Pushing Notifications as Dynamic Information Design *

Ganesh Iyer
University of California, Berkeley

Zemin (Zachary) Zhong
University of Toronto

This version: Apr 2021

*We thank the senior editor, the area editor, two anonymous reviewers, and Tony Ke, Shubhranshu Singh, and seminar participants at Columbia University, University of Toronto, University of California, Berkeley, and conference participants of the 2019 UT Dallas Frontiers of Research in Marketing Science Conference and the 2019 Marketing Science Conference for useful feedback. Previously titled "Pushing Information: Realized Uncertainty and Notification Design." Address for correspondence: Haas School of Business, University of California, Berkeley, Berkeley, CA 94720-1900. Email: giyer@haas.berkeley.edu; Rotman School of Management, University of Toronto, Toronto, ON M5S 3E6, Canada. Email: zachary.zhong@rotman.utoronto.ca.

Pushing Notifications as Dynamic Information Design

ABSTRACT

We study the dynamic information design problem of a firm seeking to influence consumer checking behavior by designing push notifications. Firm payoffs are increasing in the frequency of consumer checking. The consumer is uncertain about the arrival of information as well as its valuation. In addition to direct consumption utility, the consumer also has preferences over realized uncertainty: she experiences disutility (anxiety) from the variance of the unchecked information stock. We show that push notifications can lead to more frequent checking compared to no-push, even though it reduces the information variance. While push notifications resolve the information arrival uncertainty, they also create an endogenous impulse to check the information immediately. They can allow the firm to create a more efficient spread in the consumer's beliefs/anxiety between zero or a level enough to induce checking. We generalize push strategies in two directions: a noisy push strategy that allows the firm to add phantom notifications and a partial push strategy in which the firm can mute information arrivals. Despite consumers having rational expectations, we establish conditions under which both these strategies increase checking. We also extend the model to account for consumer self-control as well as the possibility of endogenous prices.

Keywords Information Design, Dynamic Persuasion, Push Notifications, Realized Uncertainty

JEL Codes D83, D91, L86, M31

I Introduction

With the proliferation of smartphones, tablets, and wearable devices, consumers are increasingly overwhelmed by information. With a few clicks or taps, consumers can check for the latest updates of their e-mail accounts, news feeds, and social media. The average American checks his/her phone every 12 minutes and about 80 times a day. A 2018 study by Kleiner Perkins Caulfield and Byers reports that the average amount of time spent by users on their smartphones has increased every year and now stands at 3.3 hours a day. [1]. Recent studies have also documented the proliferation of information notifications that users face on their smartphones (Sahami Shirazi et al., 2014; Pielot et al., 2014)

One important reason for this growth in information checking and consumption is that in the modern digital economy, user attention (eyeballs) is a valuable commodity. Major firms such as Facebook and Google, as well as apps on the Android and Apple platforms, make money through user engagement, i.e., by inducing consumers to spend time and check their sites and apps as frequently as possible. Consequently, social media platforms and apps use strategic information design and presentation techniques to grab user attention and time. For example, it is a common practice in the industry to build the "variable rewards" paradigm into the app design: i.e., inducing users to stay longer and to check for more information by introducing uncertainty both into the timing and the value of future information arrivals. Twitter's spinning wheel indicates that the app is loading more content when the user swipes, but whether or not relevant and interesting content will be loaded is uncertain. Users who enable push notifications on Instagram will receive numerous messages from social connections including "stories" to attract them.[2]

Examining how the firm's notification technology influences consumer behavior and how that, in turn, influences the information design strategies of firms is important for understanding the functioning of these information markets. Several studies and industry reports indicate that consumers see themselves as being progressively addicted to smartphone usage and over-checking of information at the cost of personal productivity (see Duke

[1] https://www.kleinerperkins.com/perspectives/internet-trends-report-2018/

[2] See Avery Hartmans "How app developers keep us addicted to our smartphones" *Business Insider*, 02/17/2018 for other examples.

and Montag, 2017). Furthermore, recent studies show that smartphone users suffer anxiety/withdrawal costs if they are unable to check information notifications that they know have already arrived (Tams et al., 2018). Industry analysts also point to social media users experiencing what is labeled as the fear of missing out (FOMO), the idea that individuals want to constantly check for information because they worry that they will miss out on the news generated in their social network. Indeed, in response to growing concerns about the extent of addiction to devices and the over-consumption of information, both Apple and Google have recently developed digital health initiatives to help users to manage their information consumption on mobile devices.

Our paper develops a theory of dynamic information design in the context of consumer information markets. We present a model of dynamic information consumption that captures the interaction between a strategic information provider and a forward-looking consumer who makes information checking decisions over time. Pieces of information arrive randomly over time according to a Poisson process, and they vary in their valuation (usefulness) to the consumer. For example, e-mails or updates on social media may arrive at different points in time. And they contain either relevant information or junk for the user. At each point in time, the consumer decides whether to check the information stock (e.g., unread e-mails in the inbox) by incurring some cost of checking.

The central aspect of consumer behavior that is captured in the model is preferences over "realized uncertainty." In other words, apart from the direct value of information consumption, the consumer also carries disutility for information that may have arrived but which they have left unresolved. This disutility can be interpreted as the anxiety consumers feel from the uncertainty of the information that may have arrived. Specifically, the disutility is increasing in the variance of the unresolved information stock. There are two different components of realized uncertainty in the information stock: about how many pieces of new information have arrived and about the valuation of each piece of information. This preference over realized uncertainty represents the previously described phenomenon: Consumers feel the compulsion to check their smartphones constantly due to the anxiety of not immediately consuming potentially useful information that may have arrived. And the anxiety is higher when there is uncertainty in the number of information arrivals, and when each piece of information has a higher variance.

The firm's objective is to choose an information notification design to maximize its profit, which is an increasing function of the frequency of consumer checking. The firm's payoff can therefore be seen as a function of the extent of consumer engagement. We consider the following commonly observed classes of notification design: At the one extreme, the firm can choose to adopt a passive "no-push" strategy in which the firm does not provide any notifications of information arrival. In this case, the consumer will make decisions based on the expectation of the information arrival process. The optimal strategy is to check at constant intervals of time. At the other extreme, the firm can use a genuine "push" strategy in which it commits to providing notifications to the consumer every time a new piece of information arrives. For example, the push notifications on iOS and Android for e-mail and other apps can alert consumers whenever a new piece of information arrives. The genuine push strategy reduces consumer uncertainty by resolving the uncertainty over how many pieces of new information have arrived at any point in time. Apart from these two extremes, there is a generalized "noisy push" strategy in which the firm chooses to add phantom push notifications even in the absence of true information arrival. We also generalize push notifications in an alternative direction by considering a partial push strategy in which the firm releases notifications of information arrivals only with some probability.

One of the central results of the paper is that the push strategy can be optimal for the firm even though it unambiguously reduces the consumer's uncertainty. Within the framework, push notifications are likely to lead to more frequent checking compared to no-push when the variance in the value of the information is high relative to the mean. Under the no-push strategy, the consumer's anxiety costs and the impulse to check information increases in time. In other words, there is uncertainty both in the arrivals and the value of information. Given the Poisson process, the total uncertainty and anxiety increase linearly in time. The genuine push strategy fully resolves the arrival uncertainty. But every time there is a notification, the consumer also faces realized uncertainty in that she knows that there is a piece of information whose valuation is unknown that is waiting to be consumed. This increases the anxiety costs of waiting discontinuously at the point of notification arrival and creates an endogenous impulse to check the information immediately to nullify the jump in anxiety costs. And this impulse is increasing in the variance of the information value. If the impulse is strong enough, the consumer will check more frequently than in the no-push case.

The economic mechanism underlying the optimality of push notifications depends upon how they govern the evolution of consumer beliefs and the associated anxiety. Push notifications, by definition, lead to a discretization of checking behavior. Beliefs jump discretely at the arrival of notification but remain constant between any two notifications. Therefore, notifications can induce a spread in the beliefs such that the consumer either believes that there is no information, or that there is enough amount of information to trigger checking that she otherwise would not do. Consider a simple example: Suppose every notification arrival introduces just enough of a jump in anxiety to trigger checking. In the case of no-push, the consumer's belief increases linearly in time before she checks. That implies some of the anxiety is "wasted" when the consumer is becoming progressively anxious but not sufficiently enough to trigger checking. In contrast, push notifications can introduce an efficient spread in the consumers' belief evolution, such that the consumer is either not at all anxious (when there is no notification), or anxious enough to check. When spread in the beliefs display this type of efficiency, the consumer may also locally accelerate her checking frequency by checking immediately rather than waiting for the next notification. This is more likely when the variance of the information is sufficiently large compared to the mean or when the information arrival rates are lower. And under these conditions, push notifications are preferred by the firm compared to no-push.

We then consider the case of a generalized noisy push strategy whereby even in the absence of genuine information arrival, the firm strategically adds phantom pushes with no informational value for consumers. This strategy represents the motivation of platforms to induce more consumer checking to monetize consumer attention. We characterize the optimal noisy push strategy and show that the equilibrium amount of noise, in general, falls in the interior between the extremes of no-push and genuine push. Noisy push strategy can generate higher equilibrium profits for the firm compared with the genuine push strategy. The firm wants to increase phantom pushes as much as it can get away with. But the rational consumer expects this and will have the incentive to reduce the frequency of checking.

The firm's equilibrium choice of the amount of noise balances these forces such that the consumer beliefs and anxiety do not attenuate so much that the consumer prefers to wait for an additional notification. The incentive to add phantom pushes increases if there is higher anxiety, lower information arrivals, higher information variance, and lower mean consumption value.

We also examine a generalization of push notifications in the other possible direction, namely partial push: i.e., the firm only pushes a subset of information arrivals. Specifically we consider a probabilistic partial pushing strategy in which the firm releases notifications of information arrivals with some probability. This can be seen as being equivalent to the firm muting notifications. When the optimal checking strategy is for the consumer to check every notification, then we find that the firm in equilibrium chooses to either to follow the genuine push strategy (i.e., push all notifications), or otherwise follow the no-push strategy. This occurs when the variance of the information is relatively high. When the variance becomes smaller it is optimal for the consumer to check less frequently (i.e., every two notifications). Now we find that the firm will mute some of the observed notifications and the consumer may check (even without a notification) if she has waited for a sufficient time interval. In fact, in this case as the variance decreases it is possible that the muting probability increases. This implies that the endogenous uncertainty created by muting and the variance of the unchecked information act like substitutes in equilibrium.

We extend the model to account for consumer self-control problems to explain the prevalence of consumer blocking of notifications. Consider a dual-self model, in which the long-term self has lower disutility over realized uncertainty compared with her short-term self. The long-term self can influence the short-term self's checking behavior by accepting or blocking notifications. We find that the long-term self may block notifications if push notifications lead to a higher checking frequency. We also extend the model to endogenous prices by allowing the platform to charge consumers a subscription fee on top of the payoffs from consumer checking. The firm prefers the genuine push strategy if it leads to more frequent checking because the subscription fee will then be also higher under the push strategy. Comparing the noisy push with the genuine push strategy, the firm faces the trade-off of higher profits from checking due to more push notifications, and a higher subscription fee due to less noise. Therefore, we find that the optimal noise level is the same as in the main model if checking were to be socially optimal, and lower if checking is not socially optimal yet large enough. Finally, we consider an extension in which the flow anxiety cost that is increasing in time and the analysis suggests that the firm may be more likely to prefer the no-push strategy over the push strategy when the flow anxiety cost increases at a faster rate in time.

II Related Research

The literature on information disclosure (e.g. Grossman, 1981; Milgrom, 1981) usually assumes that the sender's information disclosed to the receivers is conditional on their private information. This leads to the standard unraveling result that all quality types are separated and disclosed in equilibrium.[3] We focus on the problem of a firm pre-determining the notification design policy prior to the information arrivals. In other words, the firm is able to commit to an information policy before the state of the world is realized. Therefore, the problem we study is related to the information design/Bayesian persuasion literature (e.g. Kamenica and Gentzkow, 2011; Bergemann and Morris, 2019; Kamenica, 2018). Specifically, this paper is most closely related to the problem of dynamic Bayesian persuasion (Ely, 2017). Ely (2017) characterizes the general dynamic persuasion mechanisms and considers the problem of a sender who seeks to persuade a myopic receiver who takes action in each period based only on the current state. In contrast, in this paper, the sender faces the dynamic persuasion problem of a consumer who is strategic and forward-looking. Consumers know that their current decision to check will dynamically affect their actions in future periods, and this makes the anxiety thresholds endogenous to the firm's information policy. In addition, we also develop a demand-side foundation that rationalizes the consumers' information checking impulse and its implications for the firm's strategy.

Our model provides a psychological micro-foundation based on the disutility for

[3]The subsequent literature has proposed several ways to mitigate this information unraveling problem. These include uncertainty in information endowment (Dye, 1985), costly communication (Jovanovic, 1982; Verrecchia, 1983; Guo and Zhao, 2009), limited bandwidth (Mayzlin and Shin, 2011; Iyer and Singh, 2021), imperfect consumer recall (Lauga, 2011), and through positive competitive externalities (Lu and Shin, 2018).

realized uncertainty. At any point in time, consumers have a disutility for the information stock which might have arrived but is unresolved, and this disutility is proportional to the variance of the information stock. This characterization of realized uncertainty ties together interpretations from several strands of research in psychology. First, it can be interpreted as the anxiety arising from intolerance to uncertainty (e.g. Ladouceur et al., 2000). Recent studies in experimental economics show subjects prefer early resolution of uncertainty, even if the information is not instrumental (Falk and Zimmermann, 2016; Ganguly and Tasoff, 2017; Masatlioglu et al., 2017; Nielsen, 2017). Closely related is the phenomenon of the preference for indeterminacy: For example, Brun and Teigen (1990) show that subjects prefer to guess the outcome of a coin toss before rather than after the event occurs. Similarly, in the actual context of sports consumption (Vosgerau et al., 2006), consumers show preferences for indeterminacy and prefer live TV content compared to recorded. Our formulation provides a way to represent intolerance to uncertainty or the preference for indeterminacy. Second, the desire to resolve realized uncertainty can also be interpreted as the curiosity impulse (Loewenstein, 1994) or the "information-gap" theory, which argues that individuals will become more motivated to know something if they learn that it is knowable.

More generally, this paper is also related to the literature on belief-based utility: For example, Kőszegi and Rabin (2009) present a model in which agents are loss averse over be over changes in beliefs about present and future consumption. Closer to our analysis is Ely et al. (2015) who study the optimal information revelation plan of a principal when the agent has non-instrumental information preference that involves entertainment utility over suspense (when the next period's beliefs have greater variance) versus surprise (when the current beliefs are further away from the last period's beliefs).

III The Model

We develop a model of an information market to characterize the dynamic interaction between a consumer and an information provider. The model captures many important information consumption contexts that consumers regularly encounter, including news consumption, email usage, social-network updates, and instant messaging. The common feature characterizing these information consumption contexts is that information of varying usefulness can arrive over time, and the information provider (firm) can strategically design the format in which the information is presented to the consumer. For example, news information is generated exogenously and arrives over time, but the news provider can choose how to organize, aggregate, and transmit the information. We start by first specifying the information arrival process and its valuation.

III.A Information arrival

Nature generates information, and pieces of information arrive according to a Poisson process with a rate of $\lambda > 0$. Therefore, the number of information arrivals N in a time interval τ is given by $P(N(\tau) = k) = \dfrac{e^{-\lambda\tau}(\lambda\tau)^k}{k!}$, $k \in \mathcal{N}$, and the expected number of arrivals for time τ is given by $\mathrm{E}[N(\tau)] = \mathrm{Var}[N(\tau)] = \lambda\tau$. Furthermore, a piece of information varies in its usefulness. For example, an email waiting in the consumer's inbox may contain an invitation for a job interview, or alternatively may be a generic promotion message in which the consumer has no interest. To capture this feature, we assume that the k-th piece of information yields the consumer a random utility I_k, which is i.i.d. with mean $\mathrm{E}[I_k] = E_I$ and variance $\mathrm{Var}[I_k] = V_I$, and is independent from the information arrival process. Thus, the consumer has uncertainty about both the timing of information arrivals and the value of each piece of information. Next, we specify the consumer behavior and the information consumption utility.

