Some Notes on Costless Signaling Games

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1 Preliminaries

- Our running example is that of a decision maker (DM) consulting a knowledgeable expert for advice about what decision to make
  - The decision affects the welfare of both
  - The expert’s information is soft: no contracts are possible.

2 The Model

There are a continuum of states of nature $\theta$. States are drawn from the atomless distribution $F$ with support $[0, 1]$ and positive mass everywhere.
- The decision maker chooses an action $y \in \mathbb{R}$.
- The expert chooses a message $m \in [0, 1]$

Payoffs
- The payoff of the DM is $U(y, \theta)$.
- The payoff for the expert is $V(y, \theta, b)$ where $b$ is the expert’s bias relative to the DM.

Note that $m$ does not enter the payoffs at all!

Extensive Form
- E learns $\theta$
- E sends message $m$
- DM chooses $y$
- Payoffs are realized

Solution Concept: Perfect Bayesian Equilibrium
- A strategy for E is $\mu(\theta) : \theta \rightarrow m$
- A strategy for DM is $y(m) : m \rightarrow y$
- Let the beliefs of the decision maker following message $m$ be given by the cdf $G_m(\theta)$.
2.1 Leading Example

For the purpose of these notes, we’ll use the leading example of Crawford and Sobel (1982) to generate results. Specifically, we assume that

1. \( \theta \) is distributed uniformly on the unit interval
2. The DM’s payoff function is
   \[
   U (y, \theta) = -(y - \theta)^2
   \]
3. The expert’s payoff function is
   \[
   V (y, \theta, b) = -(y - (\theta + b))^2
   \]
   The DM’s most preferred action in state \( \theta \) is then \( y^* (\theta) = \theta \); whereas E’s most preferred action is \( y^* (\theta, b) = \theta + b \).

- Notice that if \( b = 0 \), preferences are perfectly aligned.

Since the DM must choose the best possible \( y \) given beliefs, the following lemma is helpful:

**Lemma 1** In any PBE, \( y (m) = E (\theta|m) \).

Recall that optimization required choosing \( y \) to maximize

\[
\int_0^1 -(y - \theta)^2 dG_m (\theta)
\]

Differentiating

\[
\int_0^1 -2 (y - \theta) dG_m (\theta) = 0
\]

Thus, an optimal \( y \) solves

\[
y = \int_0^1 \theta dG_m (\theta)
\]

= \( E (\theta|m) \)

This completes the proof.

Another structural property of equilibrium proves very useful:

Consider an equilibrium consisting of a finite number of equilibrium actions \( y_1 < y_2 < \ldots < y_n \). Suppose that message \( m_i \) induces action \( y_i \).

**Lemma 2** The expert should induce action \( y_i \) in the interval of state \([a_{i-1}, a_i]\) where \( a_{i-1} \) solves

\[
y_i - (a_{i-1} + b) = a_{i-1} + b - y_{i-1}
\]

and \( a_i \) solves

\[
y_{i+1} - (a_i + b) = a_i + b - y_i
\]

unless \( i = 1 \), in which case \( a_{i-1} = 0 \) or \( i = n \), in which case \( a_n = 1 \).
When $\theta = a_{i-1}$, $E$ is indifferent between inducing action $y_i$ and action $y_{i-1}$. Further, it is easy to verify that both of these actions are preferred to lower actions.

Next, notice that since $V_{12} > 0$, for $\theta > a_{i-1}$, $y_i$ is strictly preferred to $y_{i-1}$ and vice-versa for $\theta < a_{i-1}$.

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Combining all of these implications, we can conclude that $E$ can do no better than to induce action $y_i$ in interval $[a_{i-1}, a_i]$.

This argument also helps to say something useful about another structural property of equilibria consisting of only $n$ actions. Let $Y(\theta)$ be the equilibrium action induced in state $\theta$ in a given equilibrium (because of risk aversion of the DM, only one action will be induced in equilibrium).

**Definition 3** An equilibrium is said to be monotonic if for all $\theta', \theta''$ where $\theta' < \theta''$, $Y(\theta') \leq Y(\theta'')$.

With this definition, Lemma 2 then implies:

**Corollary 4** All equilibria are monotonic.

### 2.2 Benchmark Case $b = 0$

We begin by considering the benchmark where preferences are perfectly aligned.

**Proposition 5** Suppose that preferences are perfectly aligned, then full revelation is a PBE:

Proof: Consider the strategies:

$\mu(\theta) = \theta$

$y(m) = m$

$Pr(\theta = m|m) = 1$

Clearly, the expert can do no better than to get his best action in each state.

