pays the second highest bid (introduced by Vickrey)

**Proposition 1** Suppose bidders have private values, then bidding one’s valuation is a weakly dominant strategy.

**Proof.** General and informal.
Suppose you bid \( b > v_1 \), then the outcome compared to the putative eqm strategy is changed only when highest (other than 1’s) is between \( v_1 \) and \( b \). But in these circumstances you incur losses.

Suppose you bid \( b < v_1 \), then the outcome compared to the putative eqm strategy is changed only when highest bid (other than 1’s) is between \( b \) and \( v_1 \). In these circumstances you miss out on profits. ■

Less general, more mathematical:
Suppose everyone else is bidding according to the increasing bidding strategy \( \beta(v) \). Let \( F_1^{(n-1)}(y) \) be the distribution function of the highest of \( n - 1 \) draws from \( F \).

That is
\[
y = \max \{v_2, v_3, ..., v_n\}
\]
Bidder 1 wins if \( b \geq \beta(y) \), so
\[
E \pi_1(v_1, b) = \int_0^{\beta^{-1}(b)} [v_1 - \beta(y)] dF_1^{(n-1)}(y).
\]
Differentiating wrt \( b \),
\[
[v_1 - b] f_1^{(n-1)}(\beta^{-1}(b)) \frac{d}{db} [\beta^{-1}(b)] = 0
\]
Since \( f_1^{(n-1)}(\beta^{-1}(b)) \neq 0 \) and \( \frac{d}{db} [\beta^{-1}(b)] \neq 0 \), it then follows that \( b = v_1 \).

It then follows that:

**Proposition 2** The Vickrey auction is efficient when bidders have private values.

- Experimental findings sometimes differ from this.

**Revenues**

The seller’s expected revenue is simple \( E[y_2^{(n)}] \), i.e., the second highest of \( n \) draws from \( F \).

With uniform distributions, this becomes

\[
E[y_2^{(n)}] = \frac{n - 1}{n + 1}
\]

so it’s obvious that revenues are increasing in \( n \) and converge to where the seller obtains all the surplus.

We can get at revenues a different way:

What is the expected payment of a bidder with valuation \( x \)?

\[
P_{II}(x) = \Pr\{\text{win}\} \times E\{\text{payment|win}\}
= \Pr\{\text{win}\} \times \frac{E\{\text{payment} \times I_{\text{win}}\}}{\Pr\{\text{win}\}}
= \int_{y \leq x} y dF_1^{(n-1)}(y)
\]

Under the uniform distribution

\[
= \int_{0}^{x} y dy \left[ y^{n-1} \right]
= \frac{n - 1}{n} x^n
\]

The ex ante expected payment is then

\[
P_{II} = \int_{0}^{1} P_{II}(x) dF(x)
= \frac{n - 1}{n(n + 1)}
\]

Expected revenue

\[
ER = nP_{II}
= \frac{n - 1}{n + 1}
\]
More generally

\[ P_{II} (x) = \int_{y \leq x} ydF_1^{(n-1)} (y) \]

Integrate by parts

\[ P_{II} (x) = xF_1^{(n-1)} (x) - \int_0^x F_1^{(n-1)} (y) dy \]
\[ = xF^{n-1} - \int_0^x F^{n-1} dy \]

4 First Price Auctions

Bidder 1, given valuation \( v_1 \), chooses \( b \) to maximize

\[ E\pi_1 (v_1, b) = \int_0^{\beta^{-1}(b)} [v_1 - b] dF_1^{(n-1)} (y) \]
\[ = [v_1 - b] F_1^{(n-1)} (\beta^{-1} (b)) \]

Differentiate

\[ [v_1 - b] dF_1^{(n-1)} (\beta^{-1} (b)) \frac{d}{db} (\beta^{-1} (b)) - F_1^{(n-1)} (\beta^{-1} (b)) = 0 \]

Under a symmetric equilibrium, \( b = \beta (v_1) \); hence

\[ [v - \beta (v)] f_1^{(n-1)} (v) \frac{1}{\beta' (v)} - F_1^{(n-1)} (v) = 0 \]

Rewriting

\[ \beta' + \beta \times A (v) = vA (v) \]

where \( A (v) = \frac{f_1^{(n-1)} (v)}{F_1^{(n-1)} (v)} \).