III.B Consumer

Consider a consumer (female) who chooses between adopting the information provider's service or an outside option. If she adopts the service, she will have access to the information that arrives through the process described in the previous section. At any point in time, the consumer can choose to consume the information stock that has accumulated until that point by deciding to check for information. Alternatively, if she does not check, the information will keep accumulating according to the Poisson process and will be available for potential future consumption. The consumer will incur a checking cost every time she checks, which is normalized to one without loss of generality.

The consumer utility function has two components: First, the consumer gets utility from information consumption. The consumption utility of checking a piece of information is given by the random variable I_k specified above. If the consumer checks at some point in time, the total consumption utility is given by $u = \sum_{j \in J} I_j$, in which J is the set of information stock at the time of checking. Second and central to the theory of this paper, the consumer also experiences a disutility from leaving unchecked the information that might have already arrived. This disutility from realized uncertainty is what represents the consumer's anxiety cost or compulsion to check information. We assume that it is given by a flow disutility $u'_t = -\rho \text{Var}[u_t]$, in which u_t is a random variable capturing the consumption utility that could have been realized if the consumer checks at time t, and $\rho > 0$ is a parameter capturing the magnitude.[4] The disutility has several interpretations that are relevant for the information consumption contexts that motivate this analysis: First, it captures the anxiety cost resulting from uncertainty of not consuming useful information (Freeston et al., 1994; Ladouceur et al., 2000). The idea that uncertainty can induce anxiety is documented in psychology. For example, Tolin et al. (2003) show that checking compulsions are correlated with intolerance of uncertainty. We emphasize that it is the *realized* uncertainty, which could have been resolved by the consumer's actions, that causes the disutility. In other words, anxiety arises precisely because the consumer had the agency to resolve the uncertainty but chose not to. Second, it is also consistent with the psychological literature on curiosity and the information-gap mechanism (Loewenstein, 1994).

III.C Firm (Information Provider)

The firm profits from the consumer's checking behavior. In the main model, we assume that its profit is an increasing function of consumer checking. Therefore, the firm max-

[4] Note that we require that ρ is strictly positive as the theory in this paper is about how preferences over realized uncertainty affect information consumption and notification design. If $\rho = 0$ and there is no anxiety caused by realized uncertainty, then the consumer would not check in a finite time. An alternative framework for consumer checking would be the depreciation of information value over time. But in this paper, the theory focuses on the role of preferences over realized uncertainty and the associated anxiety costs.

imizes consumer checking frequency. Later, we extend the analysis by allowing the firm to charge the consumer a subscription fee of adopting the service. This set up is similar to Godes et al. (2009) and Kind et al. (2009) that model a media firm's profits arising from advertising, which are a function of consumer impressions and also potentially from consumers' subscriptions.

As mentioned, the firms profit function is based on the frequency of consumer checking which in turn depends upon the observed notifications (if any) and the inferences about the unobserved information arrivals at any point in time. This characterization of firm profits as a function of the frequency of consumer checking can be seen as the firm maximizing the extent of consumer engagement or eyeballs and can be operationalized as the extent of the consumer checking of the firms website or app. For example, a consumer may log onto the Facebook website and check either because of notification(s), or because even in the absence of a notification the consumer expects a sufficient number of information updates to have accumulated (because she has waited long enough). However, once the consumer checks the website, Facebook may show the consumer ads regardless of the number of information updates that have accumulated.

The firm's strategy is to choose an information design to influence consumer choice. It can choose from a rich set of information policies to influence consumer checking behavior. As in the Bayesian persuasion literature, the firm commits ex-ante to an information policy, and the consumer reacts rationally. We consider the following commonly observed information policies. The firm can choose to provide the consumer with access to information passively. This is the *No-Push Strategy*, where the firm does not provide any notification. Alternatively, the firm can notify the consumer, and at the extreme, it can commit to a *Genuine Push Strategy*, where the firm commits to pushing a notification to the consumer when and only when there is an information arrival. In this case, the consumer learns about the arrival of information immediately and then decides whether or not to incur the cost to check the information stock and resolve the valuation uncertainty.

However, given that the firm's objective function is increasing in the amount of checking, a more general information design problem for the firm would be to add notifications over and above the true information arrivals. In other words, the firm can adopt a *Noisy*

Push Strategy in which it allows the addition of "phantom" push notifications even in the absence of actual information arrivals. These phantom pushes are empty, and therefore they have zero informational utility for the consumer. To characterize this in the information market, assume that in addition to the true information arrival process, there also exist additional arrival processes which have zero informational value. The firm chooses whether and how many of the phantom notifications to add. For example, Facebook may allow push notifications that end up being irrelevant updates with no informational value for the consumer. Assume that the arrival of phantom pushes is a Poisson process with a rate of $k\lambda$ that is independent of the true generic information arrival process. The composite push notification process, which the consumer observes, is also Poisson with rate $(k+1)\lambda$. When $k \to \infty$, this converges to the no-push case, and when $k \to 0$, it converges to the genuine push notification case. Note that while the noisy push strategy with the additional phantom pushes can benefit the firm because it may induce more frequent checking, the rational inference of the presence of phantom pushes on the part of the consumer will also induce her to pull back on the checking behavior. Finally, we also consider a generalization of push notifications in the opposite direction, namely, a partial push strategy in which the firm only pushes a subset of the information arrivals.

Before proceeding with the analysis, we provide a justification for the relevance of the firm's payoff function. To begin with notice that there are three distinct objects of interest in the model, namely, notifications which are the decisions made by the firm to reveal whether information has arrived (but not the information valuation), information arrivals which are determined by the exogenous Poisson process and costly consumer checking which are decisions made by the consumer based on the observed notifications and the expectations about the information that remains uncertain at any point in time. Given this, the firms profit function based on the frequency of consumer checking can be interpreted as follows: The frequency of consumer checking depends upon the observed notifications (if any) and the inferences about the unobserved information arrivals at any point in time. This characterization of firm profits can be seen as the firm maximizing the extent of consumer engagement or eyeballs which can be operationalized as the extent of the consumer checking of a firms website or app. For example, a consumer will log onto the Facebook website and check either because of notifications, or because even in the absence of a notification the consumer expects a sufficient number of information updates to have accumulated because she has waited long enough. Consumer checking of the website allows Facebook to show the consumer ads regardless of the number of information updates that has arrived.

IV Analysis: Commitment to Truthful Notifications

To highlight the mechanisms that influence information consumption, we start with the baseline in which the firm chooses between a genuine push strategy and a no-push strategy. We start the analysis by characterizing the consumer's optimal information consumption behavior given the firm strategy and then the firm's equilibrium choice of the information design.

IV.A Optimal Consumer Checking Behavior

Under the no-push strategy, the consumer's realized uncertainty at time τ (since the last check) is characterized by $\mathrm{Var}[u(\tau)] = E_I^2 \mathrm{Var}[N(\tau)] + \mathrm{E}[N(\tau)]V_I$, which consists of two components: the uncertainty from how many pieces of information have arrived, and the uncertainty from how useful is each piece of information. The first component is proportional to the square of the mean utility E_I^2, and the second is captured by the variance of information utility V_I. Both increase linearly in τ. Therefore, the opportunity cost of forgoing checking also increases in the time since last check. However, in order to limit the anxiety cost, frequent checking is needed which is also costly. Thus, the trade-off between checking costs and anxiety costs determines the optimal checking frequency.

If the consumer checks every t units of time, the expected flow utility is given by $\mathrm{E}[U(t)] = \frac{1}{t}\left[\mathrm{E}[u(t)] - 1 - \int_0^t \rho \mathrm{Var}[u(\tau)]d\tau\right]$, and there exists a t_n^* that maximizes the utility. The following lemma shows that this checking policy is indeed optimal in the case of the no-push strategy.

Lemma 1 *Without push notification, the consumer's optimal checking behavior is to check every t_n^* units of time, in which $t_n^* = \sqrt{\dfrac{2}{\rho(V_I + E_I^2)\lambda}}$.*

Proof. See Appendix A. ■

Without any push notifications, the consumer's optimal checking strategy is stationary and involves a constant interval between consecutive checks as this minimizes the total anxiety flow cost. The anxiety cost is also proportional to the information arrival rate λ and anxiety parameter ρ, and so as expected, the consumer checks more often as these parameters increase. The optimal checking behavior can be seen as being consistent with the empirical patterns of consumer checking. For instance, a study by Jones et al. (2015) has found that users check instant messaging and social media apps more frequently than productivity and regular media apps, with emails coming in between. For example, the peak "revisitation" rates for social media and messaging apps peaked at 1-2 minutes while those for productivity apps and emails were at 1-16 hours. In terms of the model we can interpret social media and messaging apps to have higher λ and V_I as compared to productivity and email apps.

Consider now the genuine (truthful) push strategy: In this case, the consumer's realized uncertainty is only a function of n, the number of notifications (since last check): $\text{Var}[u(n)] = nV_I$. Thus, the consumer's checking behavior is contingent on the arrival of observed push notifications. If the consumer waits until the $n-$th notification to check, the average flow utility (since last check) is given by $\text{E}[U(n)] = \frac{\lambda}{n}\left[nE_I - 1 - \rho\frac{(n-1)n}{2\lambda}V_I\right]$. The consumer is facing a similar trade-off: a higher checking frequency (lower n) increases the checking cost, but lowers the anxiety costs. We will show in the analysis below that the consumer has no incentive to check when there is no notification, no matter how long she has waited since the last check. The consumer also has no incentive to check between two consecutive notifications. This is because that strategy is dominated by either checking at the previous notification or the subsequent one. The optimal checking behavior is therefore characterized by checking every n_p^* notifications, which can be specified as follows:

Lemma 2 *Define* $\tilde{n}_p = \arg\min_{n\in\mathcal{R}_+}\frac{\lambda}{n} + \rho\frac{(n-1)}{2}V_I = \sqrt{\frac{2\lambda}{\rho V_I}}$. *Then the consumer's optimal checking behavior with genuine push notification is to check at every* n_p^* *notifications which is as follows:*

a. $n_p^* = \lfloor\tilde{n}_p\rfloor$, *if* $\tilde{n}_p \leq \sqrt{\lfloor\tilde{n}_p\rfloor(\lfloor\tilde{n}_p\rfloor + 1)}$, *where* $\lfloor\tilde{n}_p\rfloor$ *is the floor of* $\tilde{n}_p$.

b. $n_p^* = \lceil \tilde{n}_p \rceil = \lfloor \tilde{n}_p \rfloor + 1$, *if otherwise.*

Proof. See Appendix B. ■

The optimal checking behavior of the consumer trades off the checking cost $(\dfrac{\lambda}{n})$ which is increasing in the checking frequency with the waiting cost $(\rho V_I \dfrac{(n-1)}{2})$ that is decreasing in the checking frequency. Minimizing the combination of these costs and ignoring the integer constraint gives us $\tilde{n}_p = \sqrt{\dfrac{2\lambda}{\rho V_I}}$. In general this $\tilde{n}_p$ will lie in the interval $[\lfloor \tilde{n}_p \rfloor, \lfloor \tilde{n}_p \rfloor + 1]$.

In the presence of push notifications the consumer's checking space is endogenously restricted from $\mathcal{R}_+$ to $\mathcal{N}_+$. In other words, push notifications imply a constraint on the checking choices. Notifications are by definition discrete and so the consumer will have to choose between checking every $\lfloor \tilde{n}_p \rfloor$ or $\lfloor \tilde{n}_p \rfloor + 1$ notifications. There are two countervailing effects that notifications have on consumer behavior (compared to no-push). On the one hand, push notifications from the firm unambiguously reduce realized uncertainty. Under the no-push strategy, the consumer is uncertain about i) how many pieces of information have arrived and ii) their value. Genuine push notifications resolve the uncertainty of the first component by always informing the consumer of the exact number of information arrivals at every point in time. On average, the consumer suffers less from leaving any information unchecked. Therefore, one may expect the genuine push strategy to reduce the incentive to check. However, notifications have a second potentially countervailing effect: they constrain the consumer choice space, and this can induce faster checking. The condition in part (a) of the Lemma provides insight into when that might be the case. When $\tilde{n}_p \leq \sqrt{\lfloor \tilde{n}_p \rfloor (\lfloor \tilde{n}_p \rfloor + 1)}$, the consumer will choose to check every $\lfloor \tilde{n}_p \rfloor$ rather than waiting for one more notification to arrive. In other words, the consumer's checking behavior is *locally accelerated*, and the consumer is induced to check more frequently.

The effect of notifications can also be understood by how they change the evolution of the consumer's beliefs. Under no-push, the belief about the number of arrivals and the associated anxiety is increasing linearly in time. This anxiety is not strong enough to trigger checking until $t = t_n^*$. But with push notifications, while the mean belief is the same at every point in time (the consumer is Bayesian), the belief is spread between either zero information when there is no notification, or the exact number of information arrivals. Consumer beliefs and anxiety jump discretely at the arrival of a notification, but remain constant between any two notifications. When the optimum $\tilde{n}_p$ is such that at $\lfloor \tilde{n}_p \rfloor$ the consumer's anxiety is enough to induce her to check (i.e., the condition $\tilde{n}_p \leq \sqrt{\lfloor \tilde{n}_p \rfloor (\lfloor \tilde{n}_p \rfloor + 1)}$), we have the local acceleration phenomenon in consumer checking.

Specifically, smaller the information arrival rate λ and higher the anxiety costs ρV_I the greater is possibility that the consumer accelerates her checking and chooses to check every $\lfloor \tilde{n}_p \rfloor$ notifications rather than waiting for an additional notification. In contrast, when $\tilde{n}_p > \sqrt{\lfloor \tilde{n}_p \rfloor (\lfloor \tilde{n}_p \rfloor + 1)}$, the consumer slows down her checking frequency by waiting for an additional notification.

The natural question that we can now ask is whether genuine push notifications can lead to increased checking compared to the no-push strategy, even though they reduce the amount of realized uncertainty for the consumer. We proceed to investigate this question and the resulting implications for the firm.

IV.B Optimal Firm Strategy: To Push or Not?

Comparing the case with and without push notifications, we can identify if there exist conditions under which we get the result that push notifications can lead to more frequent checking despite lowering the consumer's uncertainty. The following proposition provides the comparison:

Proposition 1 *If $V_I > \lfloor \tilde{n}_p \rfloor (E_I)^2$, $\lfloor \tilde{n}_p \rfloor \in \mathcal{N}_+$, there exists a parameter range such that the genuine push strategy leads to a higher checking frequency. This range is given by:*

$$\lambda \in \left(\rho(V_I + E_I^2) \frac{\lfloor \tilde{n}_p \rfloor^2}{2}, \rho V_I \frac{\lfloor \tilde{n}_p \rfloor (\lfloor \tilde{n}_p \rfloor + 1)}{2} \right). \tag{1}$$

In this range, the consumer optimally checks at every $n_p^* = \lfloor \tilde{n}_p \rfloor$ push notifications, and the firm's equilibrium strategy is to use the genuine push strategy over no-push. Otherwise, the firm adopts the no-push strategy.

Proof. See Appendix C. ■

Why does push notification lead to more checking? Under the no-push strategy, the consumer's anxiety cost increases linearly in time: Both the uncertainty about the number of information arrivals and the uncertainty in their valuations increase linearly in time. And when the overall anxiety cost reaches a certain threshold, the consumer "pulls the trigger" and incurs the checking cost to nullify the built-up stock of anxiety.