Clearly, the DM is optimizing given beliefs and beliefs are formed using Bayes’ rule.

So this is a legitimate PBE.

**Proposition 6** Suppose that preferences are perfectly aligned, then no revelation (babbling) is a PBE.
Proof: Consider the strategies:
\[ \mu(\theta) = \frac{1}{2} \]
\[ y(m) = \frac{1}{2} \]

Supported by beliefs:
\( \theta \) is distributed uniformly on \([0, 1]\) for all \( m \).
Since the expert’s message does not affect the action, there is no profitable devi-
ation for the expert.
Since the expert’s message is independent of \( \theta \), the DM’s beliefs use Bayes’ rule where possible.
Clearly, the DM is optimizing given beliefs.
- So there is no clear prediction about the informativeness of equilibrium even in this simple case.

Now consider an equilibrium of intermediate informativeness where the equilib-
rium consists of \( n \) actions.
From Lemma 2, we know that action \( y_i \) will be induced only for \( \theta \in [a_{i-1}, a_i] \). Hence, the DM must believe that following \( m_i \), the state is uniformly distributed in this interval. From Lemma 1, it then follows that \( y_i = \frac{a_i + a_{i-1}}{2} \). Thus, an equilibrium consists of “cut points” \( a = \{a_0, a_1, ..., a_n\} \) where \( a_0 = 0, a_1 = 1 \), satisfying both of these lemmas.

Let’s compute a size 2 equilibrium. This means we need to find a cut point \( a_1 \) solving
\[ a_1 + b - \frac{a_1}{2} = \frac{1 + a_1}{2} - (a_1 + b) \]
or
\[ a_1 = -2b + \frac{1}{2} \]
and in our benchmark case where \( b = 0 \), this means that we sensibly divide the state space into 2 equal intervals.

How about a size 3? This means we need to solve:
\[ a_1 + b - \frac{a_1}{2} = \frac{a_2 + a_1}{2} - (a_1 + b) \]
\[ a_2 + b - \frac{a_1 + a_2}{2} = \frac{1 + a_2}{2} - (a_2 + b) \]
which yields
\[ a_1 = -4b + \frac{1}{3} \]
\[ a_2 = -4b + \frac{2}{3} \]
and so on.
One can then see how to arrive at a general statement:
Proposition 7 When \( b = 0 \), there exists a PBE consisting of \( n \) equilibrium actions where \( y_i = \frac{2i-1}{2n} \).

These equilibria may be ordered by their informativeness: higher \( n \) is more informative. The limit as \( n \to \infty \) converges in payoffs to the fully revealing PBE.

3 Misaligned Preferences

Now consider the case where \( b > 0 \). In this case, E and DM do not want exactly the same action.

Proposition 8 When preferences are misaligned, full revelation is impossible.

Consider two states, \( \theta' \) and \( \theta' + b \). We know that \( y (\mu (\theta')) = \theta' \) and \( y (\mu (\theta' + b)) = \theta' + b \). But now E can profitably deviate by sending message \( \mu (\theta' + b) \) in state \( \theta' \) and thereby induce his favorite action.

In fact, one can say something even stronger.

Proposition 9 When preferences are misaligned, there exists \( \varepsilon > 0 \) such that for any two equilibrium actions \( |y' - y''| > \varepsilon \).

Suppose not. Wlog, let \( y' < y'' \). By Lemma 2, we know that there exists a state \( \theta = a \) where E is indifferent between the 2 actions. We know that for all \( \theta < a \), \( y'' \) will not be induced. Likewise, for all \( \theta > a \), \( y' \) will never be induced. Therefore, \( y' < y^* (a) \) and \( y'' > y^* (a) \). Further, since at \( a \), E is indifferent, we know that \( y' < y^* (a, b) < y'' \). Finally, since \( y^* (a, b) - y^* (a) = b \), it then follows that

\[ y' < y^* (a) < y^* (a, b) < y'' \]

so \( y'' - y' > b \).

Corollary 10 When preferences are misaligned, all equilibria consist of only finitely many equilibrium actions.

Follows from the fact that the set of rationalizable actions is \( y \in [0, 1] \) combined with the fact that all actions are at least \( b \) apart.

- So now we know that all equilibria lead to information transmission that is a partition of the state space. These are sometimes referred to as “partition equilibria”

When preferences are misaligned, there are some interesting properties to these equilibria.

Suppose that there are \( n \) equilibrium actions ordered \( y_1 < y_2 < ... < y_n \)
Lemma 11 When preferences are misaligned, the interval in which $y_i$ is induced is smaller than the interval in which $y_{i+1}$ is induced.