Multiply by \( e^{\int A} \):

\[ \beta' e^{\int A} + \beta A e^{\int A} = vAe^{\int A} \]
\[ \frac{d}{dv} (\beta e^{\int A}) = vAe^{\int A} \]

Hence

\[ be^{\int A} = \int vAe^{\int A} dv + c \]
\[ b (v) = e^{-\int A} \left( \int vAe^{\int A} dv + c \right) \]

Since \( b (0) = 0 \), then

\[ b (v) = e^{-\int A} \left( \int_0^v tA (t) e^{\int A} dt \right) \]
Now
\[ e^f A = e^f \frac{(n-1)(x_i)^{n-2} f(x)}{(F(x))^{n-1}} dx \]
\[ = e^f \frac{(n-1) f(x)}{F(x)} dx \]
\[ = e(n-1) \ln F(x) \]
\[ = F(x)^{n-1} \]

Hence
\[ b(v) = \frac{1}{F(v)^{n-1}} \int_0^v (n-1) x \frac{f(x)}{F(x)} F(x)^{n-1} dx \]
\[ = \frac{1}{F^{n-1}} \int_0^v x d [F^{n-1}] \]

Integrate by parts
\[ b(v) = v - \frac{1}{F^{n-1}} \int_0^v F^{n-1} dx \]

Compute expected payment:
\[ P_I(x) = b(x) F^{n-1}(x) \]
\[ = x F^{n-1} - \int_0^x F^{n-1} dy \]

Notice that it is exactly the same as the Second price auction.
Thus, the expected revenue to the seller in either of these auctions is simply
\[ ER = E_2^{(n)} [v] \]
i.e. the expectation of the second highest of \( n \) draws.
Thus, both types of auctions are equally good (Vickrey 1962).

- Strategic equivalence with other auction forms.
  - Difference between strat equivalence in first-price versus second-price case.

- But experiments do not bear this out.

5 The Revenue Equivalence Theorem

Consider the following set of auction contracts.

- Announce a minimum opening bid \( b_0 \).
- High bidder wins
- Rules are anonymous
• Strictly increasing symmetric bidding strategy in auction.
• Non-negative returns to bidding.

Use the revelation principle to restrict attention to direct mechanisms.

**Bidder 1’s Problem**
Given a valuation \( v \), choose a message \( \hat{v} \) to maximize

\[
\pi(\hat{v}, v) = v F^{n-1}(\hat{v}) - P_A(\hat{v})
\]

where \( P_A \) is the expected payment from pretending to be a \( \hat{v} \) type in auction form \( A \).

Optimize with respect to \( \hat{v} \):

\[
\pi_1(\hat{v}, v) = v (n-1) F^{n-2}(\hat{v}) f(\hat{v}) - P'_A(\hat{v})
\]

In equilibrium, \( v = \hat{v} \)

\[
v (n-1) F^{n-2}(v) f(v) = P'_A(v)
\]

Boundary condition: Find a type \( v_* \) solving

\[
P_A(v_*) = v_* F(v_*)^{n-1}
\]

Now solve the differential equation

\[
\int_{v_*}^{v} P'_A(x) \, dx = \int_{v_*}^{v} x dF^{n-1}(x)
\]

Notice that the RHS is independent of the auction form!

With algebra

\[
P_A(v) = v F^{n-1}(v) - \int_{v_*}^{v} F^{n-1}(x) \, dx
\]

for all \( v \geq v_* \)

Expected revenue

\[
ER = nE_v [P_A(v)]
\]

\[
= n \int_{v_*}^{1} [vf(v) + F(v) - 1] F^{n-1}(v) \, dv
\]

So we have proved the revenue equivalence theorem!

### 6 Applications
Knowing the RET can be helpful in directly computing bidding strategies.

**First-price auction**
\[ P_I (v) = \Pr \{ y \leq v \} b (v) \]
\[ vF^{n-1} (v) - \int_{v_\ast}^v F^{n-1} (x) \, dx = F^{n-1} (v) b (v) \]

So
\[ b (v) = v - \frac{1}{F^{n-1} (v)} \int_{v_\ast}^v F^{n-1} (x) \, dx. \]

**All-pay Auction**
\[ P_{AP} (v) = \gamma (v) \]

so
\[ \gamma (v) = vF^{n-1} (v) - \int_{v_\ast}^v F^{n-1} (x) \, dx \]

And so on.