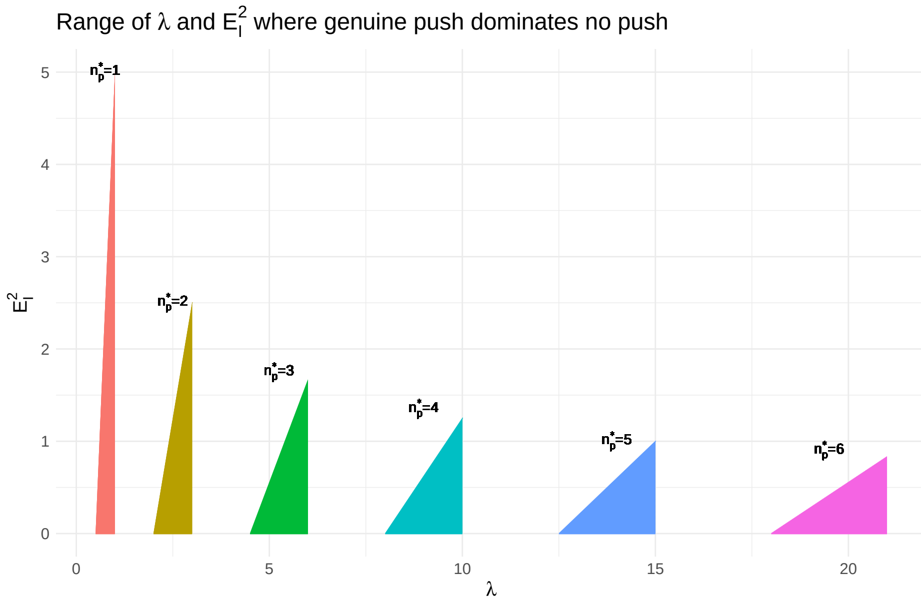
With push notifications, the consumer's anxiety cost level becomes a step function in time. Whenever there is a notification, the anxiety cost increases discontinuously. This is because the consumer now knows for sure that there is a new piece of information that has arrived and it has the associated variance of V_I. The consumer may be impelled to check because the cost of waiting increases discontinuously. Next, recall that push notifications endogenously constrain the consumer's checking choices from $\mathcal{R}_+$ to $\mathcal{N}_+$. If she does not check now, she will have to wait until the next information arrival, because checking between two notifications is never optimal. Therefore, depending upon the characteristics of the information arrival and its valuation, it is possible that push notifications can lead to accelerated checking. We discuss this below.

The strength of the impulse to check depends on the variance of information value V_I. When it is large enough (relative to the mean utility E_I), the consumer may check more often under genuine push notifications. This implies the interesting point that even though, on average, the push strategy unambiguously leads to lower expected uncertainty and anxiety costs, the consumer might still end up checking more often with push compared to the no-push strategy. Conversely, when V_I is small relative to E_I, the consumer checks less often under the push notification strategy. Therefore, the firm's optimal decision to use push notification or not depends on the relative value of V_I.

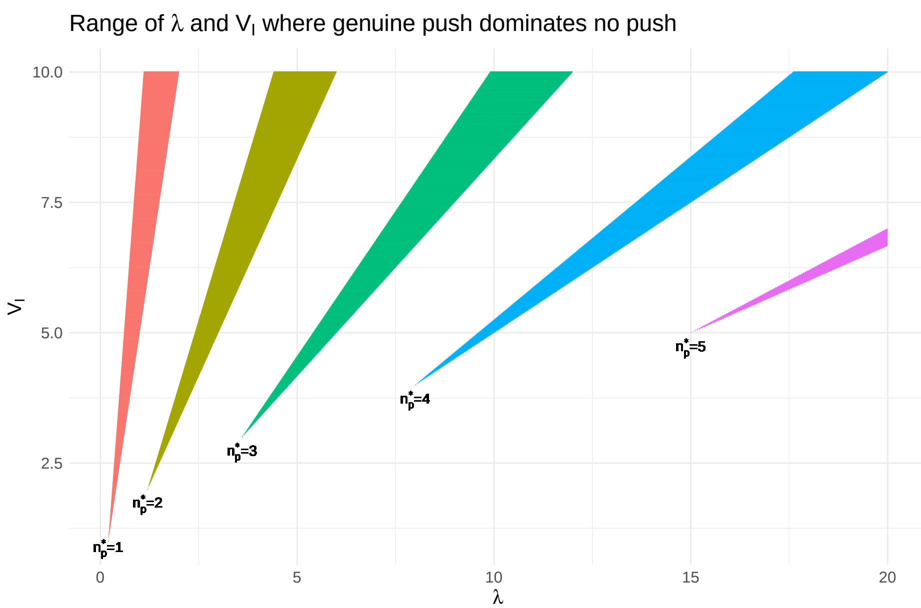
The intuition for the optimality of push notifications can be further clarified based on how they govern consumer beliefs compared to the no-push case. With no-push, the consumer's belief increases linearly in time before she chooses to check. From the firm's point of view, some of the anxiety incurred over time is inefficient and is "wasted" as the consumer is becoming progressively anxious but not sufficiently enough to trigger checking. With push notifications, the consumer's belief evolution is discretized, and this may lead to a more efficient spread in the anxiety costs from the firm's point of view. The spread can be such that the consumer is either not at all anxious (when there is no notification), or anxious just enough to be impelled to check. When push notifications induce this type of efficiency, it is possible that the consumer may locally accelerate her checking frequency by checking immediately rather than waiting for the next notification. All in all this allows the possibility that push notifications can indeed increase checking frequency despite lowering the overall level of uncertainty facing the consumer.

The condition (1) highlights the mechanisms which lead push notifications to encourage more frequent checking. The firm uses push strategies when the arrival rate (λ) is in the interval in (1). The arrival rate λ is required to be neither too low nor too high relative to the anxiety level. Suppose λ is low, and below the lower boundary of the interval, i.e., $\rho(V_I + E_I^2)\frac{\lfloor \tilde{n}_p \rfloor^2}{2}$: Information arrives infrequently, and the number of realized notifications will be relatively sparse. The consumer would wait longer for enough number of notifications to accumulate before being induced to check. But under no-push, the consumer's checking given the anxiety costs would still be higher (with some of those checks resulting in no incremental information). Now suppose λ is sufficiently high and information arrives with sufficiently high frequency. In this case, suppose the consumer were to forego checking at some n number of notifications and decides to wait for an additional notification to arrive. She knows that the wait for the next notification is not likely to be very long. This alleviates the anxiety cost and once again leads to less frequent checking.

Figure 1a represents the condition (1) and shows the range of λ and E_I^2 for which the push notification strategy dominates no-push for the firm. We can see that for any n_p^* a lower E_I^2 expands the range over which the genuine push strategy is chosen by the firm. In other words, when the expected value of the information is lower, the consumer waits longer to check autonomously under no-push. Therefore the firm is more inclined to choose push notifications. Figure 1a shows that when E_I^2 is small, there are more cases of n_p^* and larger ranges of λ for which the push notification strategy dominates. Next, for any given E_I^2 and n_p^*, there is an upper boundary of the arrival rate ($\rho V_I \frac{n_p^*(n_p^*+1)}{2}$) below which the consumer will choose to accelerate her checking to every n_p^* rather than waiting for one more notification. This accelerated checking induces the firm to prefer the push notification strategy. When the arrival rate further increases, the consumer will wait to check every



(a) λ and E_I^2



(b) λ and V_I

Figure 1: Parameter ranges in which genuine push dominates no-push

Note: Parameter ranges of λ in which the push strategy leads to a higher checking frequency, given the parameters of $\rho = 0.2$, $V_I = 5$ in Panel (a) or $E_I^2 = 1$ in Panel (b).

$n_p^* + 1$ notifications, and this induces the firm to prefer the no-push strategy.

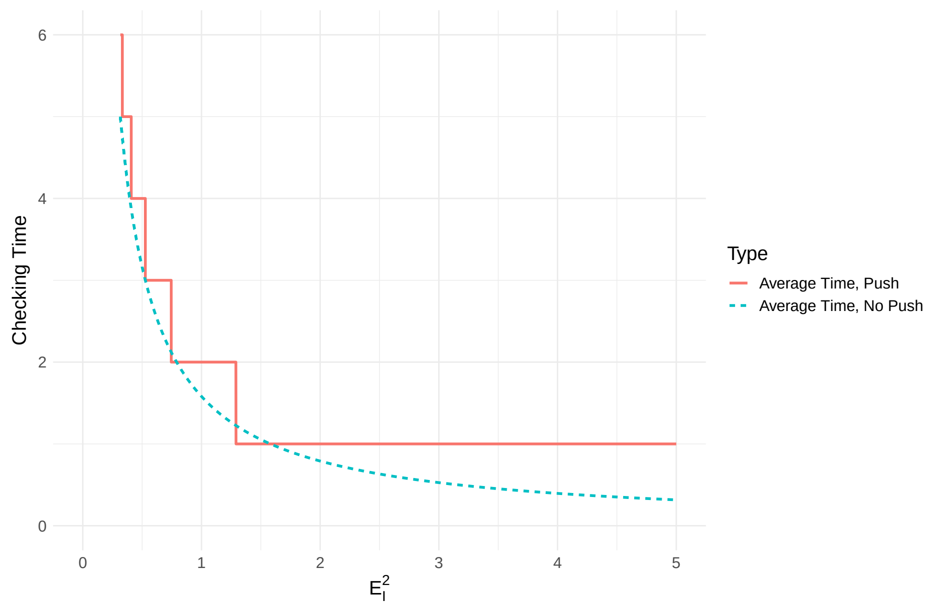
We can also see that as the arrival rate increases, the gap between any two intervals also increases. In other words, as information arrives more frequently, we have that the no-push strategy is chosen over a greater range of market conditions. Indeed as $\lambda \to \infty$ no-push is always preferred by the firm. Next, Figure 1b shows the range of λ and V_I^2 for which push notifications dominate no-push for the firm. The observations are similar: First, for any n_p^*, a higher variance expands the range of λ for which the push strategy is preferred by the firm. Second, when the variance is large, there are more cases of n_p^* that the push strategy could dominate.

Having characterized the optimal firm strategy, we move on to the consumer surplus under push notifications:

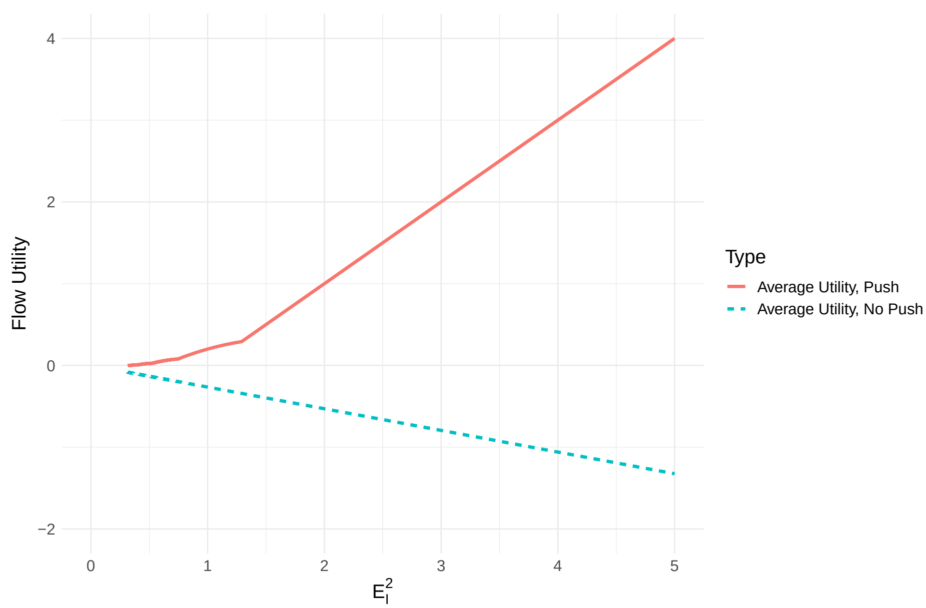
Corollary 1 *The consumer's expected utility is higher under the genuine push as compared to no-push. Therefore, if push notifications lead to more frequent checking, it is a Pareto improvement over the no-push strategy.*

Proof. See Appendix D. ■

The consumer's utility is higher under the genuine push because the anxiety cost is lower due to less uncertainty if she adopts the same checking strategy under no-push. Thus, her expected utility under the optimal checking strategy will be even higher. Corollary 1 highlights that push notification can be a Pareto improvement that benefits both the consumer and the firm when the variance of the information is sufficiently large. In that case, compared with no-push, the push strategy is socially optimal. As a numerical example, Figure 2a illustrates, the average checking time for push and no-push strategies using the following parameters: $\lambda = 1$, $\rho = 0.2$, $V_I = 3(E_I)^2$. In the range of $E_I = (1.29, 1.58)$, the consumer checks more frequently with push strategy. Figure 2b shows the consumer expected utility for the same parameter range. We observe that the consumer utility under genuine push is higher than without push notifications, as Corollary 1 suggests.



(a) Average checking time



(b) Consumer utility

Note: Numerical example of average checking time and utility for no-push and genuine push strategies, parameters given by $\lambda = 1$, $\rho = 0.2$, $V_I = 3(E_I)^2$.

Figure 2: Numerical example of checking time and utility

V Generalizations of Push Strategy

In this section we examine generalizations of the push strategy in two directions, one in which the firm adds phantom notifications and the other in the opposite direction where the firm releases notifications only for a subset of information arrivals.

V.A Noisy Push Strategy

Up until now, we have restricted the analysis to the case in which the firm is committed to a truthful information design policy of providing genuine notifications. We now consider the general case in which the firm has the ability to add phantom push notifications that do not contain any useful information for the consumer. In practice, these phantom notifications may be irrelevant updates in social media or spam emails. Given that the firm's payoffs are increasing in consumer checking, the firm may strategically adopt a noisy push strategy that could motivate increased checking by mixing genuine notifications with phantom ones which do not contain any useful information. This may be the case even if the consumer has rational expectations of the firm's strategic behavior. Suppose that the firm can add phantom pushes at a rate of $k\lambda$ that is independent of the true information arrival and let us label this as a k-noisy push process.[5] Under the noisy push strategy, the rational consumer will be skeptical and less motivated to check any given notification. The question is whether it is still possible that they may check more in the aggregate? When will noisy push be an equilibrium for the firm? The following proposition describes the equilibrium.

Proposition 2 *Under the noisy push strategy, for any k chosen by the firm, the consumer will check every n_x^* notifications given by* $n_x^* = \arg\min_{n \in \mathcal{N}_+} \frac{(k+1)\lambda}{n} + \frac{\rho(n-1)}{2(k+1)}(V_I + E_I^2 \frac{k}{1+k})$. *Further, the firm's optimal choice of k should obey the following necessary condition,*

$$\frac{(k+1)^2\lambda}{n_x^*(n_x^*+1)} = \frac{\rho}{2}V_I + \frac{k}{2(k+1)}E_I^2 \text{ , } k > 0 \text{ and } n_x^* \in \mathcal{N}_+ \tag{2}$$

And the following characterizes the noisy push equilibrium.

[5] Alternatively, suppose that the firm had access to a continuum of possible phantom information processes with Poisson arrival rate $k\lambda$ which is independent of the true information arrival and could choose k.

a. *When there exists a pair* $\{N_x^*, k^*\}$, *satisfying* Equation 2 *and* $\{N_x^*, k^*\} = \arg\max_{\frac{n_x^*}{1+k}} \{n_x^*, k\}$, *such that* $V_I > E_I^2 \frac{N_x^* - k^*}{1+k^*}$, *the firm uses a noisy push strategy with noise level* $k^* \in [0, \infty)$ *and the consumer's equilibrium checking frequency is* N_x^*. *In particular,* $k^* = 0$ *if and only if* $N_x^* = \frac{1}{2}[\sqrt{\frac{8\lambda + \rho V_I}{\rho V_I}} - 1] \in \mathcal{N}_+$, *in which case the optimal noisy push strategy degenerates to the genuine push strategy.*

b. *When* $V_I \leq E_I^2 \frac{n_x^* - k}{1+k}$ $\forall \{n_x^*, k\}$ *given by* Equation 2, *the optimal* $k^* \to \infty$, *i.e., the optimal noisy push strategy degenerates to the no-push strategy.*

Proof. See Appendix E ■

As in the case of the basic model in Lemma 2, the consumer's optimal checking behavior minimizes the checking and waiting cost up to the integer constraint. In the case of a k-noisy push process, with every notification, the consumer will infer that the probability of genuine information arrival is $\frac{1}{1+k}$. As before, the consumer's belief and anxiety cost will jump discontinuously at every notification but now to a lesser extent than in the case of genuine push (and it will not vary between any two notifications). Given the firm's choice of noise k, the consumer's equilibrium checking behavior is to check at every n_x^* notifications, which given the consumer's rationality, is weakly larger than the case of genuine push.