Recall that at cut point $a_i$, equation (2) holds. Therefore, using Lemma 1,

\[ a_i + b - \frac{a_i + a_{i-1}}{2} = \frac{a_{i+1} + a_i}{2} - (a_i + b) \]

\[ \frac{a_i - a_{i-1}}{2} + b = \frac{a_{i+1} - a_i}{2} - b \]

or

\[ a_{i+1} - a_i = a_i - a_{i-1} + 4b \]

and we’re done.

Now we’re in a position to characterize the equilibria.

Let the cut point $a_1$ be given by $\alpha$. Then

\[ a_2 = 2\alpha + 4b \]

where we have used the fact that $a_0 = 0$.

\[ a_3 = 2a_2 - \alpha + 4b \]

\[ a_3 = 3\alpha + 12b \]

and so in.

This gives a difference equation:

\[ a_i = \alpha i + 2i(i - 1) b \] (3)

Now, to figure out the largest number of cut points possible, notice that equation (3) is increasing in $\alpha$. Setting $\alpha = 0$ therefore makes $a_n$ as small as possible. If, however, $a_n > 1$ when $\alpha = 0$, then there’s no way to construct a partition with that many cut points. Therefore, the maximum number of feasible cut points is the largest integer $i$ such that

\[ 2i(i - 1) b < 1. \]

Solving for a feasible $n$ then consists of increasing $\alpha$ until $a_n = 1$. This yields:

\[ a_i = \frac{i}{n} + 2bi(i - n) \]

We can then calculate the payoffs to the DM. These are given by

\[ EU = \sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} (y_i - \theta)^2 d\theta \]

and this may be shown to be

\[ EU = -\left( \frac{1}{12n^2} + \frac{b^2}{3} (n^2 - 1) \right). \]

It is easy to show that this is increasing in $n$ and decreasing in $b$. 

4 Contract Theory

We now consider two sets of contracts. First, where the DM can make (non-negative) transfers contingent on messages but cannot commit to actions. The second, where the DM can commit to actions but cannot make non-negative transfers.

**Variation 1: Transfers without commitment**

Consider a simple scheme whereby the DM specifies a compensation amount \( t(m) \) as a function of the message. The DM cannot commit to an action. We assume a bankruptcy constraint on the part of E, so \( t(m) \geq 0 \) for all \( m \).

Consider the truthful contract. Under truthful revelation, \( Y(\theta) = \theta \).

Now to induce truth-telling by the expert requires that \( \hat{\theta} = \theta \) maximize

\[
EV(\theta, \hat{\theta}) = -\left(\hat{\theta} - (\theta + b)\right)^2 + t(\hat{\theta})
\]

Differentiating

\[
-2\left(\hat{\theta} - (\theta + b)\right) + t'(\hat{\theta}) = 0
\]

Since this must hold at \( \hat{\theta} = \theta \), we obtain the differential equation

\[
t'(\theta) = -2b
\]

And since we don’t need to compensate E to induce the highest action, the endpoint is

\[
t(1) = 0
\]

Hence

\[
t(\theta) = 2b(1 - \theta)
\]

This yields the following expected utility to the DM

\[
EU = -\int_0^1 t(\theta) \, d\theta
\]

\[
= -\int_0^1 2b(1 - \theta) \, d\theta
\]

\[
= -b
\]

Notice that when \( n = 1 \), \( EU = -\frac{1}{12} \) when there are no contracts; therefore for all \( b \geq \frac{1}{12} \), contracting this way is worse. In fact, one can show that it is always worse.

**Variation 2: No transfers, but commitment**

Consider a simple contract whereby the DM delegates all decision making rights to the expert. In this case, the expert choose \( y(\theta) = \theta + b \).

The DM’s expected payoff is then

\[
EU = \int_0^1 -(\theta + b - \theta)^2 \, d\theta
\]

\[
= -b^2
\]
Notice immediately that this is better than the contract with truth-telling and no commitment.

**Proposition 12** *Whenever E will communicate any information to the DM whatsoever, delegation is better than talk.*

To prove this, we need to show that for all \( b \leq \frac{1}{4} \)

\[
\frac{1}{12n^2} + \frac{b^2(n^2 - 1)}{3} > b^2
\]

\[
\frac{1}{12n^2} + \frac{b^2(n^2 - 4)}{3} > 0
\]

When \( n \geq 2 \), then required inequality obviously holds.
When \( n = 1 \), we need

\[
b^2 < \frac{1}{12}
\]

or

\[
b < \frac{1}{2\sqrt{3}}
\]

\[\approx .29\]

and we’re done.