### 7 Empirical Tests of the RET
- Turkish treasury auctions
- Structural estimation literature
- eBay experiments

### 8 Revenue Maximization
Using the RET, the principal’s problem is to choose \( v_\ast \) to maximize
\[ ER = v_0 F (v_\ast)^n + n \int_{v_\ast}^1 \left[ vf (v) + F (v) - 1 \right] F^{n-1} (v) \, dv \]

Rewrite this
\[ ER = v_0 F (v_\ast)^n + n \int_{v_\ast}^1 \left[ v - \frac{1 - F (v)}{f (v)} \right] dF^n (v) \, dv \]

call the term in the square brackets the marginal revenue to the seller.

Optimizing
\[ v_0 dF (v_\ast)^n - \left[ v_\ast - \frac{1 - F (v_\ast)}{f (v_\ast)} \right] dF^n (v_\ast)^n = 0 \]

Which implies
\[ v_0 = \left[ v_\ast - \frac{1 - F (v_\ast)}{f (v_\ast)} \right] \]

or MR=MC.
Further

\[ v^*_s = v_0 + \frac{1 - F(v^*_s)}{f(v^*_s)} \]

so revenue maximization and allocative efficiency are in conflict.

- For first and second price auctions this means that the optimal auction is simply to choose an opening bid of \( v^*_s \).
- This opening bid is equal for these two auction forms
- An entry fee will do the trick as well (same entry fee for all auction forms).
- A small increase in the reserve above its lowest level always raises revenues.
- Optimal reserve price is independent of \( n \).

**Negotiation**

To see the monopoly interpretation, consider the case where \( n = 1 \).

In this case, the auctioneer is simply a monopolist facing the problem of choosing an offer to maximize

\[ E\pi = p(1 - F(p)) + v_0 F(p) \]

so \( 1 - F \) is the demand curve.

Optimizing

\[ (1 - F) - pf + v_0 f = 0 \]

Dividing and rearranging

\[ p - \frac{1 - F}{f} = v_0 \]

or

\[ MR = MC \]

which is of course the same as the optimal reserve price in an \( n \) player auction.

## 9 Tort Reform

- US society is too litigious
- How to reduce incentives to sue?

Toy model: Both parties privately observe the value of winning the case. Each decides how much to spend on lawyers. Higher spender wins.

- This is just the all-pay auction we analyzed earlier.
- European system: Loser pays winner’s expenses.
  - Notice that the expected payoff from the lowest type is negative
So the RET does not hold
So what happens?

Bidder’s problem is to choose a bid $b$ to maximize

$$E\pi (\hat{v}, v) = v F (\hat{v}) - \beta (\hat{v}) (1 - F (\hat{v})) - \int_{\hat{v}}^{1} \beta (t) f (t) dt$$

Optimizing

$$vf (\hat{v}) - \beta' (\hat{v}) (1 - F (\hat{v})) + \beta (\hat{v}) f (\hat{v}) + \beta (\hat{v}) f (\hat{v}) = 0$$

In equilibrium, $\hat{v} = v$

$$vf - \beta' (1 - F) + 2\beta f = 0$$

Rewriting

$$\beta' + \beta P (v) = Q (v)$$

where $P (v) = - \frac{f}{1 - F}$ and $Q (v) = v \frac{f}{1 - F}$.

Usual trick:

$$\beta (v) = e^{-\int P} \int Q e^{P} dx$$

and since

$$\int P = \ln (1 - F) .$$

Then

$$\beta (v) = \frac{1}{1 - F} \int_{0}^{v} xf (x) dx$$

No upper bound on bidding.

Expected payment

$$EP_{EUR} (v) = \beta (v) (1 - F (v)) + \int_{0}^{1} \beta (t) f (t) dt$$

Recall that expected payment in a Vickrey auction is

$$EP_{II} (v) = \int_{0}^{v} xf (x) dx$$

Thus, for all $v$

$$EP_{EUR} (v) > EP_{II} (v)$$

Hence

$$EP_{EUR} > EP_{II}$$

so the European legal system is more expensive than the American system.
- Quayle plan: Losing party pays the winner an amount equal to his own expenses.

Here the RET applies. So the Quayle plan costs exactly the same as the current plan.

- Same incentives to initiate litigation.