Despite the fact that the consumer fully anticipates the presence of phantom pushes, it is nevertheless possible for the firm to gain by adding phantom pushes and inducing the consumer to check more. In fact, the firm will optimally choose to add noise such that the consumer is just about indifferent between checking every n_x^* notifications rather than waiting for an additional notification. And this is the interpretation of the necessary condition in 2.

Given this, $V_I \geq E_I^2 \frac{n_x^* - k^*}{1+k^*}$ then represents the condition when the equilibrium noisy push payoffs are higher for the firm than the case of no-push. So once again, the variance of the information has to be relatively large for the noisy push to be optimal for the firm. In the noisy push equilibrium, the firm chooses k^*, which induces N_x^* such that the firm maximizes the checking frequency. The proposition also makes precise the optimal amount of noise the firm can add through phantom pushes. Notice that the equilibrium is

$\{N_x^*, k^*\} = \arg\max_{\frac{n_x^*}{1+k}} \{n_x^*, k\}$. The firm wants to increase phantom pushes as much as it can get away with. However, the rational consumer expects that the probability of arrival to be $\frac{1}{1+k}$, $\forall k$, and would want to (weakly) reduce the frequency of checking. The firm's optimal choice balances these forces by progressively adding phantom pushes up to $k = k^*$ such that the consumer is just indifferent between checking at the optimal N_x^* rather than waiting for one more notification. Finally, when $V_I \leq E_I^2 \frac{n_x^* - k}{1+k}$ for all $\{n_x^*, k\}$, then the firm optimal choice of noise goes beyond bound, and the noisy push strategy becomes equivalent to and induces the same frequency of consumer checking as the no-push strategy.

Notice that k^* also obeys the condition Equation 2 which is an implicit equation whose closed-form solution is complicated. But we can still characterize the comparative statics: $\frac{\partial k^*}{\partial \rho} > 0$, $\frac{\partial k^*}{\partial \lambda} < 0$, $\frac{\partial k^*}{\partial V_I} > 0$, and $\frac{\partial k^*}{\partial E_I} < 0$.

The firm in equilibrium adds more phantom pushes when ρ is higher. In other words, the firm responds to consumer anxiety by adding more phantom pushes. Further, lower information arrival rates allow the firm to add more noise – i.e., when the true information arrival is more infrequent, the firm can maintain consumer checking behavior with even higher levels of noise. Also, a higher variance and a lower mean of the information consumption utility, as expected, lead the firm to add more phantom pushes.

We can also see that in the special case when $E_I \to 0$, we have a stronger result that the noisy push will dominate no-push for any parameter range of λ and V_I. Firms will have the incentive to use noisy push precisely when the information on average has less value. In fact, in this case, the noisy push strategy will increase the checking frequency to $\sqrt{\frac{m+1}{m}}$ where m is the number of notifications that the consumer checks. For example, when V_I is high such that $m = 1$, i.e., the consumer checks at every notification, the noisy push strategy can improve checking frequency to $\sqrt{2}$, or by 41%, relative to the no-push strategy.

Corollary 2 *In equilibrium, the noisy push strategy with k^* leads to higher consumer surplus compared with no-push strategy.*

Proof. See Appendix F ■

Compared with the no-push strategy, the noisy push strategy could increase consumer surplus, because it still gives the consumer some information to alleviate the anxiety cost of uncertainty. Similar to the case in the genuine push strategy, the consumer will enjoy a higher surplus compared with the no-push strategy.

V.B Partial (Probabilistic) Push

In the previous section, we show that the information provider may induce more checking through a noisy push strategy by adding phantom notifications to genuine notifications. Now we go in the other direction and explore the other possible generalization of push notifications: i.e., partial push or the aggregation of genuine push notifications. Partial pushing in which only some of the information arrivals are released as notifications can be seen as an aggregation strategy equivalent to the firm "slowing" down notifications. Nevertheless, this strategy also has the potential to induce consumers to endogenously check even in the absence of notifications. In what follows, we ask whether partial push can also lead to more frequent checking compared to the genuine push or no-push strategy.

Formally, consider a general probabilistic partial push strategy in which, at each instance of genuine information arrival, the firm commits to sending a notification with probability $p \in (0, 1)$. In other words, with probability $1 - p$, the notification is "muted" and aggregated. Note that when $p \to 1$, this probabilistic notification strategy converges to the genuine push strategy, while when $p \to 0$, it converges to the no-push strategy.

Because of the firm's probabilistic muting, the consumer will rationally expect that information may have potentially arrived even in the absence of notifications. Therefore, the optimal checking strategy is characterized by both the time since the last check τ and the number of notifications n that the consumer actually observes. The following lemma characterizes consumers' checking strategy under partial push notifications.

Lemma 3 *The consumer's optimal checking behavior with notification aggregation (with the push probability p) is to check whenever the state $(n, \tau) \in I_b$, in which $I_b \subset \mathbb{N} \times \mathbb{R}$ that solves the maximization problem $I_b = \arg\max_{I_b} U(I_b)$, where $U(I_b)$ is the average flow utility as defined in the proof in Appendix G. The average optimal time per check t^* is also derived in the proof.*

Proof. See Appendix G. ■

Let us explain the nature of the equilibrium. First, the checking strategy is a function of both the number of notifications n as well as the time τ since the last check (i.e., $(n, \tau) \in \mathbb{N} \times \mathbb{R}^{+}$) because when the consumer does not check the state transitions are independent of the history and the flow payoffs are only dependent on (n, τ). The boundary set of states I_b are the threshold set of states beyond which the consumer will check. When $p \in (0, 1)$ consumers expect that with the passing of time information might arrive and be muted. Thus the anxiety costs can increase even if the observed number of notifications are the same. This means that there will be a minimum waiting time $\tau_m(n) = \min(\tau, \text{s.t.}(n, \tau) \in I_b)$ since the last check given that there are n notifications. Clearly this waiting time becomes smaller with larger number of observed notifications.

Intuitively, there are two cases when a consumer checks implying two parts to the boundary set I_b. First, the consumer may check immediately after receiving a notification. Second, and new to the partial push strategy, the consumer may wait for a minimum waiting time of $\tau_m(n)$ and check even without receiving any notification (but having observed that there are n notifications before). The optimal checking strategy I_b characterizes the set of notifications and associated minimum waiting times that maximizes the consumer's expected utility $U(I_b)$ given the firm's muting strategy p. In appendix G we derive the optimization problem for the consumer as shown in (9). While the optimization problem is well-defined, there is no analytical solution to consumers' optimal checking strategy. We solve for I_b numerically for two cases, one in which the consumer checks after every notification and one in which she checks after every two notifications. Given this, we conduct a numerical grid search across the range of feasible parameters to identify the optimal p^* for the firm. In the appendix, we present the equilibrium solution under a wide range of feasible parameters. The equilibrium presents the following results:

Result 1 *When the consumer checks every notification, the optimal push probability* p^* *takes a corner solution of* $p^* \in \{0, 1\}$.

In example 1, appendix G, we first describe the case when the consumer checks at every notification when $\lambda = 1, \rho = 0.2$.[6] The feasibility requirement for the consumer to check at every notification in equilibrium rather than waiting is $\lambda \leq \rho V_I$ as $p = 1$ (see Proposition 1), and this implies a feasible range $V_I \geq 5$. Figure 3 in the appendix specifies the equilibrium over a range of feasible V_I and E_I,. When V_I equals 5, the consumer is just indifferent from checking every notification and every two notifications under the genuine push strategy, which implies that the previously described local acceleration effect of push notification is the most effective. Thus, the optimal pushing probability is 1 for a wide range of E_I. As V_I increases, the checking frequency under the no-push strategy increases, while the average checking frequency under the genuine push strategy stays constant (i.e., equals the arrival rate $\lambda = 1$). Thus, implies that the firm will be inclined to mute notifications. Figure 3 shows that when $V_I \geq 8$, there will exist ranges of E_I in which the optimal pushing probability is 0, i.e., the firm adopts the no-push strategy.

To summarize, the results show the optimal muting strategy converges to either the genuine push or the no push strategy. Intuitively this is because, on the one hand, if the consumer checks at every notification, choosing a muting strategy $p \in (0, 1)$ implies a slowing down in the pushing of notifications. This reduces the expected frequency of notification arrival and the checking frequency. On the other hand, if the consumer were to check more frequently under no push than under genuine push ($p^* \to 1$), then it will be optimal for the firm to mute all notifications.

In contrast to the above case, if the consumer checks less often, then this opens up the possibility that it may be optimal for the firm to "slow down" the pushing of notifications by muting some notifications. Therefore, in example 2 of Appendix G, we consider the case in which consumers check every two notifications. We first note the existence of an interior p equilibrium.

Result 2 *When the consumer checks every two notifications under genuine push, then there will exist possible parameter ranges in which the optimal pushing probability is an interior* $p^* \in (0, 1)$.

[6] Note that varying λ and ρ does not affect the qualitative insights from the results that we discuss below, rather they will affect the feasible range of V_I that will be relevant for the consumer policy.

Continuing with the same example of $\lambda = 1, \rho = 0.2$ we can now derive the feasible range of $V_I \in (2, 5)$ by comparing the case of the consumer checking every two notifications with the cases of checking every one or every three notifications. Figure 4 in Appendix G shows the equilibrium over the range of feasible V_I and E_I. When V_I approaches 5 from below the optimal push probability $p^* \to 1$ and so similar to the result in example 1 (Figure 3), the firm uses the genuine push strategy. But as V_I decreases further, the firm finds it optimal to mute notifications, and the muting probability increases because this induces the consumer to not wait too long. Intuitively, the endogenous uncertainty created by muting and the variance of the unchecked information act as substitutes in equilibrium. Consistent with this intuition, notice that as V_I continues to decrease, the firm increases the amount of muting (i.e., decreases p^*) over a range of E_I's. As V_I decreases even further, the muting incentives become so strong enough that $p^* = 0$ and the firm prefers full muting, or in other words, the no-push strategy.

Note that in Figure 4, the interior equilibrium in which muting leads to higher checking frequency exists in the example when $V_I \in (3.8, 5)$ for at least some E_I. The interior solution is particularly prevalent when V_I approaches the upper bound of the feasible range 5, in which case the consumer, while checking every two notifications, is almost indifferent to the policy of checking every notification. In this case, the firm optimally mutes a small percentage of notifications, such that the consumer is induced to check even after one notification as long as she has waited for a sufficiently long time interval.

VI Extensions and Robustness

In this section we analyze three extensions to the basic model which expand the theory and examine the robustness of the results.

VI.A Self-Control Problems and Consumer Blocking

The model that we have analyzed thus far is one with a consumer who has time-consistent preferences over the realized uncertainty: i.e., at any time t the consumer has the same anxiety parameter ρ for the current period t anxiety and the expected future anxiety (evaluated at time t). However, the information consumption situations that motivate this paper may naturally involve the tension between short-term impulses and long-run preferences. The consumer may be subject to self-control problems and have the preference to check more frequently than what her long-term selves might wish to do. The basic model does not capture the self-control problem that would lead to such information addiction. In our setup, the preferences for realized uncertainty predicts that (rational) the consumer will always be better off under push notification compared with no-push, even when the notifications are noisy. However, we commonly observe information consumption situations where consumers have the incentive to actively block notifications. The model described below links consumer self-control problems to their blocking behavior and then investigates the effect on firm incentives.[7]

Considering self-control problems provides a rationale for why the consumer may strategically block notifications even when they reduce the overall variance in the realized information. To capture the self-control problem in information consumption in a parsimonious manner, we extend the basic model by adopting a dual-self framework (e.g. Thaler and Shefrin, 1981; Fudenberg and Levine, 2006). Accordingly, assume that there is a long-term self and a short-term self. The short-term self is endowed with the same preference as in our basic model. However, the long-term self has lower anxiety over future realized uncertainties and evaluates these uncertainties with parameter $\hat{\rho} < \rho$. Another interpretation of this setup is that the long-term self is subject to less temptation (of checking), but otherwise has the same preference as the short-term self. The long-term self is also sophisticated. She correctly predicts that the short-term self will decide when to check under parameter ρ.

The timeline is as follows. At the beginning $T = 0$, both push and no-push strategies made available. The long-term self chooses an information policy, i.e., push technology, for the short-term self. For example, it may happen at the point when the consumer installs an app or the first time the operating system asks whether the consumer wants to block push

[7]Thus the analysis adds to the research stream that investigates the responses of firms to consumer self-control problems typically using the hyperbolic discounting framework. These include, for example, the design of pricing contracts (Della Vigna and Malmendier, 2006), product and size decisions (Jain, 2012a), or sales agent contracts (Jain, 2012b).

notifications from the app, or whether the consumer wants a pre-set notification frequency. After this initial decision by the long-term self, the short-term self chooses when to check for information in real-time in every consumption period conditional on the information policy. The decisions of the consumer therefore become an intra-personal game between the long-term self's decision at $T = 0$ and the short-term self who makes decisions in every consumption period.

The following proposition characterizes the result:

Proposition 3 *If the push notification strategy leads to more frequent checking than no-push, there exists a threshold $\hat{\rho^*}$, such that the long-term self will choose to block notifications by committing to no-push if $\hat{\rho} < \hat{\rho^*}$. Otherwise, the long-term self always commits to using push notifications.*

Proof. See Appendix H. ■

The long-term self prefers the short-term self to check less often as it cares less about the immediate anxiety of delaying checking. If push notifications lead to more frequent checking, its benefit of reducing the anxiety cost may not compensate for the increased costs of checking. In the extreme case of $\hat{\rho} \to 0$, the long-term self does not care about future self's anxiety. The only objective is to minimize checking frequency. Therefore, the long-term self will block notifications if receiving notifications induces her future self to check more often. However, if push notifications reduce checking frequency, the long-term self will welcome that since the consumer can save checking cost while suffering less from anxiety. Proposition 3 implies that if the firm allows consumers to choose a push notification policy, we may expect some sophisticated and unconcerned (i.e., lower $\hat{\rho}$) consumers to block notifications.

Specifically, the threshold $\hat{\rho}$ below which the consumer will block push notifications can be calculated to be $\hat{\rho^*} = \left[\frac{1}{2}(V_I + E_I^2)\lambda t^* - \frac{(n^* - 1)}{2} V_I\right]^{-1} (\frac{\lambda}{n^*} - \frac{1}{t^*})$. The first term is the inverse of the difference in the average flow disutility between push and no-push strategies. The second term is the difference in average checking costs. At $\hat{\rho^*}$, the two differences cancel out, and the long-term self is indifferent between blocking or not. In terms of comparative statics, locally, we may treat n^* as a constant. Then $\hat{\rho^*}$ is decreasing in E_I^2, implying that the long-term self is less likely to block if the mean utility is high (and thus the anxiety cost is high under the no-push strategy). Similarly, $\hat{\rho^*}$ is decreasing in V_I when n^* is sufficiently small. This suggests that the long-term self is less likely to block if the variance for the information is higher and when the equilibrium checking is more frequent. Finally, $\hat{\rho^*}$ is decreasing in the arrival frequency λ when n^* is sufficiently large: $n^* > 2\sqrt{\frac{2\lambda}{(V_I + E_I^2)\rho}}$. It implies that the long-term self is less likely to block when information arrives frequently while the checking is sparse. The reason is that under these conditions, push notifications are less likely to induce more frequent checking.

VI.B Endogenous price

In the main model, there is no price for adopting the service of the information provider. Although this is consistent with the context of many information providers such as e-mails and social media, many information providers, such as newspapers and cable channels, charge consumers subscription prices. This section allows for the endogenous choice of price by the firm.

Suppose that at the beginning of the game, the firm sets a subscription fee p (per unit of time) for its service. The consumer has to pay the fee to check for information. Otherwise, the consumer gets the outside option with utility normalized to zero. The firm's flow profit is given by $\pi = p + \frac{\pi_c}{t}$ in which t is the average checking time for the consumer, π_c is a parameter for the profit per check, and p is the subscription fee. The consumer's decision is to adopt the firm's information platform if $E[U] \geq p$ in which U is the utility flow under the optimal checking strategy. And the firm's decision is to choose p as well as its information design to maximize π. Notice that if $\pi_c > 1$, the consumer's checking yields more profit for the firm than it costs the consumer, implying that checking is socially optimal. Given that the firm is a monopoly, it will set a subscription fee that equals the expected consumer flow utility to extract all the surplus.

Corollary 3 *With the subscription fee, the firm prefers the genuine push strategy or the noisy push to the no-push strategy when they lead to more frequent checking.*

The intuition for Corollary 3 follows directly from Corollary 1, and Corollary 2.

Compared with the no-push strategy, if the genuine push or the noisy push strategy leads to a Pareto improvement without endogenous price, then the firm can charge a higher price *and* make a higher profit from more checking. Comparing the genuine push and noisy push strategies, the similar results can hold if consumer checking is socially beneficial. Otherwise, the firm may choose a lower level of noise.

Proposition 4 *With subscription fee and $V_I > E_I^2$, if checking is socially optimal ($\pi_c > 1$), the firm's optimal level of noise k_p^* is the same as the optimal noisy push that maximizes the checking frequency, given by k^* in Proposition 2. Otherwise, the firm prefers a smaller yet strictly positive level of noise ($0 < k_p^* \leq k^*$) if the profit per check is not too small $\pi_c > \frac{E_I^2}{V_I}$ and $n_p^* > 1$.*

Proof. See Appendix I. ■

When the consumer checks every notification ($n_p^* = 1$), the optimality of noisy push takes a simple form:

Corollary 4 *If under genuine push strategy the consumer checks every notification, the firm prefers genuine push strategy (i.e. $k_p^* = 0$) if $\pi_c < 1$ and optimal noisy push strategy (i.e. $k_p^* = k^*$) if $\pi_c > 1$.*

Proof. See Appendix I. ■

In deciding whether to add noise to push, the firm faces a trade-off. Adding more noise can lead to higher profits from increased checking, but it will also lower consumer surplus due to higher checking costs. The anxiety cost is also weakly lower under noisy push because the consumer checks slightly more frequently. If checking is socially optimal, the profits from more frequent checking can offset the losses from lower subscription fees due to increased checking costs. If not, the optimal level of noise may be lower but can still be positive as long as the profit per check is not too small.

VI.C Increasing Anxiety over Time

In the main model, the flow anxiety cost is a function of only the underlying realized uncertainty (i.e., $u_t' = -\rho \text{Var}[u_t],$). Therefore under push notifications, the flow anxiety cost was constant for any point in time between two consecutive notifications. However, one may argue that there are situations in which the flow anxiety cost could be increasing in time even for the same realized uncertainty because consumer preferences are such that they get increasingly impatient with time. In other words, the anxiety cost parameter can be a function of time (since the last check), i.e., $\rho(t)$. Thus $\rho(t)$ can be interpreted as the consumer's time-varying anxiety valuation.

Specifically, we consider $\rho(t) = \rho + bt$, in which $b \geq 0$ is a parameter that captures how the consumer's flow anxiety preference changes and increases in time. We consider the case of increasing consumer anxiety over time, i.e., $b > 0$.

The case in the base model is equivalent to $b = 0$. Compared to the base model the consumer's checking frequency under no-push will increase because the anxiety cost increases as time elapses. The consumer's flow utility when checking every t units of time is given by:

$$\begin{aligned} \mathrm{E}[U_b(t)] &= \frac{1}{t}[E_I\lambda t - 1 - \int_0^t \rho(\tau)\mathrm{Var}[u(\tau)]\mathrm{d}\tau] \\ &= E_I\lambda - \frac{1}{t} - (V_I + E_I^2)\lambda(\frac{\rho t}{2} + \frac{bt^2}{3}) \end{aligned}$$

We have that $t_b^* = \arg\max_t -\frac{1}{t} - (V_I + E_I^2)\lambda(\frac{\rho t}{2} + \frac{bt^2}{3})$. Taking the F.O.C., we have t_b^* satisfies $\frac{1}{t^2} - (V_I + E_I^2)\lambda(\frac{\rho}{2} + \frac{2bt}{3}) = 0$. It can be shown that there exists a unique real root of the third degree polynomial in t, although the explicit form is complicated. But using the implicit function theorem it can be shown that $\frac{\partial t_b^*}{\partial b} < 0$, i.e., the consumer checks more frequently when the flow anxiety cost increases at a faster rate over time.

Under the genuine push strategy, the consumer's optimal checking strategy is as follows: When V_I is sufficiently large, checking every notification is still an optimal strategy, and in this case, as before, there will be no anxiety cost carried between notifications (the consumer's utility is given by $E[U_b(n = 1)] = E_I\lambda - \lambda$). Consequently, the consumer's checking frequency will not change compared to the base case (it is still given by $\frac{1}{\lambda}$). Therefore, in the parameter range where checking every notification is optimal, comparing push vs. no push, the firm is more likely prefer no-push over push when compared to the base case of constant flow anxiety ($b = 0$).

Note that the condition in which checking every notification is optimal is different from what is specified in Proposition 1. This is because the payoff when the consumer does not check at every notification is different. If the consumer does not check at every notification, the alternative checking strategy is more complicated. The checking policy is characterized by a time t_m, in which the consumer does not check at the first notification, but checks either after t_m units of time (as long as there is no second notification within time t_m), or checks immediately at the second notification if it arrives before t_m. The optimal t_m is given by the following:

$$\max_{t_m} \frac{1}{t(t_m)}[e^{-\lambda t_m}(E_I - 1 - \int_0^{t_m} \rho(t)V_I \mathrm{d}t) + (1 - e^{-\lambda t_m})(2E_I - 1 - \int_0^{t_m} (\int_0^t \rho(\tau)V_I \mathrm{d}\tau)e^{-\lambda t}\lambda t \mathrm{d}t)]$$

in the maximization $t(t_m) = \frac{1}{\lambda} + e^{-\lambda t_m} t_m + \int_0^{t_m} te^{-\lambda t}\lambda t \mathrm{d}t$ is the average checking time, and inside the brackets, the first part captures the average utility when the consumer waits until t_m units of time to check, while the second part captures the average utility when the consumer checks immediately at the second notification that arrives before t_m. Note when $t_m \to 0$, the policy converges to checking at every notification. When $t_m \to \infty$, the policy converges to checking at every two notifications. The condition that the consumer checking at every notification is optimal is given by $\mathrm{E}[U_b(t_m)] \leq E[U_b(n = 1)] = E_I\lambda - \lambda$.

The structure of this problem is somewhat similar to that of the muting strategy. While the analytical solution for the optimal checking time is not tractable, we can still specify how b affects consumer checking behavior. In general, $\mathrm{E}[U_b(t_m)]$ is decreasing in b. Suppose b decreases and the consumer maintains the same checking strategy. Then the consumer's utility will strictly increase because the only difference is that the consumer will be faced with less anxiety. If the consumer chooses the optimal checking strategy under the new b, the utility will be even higher. Therefore, the condition that $E[U_b(t_m)] \leq E_I\lambda - \lambda$ is more likely to hold when b is large, implying that the consumer is more likely to check immediately at every notification.

VII Conclusion

The consumption of information through smartphone apps, tablets, and other digital media is one of the central aspects of the digital consumer economy. Recent studies have pointed to the consistent and significant increases in the amount of time Americans spent on their mobile devices. A 2019 estimate suggests adults have spent more time on their smartphones ($>$ 3.5 hours a day) than watching TV. Thus user engagement and the checking of content is a valuable commodity, and firms would like to design their information revelation policies to maximize the amount of checking.

Our paper introduces the theory of information design to consumer information consumption and marketing strategy. Specifically, we study the dynamic design of information notifications by a firm that faces a consumer who has consumption utility as well as disutility for realized uncertainty of the information stock or anxiety costs. The consumer is uncertain both about the arrival of information as well as the value of the arrived information. The firm can strategically design the presentation of notifications to consumers in order to maximize consumer checking. Push notifications, by definition, resolve the arrival uncertainty faced by consumers and leave behind only the uncertainty in the valuation of the information stock. Surprisingly, despite the fact that push notifications reduce consumer uncertainty, we find that they can lead to more frequent checking as compared to no-push. Push notifications create an endogenous impulse to check, and discontinuously increases the anxiety costs. They allow the firm to create a more efficient spread in consumer beliefs and the associated anxiety such that the consumer either has no anxiety or just close to enough anxiety so as to make them check for information. Specifically, push notifications are preferred by the firm when the variance of the information is relatively high compared to the mean valuation, and when the information arrival rates are lower.

The firm has the incentive to strategically add noise to its notification design by mixing phantom pushes along with genuine pushes. This generalized noisy push strategy increases the checking frequency even when the consumer is fully rational. Slower true information arrivals and higher relative variance of the information enhance the firm's ability to introduce noise. We also find that while push notifications enhance consumer welfare compared with the no-push case, the noisy push strategy may lower consumer welfare compared to when the firm is committed to truthful notifications.

We also examine a generalization in the other possible direction by considering probabilistic pushing in which the firm mutes notifications by releasing them with some probability. When the variance of information is sufficiently high such that the consumer checks every notification, the firm in equilibrium chooses between the extremes of either genuine push, or else the no-push strategy. It is only when it is optimal for the consumer check less often that the firm will mute with an interior probability so that the consumer may check even without a notification. The endogenous uncertainty created by muting induces the consumer to check if enough time has passed without any new notifications.

In an extension we show the linkage of consumer self-control problems through a dual-self framework to the incentive of consumers to block notifications. We also extend the model to endogenous prices by allowing the platform to charge consumers a subscription fee. Finally, we consider a plausible characterization of consumer preferences in which the flow anxiety cost that is increasing in time to show that the firm may be more likely to prefer the no-push strategy over the push strategy when the flow anxiety cost increases at a faster rate.

Our analysis has broad implications across information consumption contexts. Information providers such as news agencies, email, messaging services, as well as social media, all make profits based on consumer engagement and checking of information. It is common for information providers to use push technologies to consumers' smartphones, tablets, and computers. Understanding how consumers respond to different push technologies has implications both for the design of the optimal push strategy as well as its implications for consumer welfare. As we have already argued, it is hard to overstate the magnitude of information consumption on smartphones and mobile devices not only in people's personal lives but also on their work productivity. A recent study estimates that employees in the U.S. firms spent five hours per week on non-productive activities on their smartphones, and \$15.5 billion loss of productivity per week, or over \$800 billion per year. Even small changes to checking behavior can have significant productivity implications. Indeed, both Google and Apple have displayed incentives to regulate push notifications from information providers. Our analysis provides a framework for platforms and policymakers to consider the welfare implications of push technologies.

There are some interesting avenues for future analysis. Some alternative firm payoff functions can be contemplated in news or information provision markets. For example, on media websites or on television news arrives over time and the firm may be able to explicitly bundle paid ads along with each information arrival. In this case there may be a different set of considerations such as the value of news information depreciating over time if not checked immediately. In such a market the firm's objective might be to keep the consumers continuously viewing the news program for as long as possible so that they do not miss any notifications (ads). It will also be useful to consider situations in which the consumers' anxiety may attenuate over time because of forgetting or reduced salience. This type of situation may be relevant for low involvement markets, and in this case the firm will face the challenge of sustaining consumer interest over time. In this context and more broadly it will be interesting to understanding the interaction of the product and promotional design with the information notification design adopted by firms.

References

Bergemann, D. and Morris, S. (2019). Information design: A unified perspective. *Journal of Economic Literature*, 57(1):44–95. II

Brun, W. and Teigen, K. H. (1990). Prediction and postdiction preferences in guessing. *Journal of Behavioral Decision Making*, 3(1):17–28. II

Della Vigna, S. and Malmendier, U. (2006). Paying not to go to the gym. *American Economic Review*, 96(3):694. 7

Duke, É. and Montag, C. (2017). Smartphone addiction, daily interruptions and self-reported productivity. *Addictive Behaviors Reports*, 6:90–95. I

Dye, R. A. (1985). Disclosure of nonproprietary information. *Journal of Accounting Research*, pages 123–145. 3

Ely, J., Frankel, A., and Kamenica, E. (2015). Suspense and surprise. *Journal of Political Economy*, 123(1):215–260. II

Ely, J. C. (2017). Beeps. *American Economic Review*, 107(1):31–53. II

Falk, A. and Zimmermann, F. (2016). Beliefs and utility: Experimental evidence on preferences for information. CESifo Working Paper Series. II

Freeston, M. H., Rhéaume, J., Letarte, H., Dugas, M. J., and Ladouceur, R. (1994). Why do people worry? *Personality and Individual Differences*, 17(6):791–802. III.B

Fudenberg, D. and Levine, D. K. (2006). A dual-self model of impulse control. *American Economic Review*, 96(5):1449–1476. VI.A

Ganguly, A. and Tasoff, J. (2017). Fantasy and dread: The demand for information and the consumption utility of the future. *Management Science*, 63(12):4037–4060. II

Godes, D., Ofek, E., and Sarvary, M. (2009). Content vs. advertising: The impact of competition on media firm strategy. *Marketing Science*, 28(1):20–35. III.C

Grossman, S. J. (1981). The informational role of warranties and private disclosure about product quality. *The Journal of Law and Economics*, 24(3):461–483. II

Guo, L. and Zhao, Y. (2009). Voluntary quality disclosure and market interaction. *Marketing Science*, 28(3):488–501. 3

Iyer, G. and Singh, S. (2021). Persuasion contest: Disclosing own and rival information. 3

Jain, S. (2012a). Marketing of vice goods: A strategic analysis of the package size decision. *Marketing Science*, 31(1):36–51. 7

Jain, S. (2012b). Self-control and incentives: An analysis of multiperiod quota plans. *Marketing Science*, 31(5):855–869. 7

Jones, S. L., Ferreira, D., Hosio, S., Goncalves, J., and Kostakos, V. (2015). Revisitation analysis of smartphone app use. In *Proceedings of the 2015 ACM International Joint Conference on Pervasive and Ubiquitous Computing*, pages 1197–1208. IV.A

Jovanovic, B. (1982). Truthful disclosure of information. *The Bell Journal of Economics*, pages 36–44. 3

Kamenica, E. (2018). Bayesian persuasion and information design. *Annual Review of Economics*. II

Kamenica, E. and Gentzkow, M. (2011). Bayesian persuasion. *American Economic Review*, 101(6):2590–2615. II

Kind, H. J., Nilssen, T., and Sørgard, L. (2009). Business models for media firms: Does competition matter for how they raise revenue? *Marketing Science*, 28(6):1112–1128. III.C

Kőszegi, B. and Rabin, M. (2009). Reference-dependent consumption plans. *American Economic Review*, 99(3):909–936. II

Ladouceur, R., Gosselin, P., and Dugas, M. J. (2000). Experimental manipulation of intolerance of uncertainty: A study of a theoretical model of worry. *Behaviour Research and Therapy*, 38(9):933–941. II, III.B

Lauga, D. (2011). Persuasive advertising with sophisticated but impressionable consumers. *Working Paper*. 3

Loewenstein, G. (1994). The psychology of curiosity: A review and reinterpretation. *Psychological Bulletin*, 116(1):75. II, III.B

Lu, M. Y. and Shin, J. (2018). A model of two-sided costly communication for building new product category demand. *Marketing Science*, 37(3):382–402. 3

Masatlioglu, Y., Orhun, A. Y., and Raymond, C. (2017). Intrinsic information preferences and skewness. Working Paper. II

Mayzlin, D. and Shin, J. (2011). Uninformative advertising as an invitation to search. *Marketing Science*, 30(4):666–685. 3

Milgrom, P. R. (1981). Good news and bad news: Representation theorems and applications. *The Bell Journal of Economics*, pages 380–391. II

Nielsen, K. (2017). Preferences for the resolution of uncertainty and the timing of information. Working paper. II

Pielot, M., Church, K., and De Oliveira, R. (2014). An in-situ study of mobile phone notifications. In *Proceedings of the 16th international conference on Human-computer interaction with mobile devices & services*, pages 233–242. ACM. I

Sahami Shirazi, A., Henze, N., Dingler, T., Pielot, M., Weber, D., and Schmidt, A. (2014). Large-scale assessment of mobile notifications. In *Proceedings of the SIGCHI Conference on Human Factors in Computing Systems*, pages 3055–3064. ACM. I

Tams, S., Legoux, R., and Léger, P.-M. (2018). Smartphone withdrawal creates stress: A moderated mediation model of nomophobia, social threat, and phone withdrawal context. *Computers in Human Behavior*, 81:1–9. I

Thaler, R. H. and Shefrin, H. M. (1981). An economic theory of self-control. *Journal of Political Economy*, 89(2):392–406. VI.A

Tolin, D. F., Abramowitz, J. S., Brigidi, B. D., and Foa, E. B. (2003). Intolerance of uncertainty in obsessive-compulsive disorder. *Journal of Anxiety Disorders*, 17(2):233–242. III.B

Verrecchia, R. E. (1983). Discretionary disclosure. *Journal of Accounting and Economics*, 5:179–194. 3

Vosgerau, J., Wertenbroch, K., and Carmon, Z. (2006). Indeterminacy and live television. *Journal of Consumer Research*, 32(4):487–495. II

A Proofs

A Proof of Lemma 1

The informational utility for time τ (if the consumer checks) is given by $u(\tau) = \sum_{k=1}^{N(\tau)} I_k$. We have the expected utility and its variance are given by

$$\mathrm{E}[u(\tau)] = \mathrm{E}[I_k]\mathrm{E}[N(\tau)] = E_I \lambda \tau \tag{3}$$

and

$$\begin{aligned}
\mathrm{Var}[u(\tau)] &= \mathrm{Var}[\sum_{k=1}^{N(\tau)} I_k] \\
&= \mathrm{E}[N(\tau)]\mathrm{Var}[I_k] + (\mathrm{E}[I_k])^2 \mathrm{Var}[N(\tau)] \\
&= (V_I + E_I^2)\lambda\tau
\end{aligned} \tag{4}$$

Both are linear in τ. Therefore, with disutility flow of $\rho \mathrm{Var}[\sum_{k=1}^{N(\tau)} I_k]$, the consumer is getting "anxious" in a linear and increasing manner about unchecked information.

We first show that the optimal checking strategy is a constant frequency of checking. First, suppose the consumer checks at at a sequence of times $t_1, t_2, t_3, ...$, we first want to show that this checking strategy is not optimal unless $t_n = nt$, in which t is a constant. This can be proved by contradiction. Suppose the strategy is not given by $t_n = nt$. Then there must exist some t_m, t_{m+1}, t_{m+2} s.t. $t_{m+1} \neq \frac{1}{2}(t_m + t_{m+2})$. It can be shown that if we let $t_{m+1} = \frac{1}{2}(t_m + t_{m+2})$, the consumer's utility will be higher. Comparing the two strategies, the only difference is the anxiety cost between between t_m and t_{m+2}. The total anxiety cost of the original strategy is given by $\frac{1}{2}\rho(V_I + E_I^2)\lambda[(t_{m+1} - t_m)^2 + (t_{m+2} - t_{m+1})^2]$, which is minimized when $t_{m+1} = \frac{1}{2}(t_m + t_{m+2})$ (the derivative with respect to t_{m+1} is zero given t_m and t_{m+2}).

Next, consider a randomized strategy in which the consumer checking follows a stochastic process (say a Poisson process with some rate λ_c). Any possible realization of the checking process will be a sequence of checking times which we can denote as $t_{1c}, t_{2c}, t_{3c}, ..., .$ And in general for any such realized sequence there may exist some $t_{mc}, t_{(m+1)c}, t_{(m+2)c}$ such that $t_{(m+1)c} \neq \frac{1}{2}(t_{mc} + t_{(m+2)c})$, in which case the total anxiety cost will not be minimized.

Therefore the optimal strategy requires that $t_n = nt$, i.e., the consumer checks every t units of time. Given this the expected flow utility is given by:

$$\begin{aligned}
\mathrm{E}[U(t)] &= \frac{1}{t}\left[\mathrm{E}[u(t)] - 1 - \int_0^t \rho \mathrm{Var}[u(\tau)] d\tau\right] \\
&= E_I\lambda - \frac{1}{t} - \frac{1}{2}\rho(V_I + E_I^2)\lambda t
\end{aligned} \tag{5}$$

in which the first term is flow consumption utility of information, the second term is flow cost of checking, and the third term is average disutility of carrying realized uncertainty for the period t. It is maximized when $\frac{1}{t} = \frac{1}{2}\rho(V_I + E_I^2)\lambda t$ or $t^* = \sqrt{\frac{2}{\rho(V_I + E_I^2)\lambda}}$. For the optimal expected utility, we have that $\mathrm{E}[U(t^*)] = E_I\lambda - \sqrt{2\rho(V_I + E_I^2)\lambda}$.

B Proof of Lemma 2

We first show that the optimal checking strategy is to check every n_p^* notifications instead of checking between notifications with positive probability. There is no incentive to check between notifications because i) the level of anxiety ($\rho\mathrm{Var}(u_t)$) is constant for all t between two adjacent notifications, and ii) the information arrival process is Poisson and so the expected time for next arrival is also constant. Therefore, checking at any time after the notification arrival is inferior to checking exactly at the notification arrival.

Formally, let $\mathrm{E}[U(n)]$ as the expected flow utility if the consumer uses a policy of checking every n notifications. Define $n_p^* \in \mathcal{N}_+$ as optimal number of notifications to check that maximizes the utility flow. Notice that the expected waiting time for each arrival follows an exponential distribution and has expectation of $\frac{1}{\lambda}$. The expected utility flow is given by

$$\begin{aligned}\mathrm{E}[U(n)] &= \frac{\lambda}{n}\left[nE_I - 1 - \rho\frac{(n-1)n}{2\lambda}V_I\right] \\ &= E_I\lambda - \frac{\lambda}{n} - \rho\frac{(n-1)}{2}V_I \end{aligned} \tag{6}$$

in which the first term is flow consumption utility of information, the second term is flow cost of checking, and the third term is average disutility of carrying realized uncertainty. Maximizing this expected flow utility we have that $\tilde{n_p} = \arg\min_{n\in\mathcal{R}_+} \frac{\lambda}{n} + \rho\frac{(n-1)}{2}V_I$. By definition any $\tilde{n_p} \in [\lfloor\tilde{n_p}\rfloor, \lceil\tilde{n_p}\rceil]$, where $\lceil\tilde{n_p}\rceil = \lfloor\tilde{n_p}\rfloor + 1$. The consumer has to decide whether to check every $\tilde{n_p} \in \lfloor\tilde{n_p}\rfloor$, or every $\lfloor\tilde{n_p}\rfloor + 1$ notifications. The consumer will choose to check every $\tilde{n_p} \in \lfloor\tilde{n_p}\rfloor$ notifications if $\frac{\lambda}{\lfloor\tilde{n_p}\rfloor} + \rho\frac{\lfloor\tilde{n_p}\rfloor - 1}{2}V_I \leq \frac{\lambda}{\lfloor\tilde{n_p}\rfloor + 1} + \rho\frac{\lfloor\tilde{n_p}\rfloor}{2}V_I$. This implies $n_p^* = \lfloor\tilde{n}_p\rfloor$, if $\tilde{n}_p \leq \sqrt{\lfloor\tilde{n}_p\rfloor(\lfloor\tilde{n}_p\rfloor + 1)}$.

We want to show that the strategy of checking every n_p^* notifications yields higher expected flow utility than potential alternate strategies: i.e., i) the strategy of waiting for τ or ii) a probability mixture over a set of the number of notifications after which the consumer checks.

Consider an alternative checking strategy that checks exactly at the n-th notification with some probability $p < 1$, and checks at some τ units of time after the n-th notification but before the $n-1$-th notification with probability $1-p$. The expected utility flow of the latter strategy is given by:

$$
\begin{aligned}
\mathrm{E}[U(n,\tau)] =& \frac{\frac{n}{\lambda}}{\frac{n}{\lambda}+(1-p)\tau}\mathrm{E}[U(n)] + \frac{(1-p)\tau}{\frac{n}{\lambda}+(1-p)\tau}\frac{[nE_I - 1 - \rho(\frac{n(n-1)}{2\lambda}+\tau)V_I]}{\frac{n}{\lambda}+\tau} \\
<& \frac{\frac{n}{\lambda}}{\frac{n}{\lambda}+(1-p)\tau}\mathrm{E}[U(n)] + \frac{(1-p)\tau}{\frac{n}{\lambda}+(1-p)\tau}\frac{[nE_I - 1 - \rho(\frac{n(n-1)}{2\lambda}+\tau)V_I]}{\frac{n}{\lambda}} \\
<& \frac{\frac{n}{\lambda}}{\frac{n}{\lambda}+(1-p)\tau}\mathrm{E}[U(n)] + \frac{(1-p)\tau}{\frac{n}{\lambda}+(1-p)\tau}\frac{[nE_I - 1 - \rho(\frac{n(n-1)}{2\lambda})V_I]}{\frac{n}{\lambda}} \\
=& \mathrm{E}[U(n)] \leq \mathrm{E}[U(n_p^*)] \quad \text{by definition}
\end{aligned}
$$

Next it can also be shown that the consumer will not check at a combination of different numbers of notifications, since the average utility flow is just a linear combination of strategies of different n which is lower than the strategy of n_p^* with highest utility flow. Suppose the consumer chooses to alternate between a set of checking strategies ($s \in S$) of checking every n_s notifications with probability p_s. And suppose s^* maximizes the expected utility flow $\mathrm{E}[U(n_{s^*})]$. The utility flow will be given by

$$
\begin{aligned}
\mathrm{E}[U(S)] =& \sum_{s\in S} p_s \mathrm{E}[U(n_s)] \\
<& \mathrm{E}[U(n_{s^*})] \\
\leq& \mathrm{E}[U(n_p^*)]
\end{aligned}
$$

the last inequality is given by definition of n_p^*.

C Proof of Proposition 1

We want to identify the parameter range where the consumer checks more frequently under push. The expected time between any two checks under push is given by $\frac{n_p^*}{\lambda}$, where n_p^* is the optimal number of notifications as given in Lemma 2, while the expected time between two checks under no-push is given by $t_n^* = \sqrt{\frac{2}{\rho(V_I + E_I^2)\lambda}}$ as given in Lemma 1.

To find the range first note that if a consumer prefers checking every $m \in \mathcal{N}_+$ notifications to checking every $m+1$ notifications, she will check at most m notifications.[8] The condition for the

[8]This is because the average anxiety cost increases linearly in the number of notifications, while the average checking cost $\frac{\lambda}{m}$ decreases more slowly over m.

consumer to prefer to check (at most) every m notification is that:

$$\frac{\lambda}{m} + \rho\frac{m-1}{2}V_I < \frac{\lambda}{m+1} + \rho\frac{m}{2}V_I$$

or $\lambda < \rho V_I \frac{m(m+1)}{2}$.

On the other hand, the condition that the consumer will check more frequently if they check every m notifications, compared with no-push, is as follows:

$$\frac{m}{\lambda} < \sqrt{\frac{2}{\rho(V_I + E_I^2)\lambda}}$$

or $\lambda > \rho(V_I + E_I^2)\frac{m^2}{2}$.

Combining the two conditions, we have that some λ always exists as long as $V_I > mE_I^2$, as desired. Because these conditions hold for every m, (and not $m+1$) they also hold for $n_p^* = \lfloor \tilde{n}_p \rfloor$. Thus the equilibrium range for which the push strategy dominates for the firm is given by $\lambda \in (\rho(V_I + E_I^2)\frac{\lfloor \tilde{n}_p \rfloor^2}{2}, \rho V_I \frac{\lfloor \tilde{n}_p \rfloor(\lfloor \tilde{n}_p \rfloor + 1)}{2})$, as in Proposition 1.

A special case is when $V_I > E_I^2$, there is always an interval $\lambda \in (\frac{\rho(V_I + E_I^2)}{2}, \rho V_I)$ such that the consumer checking every notification, and the checking frequency is higher under push notification compared with no-push.

D Proof of Corollary 1

To prove that the push strategy leads to higher consumer utility, note the following: At any point in time since the last check, the push strategy leads to lower expected anxiety cost. Given that, a consumer under push strategy can always use the same checking strategy under no-push strategy to get strictly higher utility. And the consumer's utility using optimal checking strategy under push strategy should be even higher by definition. Notice that under no-push, the expected anxiety cost at time τ (since last check) is given by $\rho Var_n[u(\tau)] = \rho(V_I + E_I^2)\lambda\tau$. Under push strategy, the same expected anxiety cost at time τ (since last check) is given by $\rho Var_p[u(\tau)] = \rho V_I \lambda\tau < \rho Var_n[u(\tau)]$.

E Proof of Proposition 2

Following an analysis similar to Lemma 2, we will first show that the optimal checking behavior can be characterized by checking every n_x^* notifications. The optimal number of notifications is given as follows: Upon receiving some n notifications, let there are m genuine notifications. And the variance of realized uncertainty is given by $\mathrm{Var}[\sum_{j=1}^{m} I_j] = \mathrm{E}[m]\mathrm{Var}[I_j] + (\mathrm{E}[I_j])^2\mathrm{Var}[m] = \frac{n}{1+k}V_I + E_I^2\frac{nk}{(1+k)^2}$, in which the first term comes from the variance in the information itself, and the second term comes from the uncertainty of the number of information arrivals, given by the variance of binomial distribution with n notifications, and the probability $\frac{1}{1+k}$ of them being genuine.[9]

Therefore, the expected utility flow of checking per n notifications is given by

$$\begin{aligned} \mathrm{E}[U(n)] &= \frac{(k+1)\lambda}{n}\left[\frac{n}{1+k}E_I - 1 - \frac{\rho}{\lambda(k+1)}\sum_{i=0}^{n-1}\{\frac{i}{1+k}V_I + E_I^2\frac{ik}{(1+k)^2}\}\right] \\ &= \lambda E_I - \frac{(k+1)\lambda}{n} - \frac{\rho(n-1)}{2(k+1)}(V_I + E_I^2\frac{k}{1+k}) \end{aligned}$$

Maximizing $E[U(n)]$ is as before equivalent to minimizing the sum of the flow costs. Let $\tilde{n}_x = \arg\min_{n\in\mathcal{R}_+}\frac{(k+1)\lambda}{n} + \frac{\rho(n-1)}{2(k+1)}(V_I + E_I^2\frac{k}{1+k})$. Therefore, the optimal checking frequency n_x^* for any k that is chosen by the form is given by:

$$n_x^* = \arg\min_{n\in\mathcal{N}_+}\frac{(k+1)\lambda}{n} + \frac{\rho(n-1)}{2(k+1)}(V_I + E_I^2\frac{k}{1+k}) \tag{7}$$

As expected, when $k \to 0$, $n_x^* = \arg\min_{n\in\mathcal{N}_+}\frac{\lambda}{n} + \frac{\rho(n-1)}{2}V_I$ which is the checking frequency for the genuine push strategy as derived in Lemma 2. When $k \to \infty$, implying that all the push notifications are phantom ones, $n_x^* \to (k+1)\sqrt{\frac{2\lambda}{\rho(V_I + E_I^2)}}$ which is the equivalent to the checking time of $\sqrt{\frac{2}{\rho(V_I + E_I^2)\lambda}}$, the same as the checking frequency under the no-push strategy.

The optimal number of notifications to check under noisy push, n_x^*, requires the necessary condition:

$$\frac{(k+1)\lambda}{n_x^*} + \frac{\rho(n_x^*-1)}{2(k+1)}(V_I + E_I^2\frac{k}{1+k}) \le \frac{(k+1)\lambda}{n_x^*+1} + \frac{\rho(n_x^*)}{2(k+1)}(V_I + E_I^2\frac{k}{1+k})$$

solving it for λ, we have $\lambda \le \frac{\rho n_x^*(n_x^*+1)}{2(k+1)^2}(V_I + E_I^2\frac{k}{1+k})$.

We note that the checking frequency is increasing in k for any given n_x^*. For any $n_x^* \in \mathcal{N}_+$ there will be a range of k for which n_x^* is optimal. Given this the (local) optimal k^* results from increasing k such that the inequality of the necessary condition for n_x^* is binding. Therefore we have the necessary condition to be:

$$\frac{(k+1)^2\lambda}{n_x^*(n_x^*+1)} = \frac{\rho V_I}{2} + \frac{\rho k E_I^2}{2(k+1)} \tag{8}$$

[9] Alternatively, the variance can be written as $\mathrm{Var}[\sum_{j=1}^n \hat{I}_j]$, in which $\hat{I}_j$ equals I_j with probability $\frac{1}{1+k}$ and 0 with probability $\frac{k}{1+k}$. Thus, $\mathrm{Var}[\sum_{j=1}^n \hat{I}_j] = n\mathrm{Var}[\hat{I}_j] = n\{E_k[\mathrm{Var}[I_j]] + \mathrm{Var}_k[E[I_j]]\} = n\{\frac{1}{1+k}V_I + \frac{k}{(1+k)^2}E_I^2\}$, which is the same as the variance given above.

Equations (7) and (8) together can potentially determine different possible combinations of the choice of k by the firm and n_x^* by the consumer. The equilibrium pair denoted by $\{N_x^*, k^*\}$ is a specific combination of number of notifications to check and the level of noise, where $\{N_x^*, k^*\} = \arg\max_{\frac{n_x^*}{1+k^*}} \{n_x^*, k^*\}$ with $k^* \in [0, \infty)$. In other words, the firm will choose the k^* such that it along with the associated N_x^* will generate the (global) maximization of the checking frequency. We need to show that this dominates the checking frequency under no-push. The checking frequency under noisy push is higher than no-push when:

$$\frac{N_x^*}{(1+k^*)\lambda} < \sqrt{\frac{2}{\rho(V_I + E_I^2)\lambda}}$$

solving it for λ, we have $\lambda > \frac{\rho {N_x^*}^2}{2(k^*+1)^2}(V_I + E_I^2)$.

Now note that with the optimal k^*, the condition that $\lambda \leq \frac{\rho n_x^*(n_x^*+1)}{2(k+1)^2}(V_I + E_I^2 \frac{k}{1+k})$ will hold because it takes equality (and this also implies that $n_x^* = \lfloor \tilde{n}_x \rfloor$). The only condition we need to consider is $\lambda > \frac{\rho {N_x^*}^2}{2(k^*+1)^2}(V_I + E_I^2)$, or $\frac{\rho N_x^*(N_x^*+1)}{2(k^*+1)^2}(V_I + E_I^2 \frac{k^*}{1+k^*}) > \frac{\rho {N_x^*}^2}{2(k^*+1)^2}(V_I + E_I^2)$, which can be reduced to $V_I > E_I^2 \frac{N_x^* - k^*}{1+k^*}$.

The particular case in which $k^* = 0$ is given by the following equality: $\lambda = \frac{\rho N_x^*(N_x^*+1)}{2} V_I$, i.e. when the consumer is just indifferent from checking every N_x^* notifications and every $N_x^* + 1$ notifications.

When there is no finite $\{N_x^*, k^*\}$ that satisfies the condition that the checking frequency under noisy push is higher than no-push (i.e., when $V_I \leq E_I^2 \frac{n_x^* - k^*}{1+k^*}$), the optimal k^* is simply $k^* \to \infty$ which results in a checking frequency that converges to the case of no-push. From (8), if we let $k \to \infty$, we have that $\lim_{k\to\infty} [\frac{(k+1)}{n_x^*}]^2 \lambda = \frac{\rho}{2}(V_I + E_I^2)$, or equivalently, the checking frequency $\lim_{k\to\infty} \frac{n_x^*}{(k+1)\lambda} = \sqrt{\frac{2}{\rho(V_I + E_I^2)\lambda}}$, which is the same as the checking frequency in the no-push case.

F Proof of Corollary 2

To prove that the noisy push strategy leads to higher consumer utility than no-push, we show the following: At any point in time since the last check, the noisy push strategy leads to, on average, lower anxiety cost. Given that, a consumer under noisy push strategy can always use the same checking strategy under no-push strategy to get at least as much utility. And the consumer's utility using optimal checking strategy under noisy push strategy should be even higher.

Notice that under no-push, the expected anxiety cost at time τ (since last check) is given by $\rho\text{Var}_n[u(\tau)] = \rho(V_I + E_I^2)\lambda\tau$. Under noisy push strategy, the same expected anxiety cost at time τ (since last check) is given by the following: the expected number of push notifications are $n(\tau) = (1+k)\lambda\tau$. From the Proof of Proposition 2 in Appendix E, we have that the variance is given by

$$\begin{aligned}
\text{Var}_x[u(\tau)] =& \frac{n(\tau)}{1+k}V_I + E_I^2\frac{n(\tau)k}{(1+k)^2} \\
&= V_I\lambda\tau + E_I^2\lambda\tau\frac{k}{1+k} \\
&< (V_I + E_I^2)\lambda\tau
\end{aligned}$$

Therefore, we have that the expected anxiety cost at time τ under noisy push satisfies $\rho\text{Var}_x[u(\tau)] < \rho\text{Var}_n[u(\tau)]$, which implies that the anxiety cost is on average higher for no-push. As a sanity check, when $k \to \infty$, the anxiety costs under noisy push and push converge as desired.

G Proof of Lemma 3

First, define a state to be $(n, \tau) \in \mathbb{N} \times \mathbb{R}^+$, in which n is the number of observed notifications and τ is the time since the last check. The optimal consumer checking strategy is a function of the state, because the state transitions (absent checking) are independent of the history and the flow payoffs are only dependent on the state.

Define the optimal state as $I \subset \mathbb{N}\times\mathbb{R}^+$ such that the consumer checks if and only if $(n, \tau) \in I$. Clearly, $\neg I \neq \emptyset$ because $(0, 0) \in \neg I$ (otherwise the utility will $\to -\infty$ because the consumer will keep checking continuously). I will also have the following property: if $(n, \tau) \in I$ then $(n + n', \tau + \epsilon) \in I$, $\forall\ (n', \epsilon) \in \mathbb{N} \times \mathbb{R}^+$. We therefore consider the boundary set of I as I_b, which is given by $(n, \tau) \in I_b$ if $(n, \tau) \in I$ and $\{(n-1, \tau), (n, \tau - \epsilon)\} \cap \neg I \neq \emptyset$, $\forall\ \epsilon > 0$.

We can further define $\tau_m(n) = \min(\tau, \text{s.t.}(n, \tau) \in I_b)$. This is the minimum waiting time since the last check after which consumer checks given that there are n notifications. For example, $\tau_m(0)$ is the waiting time since last check without any notifications. $\tau_m(n)$ will be weakly decreasing in n, and when n is large enough, $\tau_m(n) = 0$ (otherwise the consumer may wait even with an infinite number of notifications). Let $n^* = \min_n\{n; \tau_m(n) = 0\}$.

To calculate the utility, we note that there are two cases in which the consumer checks. First, a consumer may check after waiting a period of time (and absent any immediate notification). Second, a consumer may check immediately after receiving a notification.

In the first case, the consumer will arrive at the state $(n, \tau_m(n))$ because of optimally waiting for a period of time $\tau_m(n)$ since the last check during which there are n notifications. The probability of arriving at such a state is given by the probability that there are exactly n notifications during $(0, \tau_m(n)]$, which is given by $p_{N(\tau_m(n))}(n) = \frac{(p\lambda\tau_m(n))^n e^{-p\lambda\tau_m(n)}}{n!}$. We call this set of such states as $I_{b1} = \{(n, \tau_m(n)); n < n^*\}$.

In the second case, the consumer will arrive at the state (n, τ_n), $\tau_n \in (\tau_m(n), \tau_m(n-1))$ because of the the immediate arrival of one notification. This probability is given by the probability of there being $n-1$ arrivals in the interval $(0, \tau_n)$ times the probability of one additional arrival at the time τ_n, which is given by $\int_{\tau_m(n)}^{\tau_m(n-1)} p_{N(\tau_n)}(n-1)p\lambda d\tau_n$. We call the set of such states as $I_{b2} = \{(n, \tau_n), \tau_n \in (\tau_m(n), \tau_m(n-1)); 1 \leq n \leq n^*\}$.

The average time per check is therefore given by

$$t^*(I_b) = \sum_{n \leq n^*} [p_{N(\tau_m(n))}(n)\tau_m(n) + \int_{\tau_m(n)}^{\tau_m(n-1)} p_{N(\tau_n)}(n-1)p\lambda\tau_n d\tau_n]$$

.

To solve for the optimal checking strategy, notice that the average utility for any $(n, \tau) \in I_{b1}$ in the first case is given by the following:

$$E_1[U(n, \tau)] = \frac{1}{\tau}[E_1[n, \tau] - 1 - \frac{\rho}{2}V_1[n, \tau]\tau]$$

in which $E_1[n, \tau] = (n + (1-p)\lambda\tau)E_I$, and $V_1[n, \tau] = (1-p)\lambda(V_I + E_I^2)\tau + nV_I$.

In the second case, for any $(n, \tau) \in I_{b2}$, the average utility is given by

$$E_2[U(n, \tau)] = \frac{1}{\tau}[E_2[n, \tau] - 1 - \frac{\rho}{2}V_2[n, \tau]\tau]$$

in which $E_2[n, \tau] = (n + (1-p)\lambda\tau)E_I$, and $V_2[n, \tau] = (1-p)\lambda(V_I + E_I^2)\tau + (n-1)V_I$.

And the optimal checking strategy is given by the following optimization problem.

$$\arg\max_{I_b} \frac{1}{t^*(I_b)} \sum_{n<n^*} \left[p_{N(\tau_m(n))}(n)\tau_m(n)E_1[U(n, \tau_m(n))] + \int_{\tau_m(n)}^{\tau_m(n-1)} p_{N(\tau_n)}(n-1)p\lambda\tau_n E_2[U(n, \tau_n)]d\tau_n \right] \tag{9}$$

in which $\tau_m(n) = \min(\tau, \text{s.t.}(n, \tau) \in I_b)$ as we defined earlier. We call the average utility given checking strategy I_b as $U(I_b)$. This extreme value problem is well defined because the possible set of I_b is finite (otherwise, there is some probability that the consumer will wait forever or wait even after infinite number of notifications, and the utility goes to $-\infty$) and therefore compact.

Example 1 The simplest case is that under notification aggregation, the consumer checks at every notification. If there's no notification, the consumer waits until time τ_0 to check. In what follows, we derive the specification of τ_0 and numerically compare the notification aggregation/muting strategy with push/no-push strategies.

There are two cases in which a consumer checks. In the first case, the consumer waits until $\tau = \tau_0$ and no notification has arrived, with a probability given by $p_{N(\tau_m(0))}(0) = \frac{(p\lambda\tau_m(0))^0 e^{-p\lambda\tau_m(0)}}{0!} = e^{-p\lambda\tau_0}$. The expected flow utility in this case is given by

$$E_1[U(0,\tau_0)] = \frac{1}{\tau_0}[E_1[0,\tau_0] - 1 - \frac{\rho}{2}((1-p)\lambda(V_I + E_I^2)\tau_0)\tau_0]$$

In the second case, when the consumer checks at the immediate arrival of one notification, the probability is $\int_0^{\tau_0} e^{-p\lambda\tau} p\lambda d\tau = 1 - e^{-p\lambda\tau_0}$. The expected utility flow at any point $\tau \in (0, \tau_0)$ is given by

$$E_2[U(1,\tau)] = \frac{1}{\tau}[E_2[1,\tau] - 1 - \frac{\rho}{2}((1-p)\lambda(V_I + E_I^2)\tau)\tau]$$

The average time per check is therefore given by $t^*(\tau_0) = e^{-p\lambda\tau_0}\tau_0 + \int_0^{\tau_0} e^{-p\lambda\tau} p\lambda\tau d\tau = e^{-p\lambda\tau_0}\tau_0 + \frac{1}{p\lambda}[1 - e^{-p\lambda\tau_0}(1 + p\lambda\tau_0)] = \frac{1}{p\lambda}(1 - e^{-p\lambda\tau_0})$.

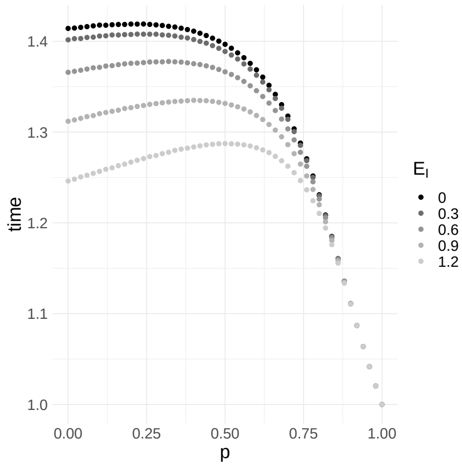
Furthermore, τ_0 is given by the following maximization problem:

$$\arg\max_{\tau_0} \frac{1}{t^*(\tau_0)}\left[e^{-p\lambda\tau_0}E_1[U(0,\tau_0)]\tau_0 + \int_0^{\tau_0} e^{-p\lambda\tau} p\lambda\tau E_2[U(1,\tau)]d\tau\right] =$$

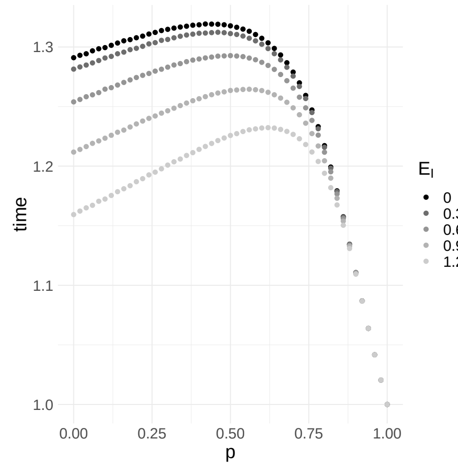
$$\arg\max_{\tau_0} \frac{1}{t^*(\tau_0)}\left[-1 - \frac{\rho}{2}(1-p)\lambda(V_I + E_I^2)[e^{-p\lambda\tau_0}(-\frac{2\tau_0}{p\lambda} - \frac{2}{p^2\lambda^2}) + \frac{2}{p^2\lambda^2})]\right] \tag{10}$$

We solve for τ_0 using a dense grid search as follows. We set the parameter $\lambda = 1$ and $\rho = 0.2$. For this case the feasible lower bound of the variance is $V_I = 5$ (from the condition that $\lambda < \rho V_I$ in Proposition 1). We test $V_I \in \{5, 6, 7, 8, 9, 10\}$, and $E_I \in \{0, 0.3, 0.6, 0.9, 1.2\}$. We solve for all possible pushing probabilities $p \in (0, 1)$ at step increments of 0.02. For every set of parameters $(\lambda, \rho, V_I, E_I, p)$, we solve for optimal τ_0 that solves the maximization problem in (10) through a search of τ_0 within the set of $\tau_0 \in (0, 100)$ with a step of 0.001 that maximizes the utility.

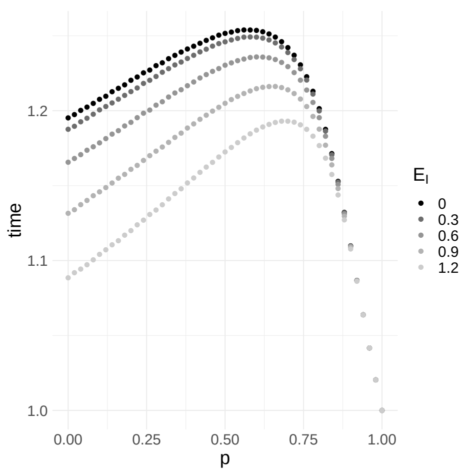
We plot the average checking time as a function of the notification probability p. Note that when $p \to 0$, the strategy converges to the no-push strategy, while when $p \to 1$ it converges to the genuine push strategy. Figure 3 shows that across a wide feasible parameter range, any interior $p \in (0, 1)$ leads to a higher checking time (lower checking frequency) than either genuine push or no push strategy, implying the consumer's optimal checking strategy is a corner solution.



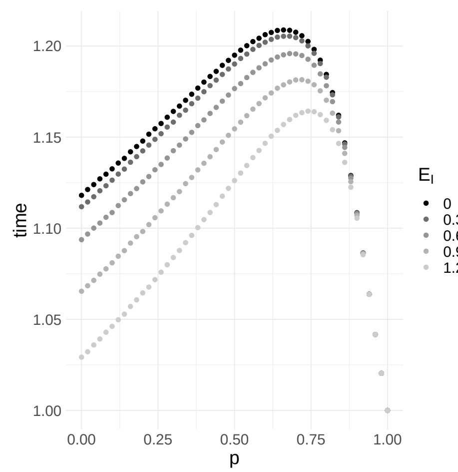
(a) $V_I = 5$



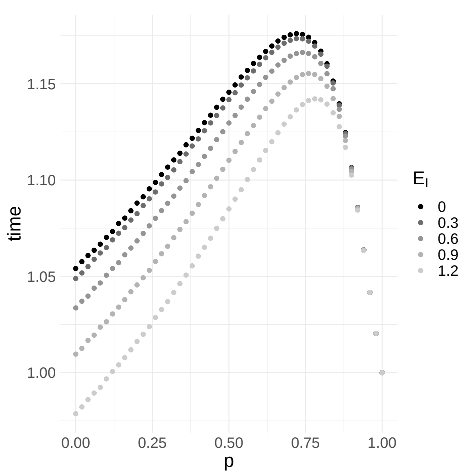
(b) $V_I = 6$



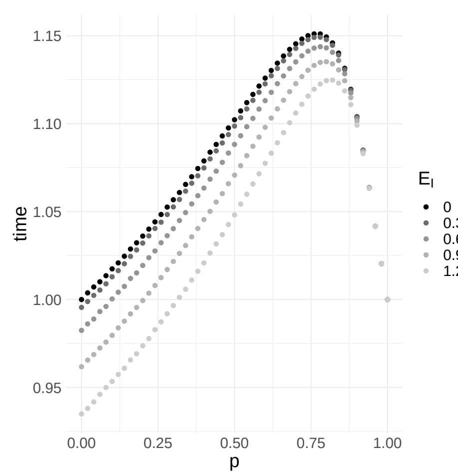
(c) $V_I = 7$



(d) $V_I = 8$



(e) $V_I = 9$



(f) $V_I = 10$

Note: Numerical example of average checking times for the muting strategy, parameters given by $\lambda = 1$, $\rho = 0.2$, and the consumer checking at every notification.

Figure 3: Numerical example of checking time under muting strategy

Example 2 We next examine the case in which the consumer checks every two notifications. After the last check, if there's no notification, the consumer waits until time τ_0 to check, and if there is one notification, the consumer waits until time τ_1 to check.

There are four cases in which a consumer checks. In the first case, when the consumer waits until $\tau = \tau_0$, the probability is given by $p_{N(\tau_m(0))}(0) = \frac{(p\lambda\tau_m(0))^0 e^{-p\lambda\tau_m(0)}}{0!} = e^{-p\lambda\tau_0}$. The expected flow utility in this case is given by

$$E_1[U(0,\tau_0)] = \frac{1}{\tau_0}[E_1[0,\tau_0] - 1 - \frac{\rho}{2}((1-p)\lambda(V_I + E_I^2)\tau_0)\tau_0]$$

The second case is when there is exactly one notification arrival before τ_1 and the consumer checks at $\tau = \tau_1$. The probability is given by $p_{N(\tau_m(1))}(1) = \frac{(p\lambda\tau_m(1))^1 e^{-p\lambda\tau_m(1)}}{1!} = p\lambda\tau_1 e^{-p\lambda\tau_1}$. The expected flow utility in this case is given by

$$E_1[U(1,\tau_1)] = \frac{1}{\tau_1}[E_1[1,\tau_1] - 1 - \frac{\rho}{2}((1-p)\lambda(V_I + E_I^2)\tau_1 + V_I)\tau_1]$$

The third case is when there is one notification arrival in $\tau \in (\tau_1, \tau_0)$, and the consumer checks immediately after the notification. The average utility is given by

$$\begin{aligned} E_2[U(1,\tau)] &= \frac{1}{\tau}[E_2[1,\tau] - 1 - \frac{\rho}{2}V_2[1,\tau]\tau] \\ &= \frac{1}{\tau}[(1 + (1-p)\lambda\tau)E_I - 1 - \frac{\rho}{2}((1-p)\lambda(V_I + E_I^2)\tau)\tau] \end{aligned}$$

The fourth case is when there are two notifications at $\tau < \tau_1$ and the consumer checks immediately at the second notification. The average utility is given by

$$\begin{aligned} E_2[U(2,\tau)] &= \frac{1}{\tau}[E_2[2,\tau] - 1 - \frac{\rho}{2}V_2[2,\tau]\tau] \\ &= \frac{1}{\tau}[(2 + (1-p)\lambda\tau)E_I - 1 - \frac{\rho}{2}((1-p)\lambda(V_I + E_I^2)\tau + V_I)\tau] \end{aligned}$$

The average time per check is given by $t^*(\tau_0,\tau_1) = \tau_0 e^{-p\lambda\tau_0} + \tau_1 p\lambda\tau_1 e^{-p\lambda\tau_1} + \int_{\tau_1}^{\tau_0} e^{-p\lambda\tau} p\lambda\tau d\tau + \int_0^{\tau_1} e^{-p\lambda\tau} p\lambda\tau p\lambda\tau d\tau = \tau_0 e^{-p\lambda\tau_0} + \tau_1 p\lambda\tau_1 e^{-p\lambda\tau_1} - e^{-p\lambda\tau_0}(\tau_0 + \frac{1}{p\lambda}) + e^{-p\lambda\tau_1}(\tau_1 + \frac{1}{p\lambda}) - e^{-p\lambda\tau_1}(\tau_1^2 p\lambda + 2\tau_1 + \frac{2}{p\lambda}) + \frac{2}{p\lambda}$.

The optimal τ_0, τ_1 solves the following problem:

$$\arg\max_{\tau_0,\tau_1} \frac{1}{t^*(\tau_0,\tau_1)} \begin{bmatrix} e^{-p\lambda\tau_0}\tau_0 E_1[U(0,\tau_0)] & +p\lambda\tau_1 e^{-p\lambda\tau_1}\tau_1 E_1[U(1,\tau_1)] \\ +\int_{\tau_1}^{\tau_0} e^{-p\lambda\tau} E_2[U(1,\tau)]p\lambda\tau d\tau & +\int_0^{\tau_1} p\lambda\tau e^{-p\lambda\tau} E_2[U(2,\tau)]p\lambda\tau d\tau \end{bmatrix}$$

We solve this optimization problem numerically by grid search using a similar procedure as in the first example. The feasible range of V_I is between $V_I \in (2, 5)$ (by comparing consumer checking every two notifications with checking every one or every three notifications). So we check all combinations of $V_I \in \{1.8, 2.3, 2.8, 3.3, 3.8, 4.3, 4.8\}$, $E_I \in \{0, 0.3, 0.6, 0.9, 1.2\}$, and $p \in (0, 1)$ by a step of 0.05. We solve for τ_0, τ_1 by searching the combination within the set of $\{\tau_0, \tau_1\} \in (0, 50)^2$ with a step of 0.01 in each dimension.

Figure 4 shows that there are cases (for example, $V_I = 4.8, E_I^2 = 0, \lambda = 1, \rho = 0.2$) where the optimal notification probability p is an interior solution (around $p = 0.9$).

H Proof of Proposition 3

The short-term self will want to check more frequently under push notification if the following conditions hold $V_I > n_p^* E_I^2$, and $\lambda \in \left(\rho(V_I + E_I^2)\frac{n_p^{*2}}{2}, \rho V_I \frac{n_p^*(n_p^*+1)}{2}\right)$, where $n_p^* = \lfloor \tilde{n}_p \rfloor$ and $\tilde{n}_p = \arg\min_{n \in \mathcal{R}_+} \frac{\lambda}{n} + \rho\frac{(n-1)}{2}V_I$.

The utility with no-push is given by

$$\mathrm{E}[U(t^*)] = E_I\lambda - \frac{1}{t^*} - \frac{1}{2}\rho(V_I + E_I^2)\lambda t^*$$

in which $t^* = \sqrt{\frac{2}{\rho(V_I + E_I^2)\lambda}}$

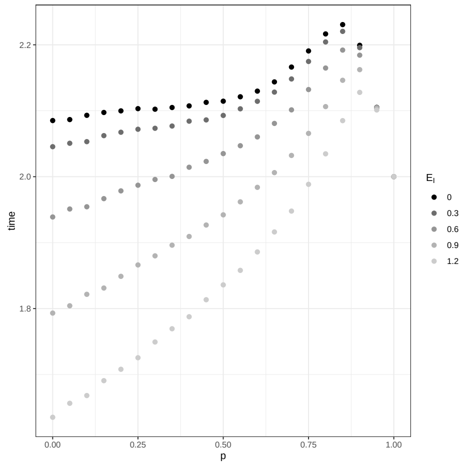
And the utility with genuine push is given by

$$\mathrm{E}[U(n_p^*)] = E_I\lambda - \frac{\lambda}{n_p^*} - \rho\frac{(n_p^*-1)}{2}V_I$$

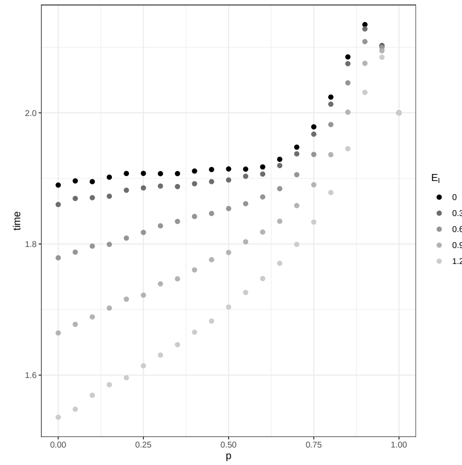
The long-term self takes the checking strategies (t^* and n_p^*) as given but evaluates the utilities using the parameter $\hat{\rho} < \rho$:

$$\mathrm{E}[\hat{U}(t^*)] = E_I\lambda - \frac{1}{t^*} - \frac{1}{2}\hat{\rho}(V_I + E_I^2)\lambda t^*$$

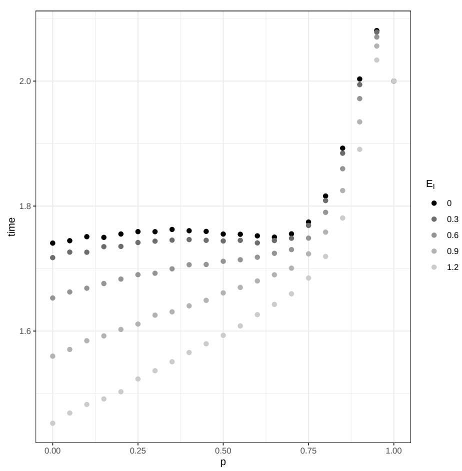
$$\mathrm{E}[\hat{U}(n_p^*)] = E_I\lambda - \frac{\lambda}{n_p^*} - \hat{\rho}\frac{(n_p^*-1)}{2}V_I$$



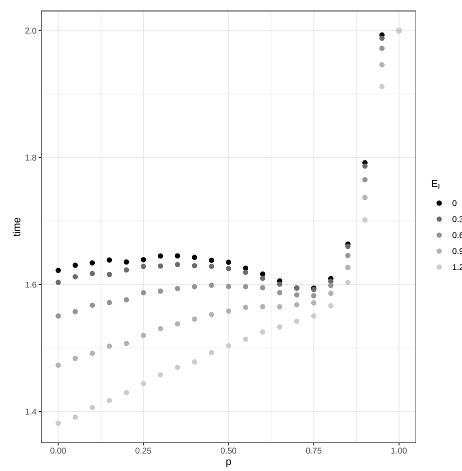
(a) $V_I = 2.3$



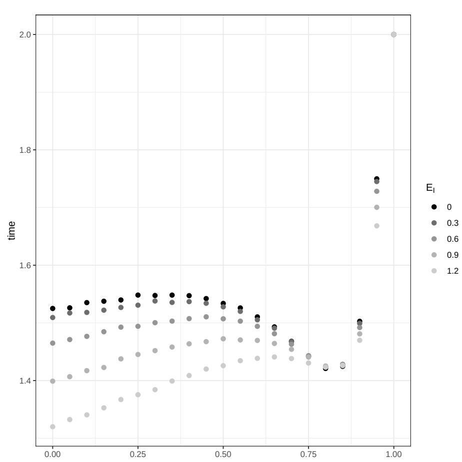
(b) $V_I = 2.8$



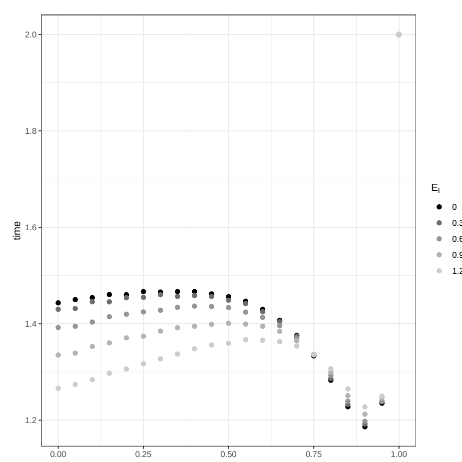
(c) $V_I = 3.3$



(d) $V_I = 3.8$



(e) $V_I = 4.3$



(f) $V_I = 4.8$

Note: Numerical example of average checking time for the muting strategy, parameters given by $\lambda = 1$, $\rho = 0.2$, and the consumer checking at every other notification when $p \to 1$.

Figure 4: Numerical example 2 of checking time under muting strategy

The long-term self prefers no-push (i.e., to block push) if $\mathrm{E}[\hat{U}(t^*)] > \mathrm{E}[\hat{U}(n_p^*)]$. Note that both $\mathrm{E}[\hat{U}(t^*)]$ and $\mathrm{E}[\hat{U}(n_p^*)]$ are linear in $\hat{\rho}$. Because we know that in the parameter range specified above, the consumer checks more frequently under push, we have $\frac{1}{t^*} < \frac{\lambda}{n_p^*}$. When $\hat{\rho} = 0$, we have $\mathrm{E}[\hat{U}(t^*)] > \mathrm{E}[\hat{U}(n^*)]$. At $\hat{\rho} = \rho$, by Corollary 1, we know that $\mathrm{E}[\hat{U}(t^*)] < \mathrm{E}[\hat{U}(n^*)]$. Given that both terms are decreasing linearly in $\hat{\rho}$, there must be a unique $\hat{\rho^*} \in (0, \rho)$, such that $\mathrm{E}[\hat{U}(t^*)] > \mathrm{E}[\hat{U}(n^*)]$ when $\hat{\rho} \in (0, \hat{\rho^*})$ and $\mathrm{E}[\hat{U}(t^*)] < \mathrm{E}[\hat{U}(n^*)]$ when $\hat{\rho} \in (\hat{\rho^*}, \rho)$. The threshold $\hat{\rho^*}$ can be calculated as:

$$
\begin{aligned}
\hat{\rho^*} &= \left[\frac{1}{2}(V_I + E_I^2)\lambda t^* - \frac{(n^*-1)}{2}V_I\right]^{-1}\left(\frac{\lambda}{n^*} - \frac{1}{t^*}\right) \\
&= \left[\sqrt{\frac{(V_I + E_I^2)\lambda}{2\rho}} - \frac{(n^*-1)}{2}V_I\right]^{-1}\left(\frac{\lambda}{n^*} - \sqrt{\frac{\rho(V_I + E_I^2)\lambda}{2}}\right)
\end{aligned}
$$

The first term is the inverse of the difference in average flow disutility between push and no-push strategies. The second term is the difference in average checking costs. At $\hat{\rho^*}$, the two differences cancel out, and the long-term self is indifferent from blocking or not.

In terms of comparative statics, locally we may treat n^* as a constant. Clearly $\hat{\rho^*}$ is decreasing in E_I^2 because both terms are decreasing in E_I^2. It implies that the long-term self is less likely to block if the mean utility is high (and implying the anxiety cost is high under the no-push strategy). Similarly, $\hat{\rho^*}$ is decreasing in V_I when n^* is sufficiently small: $\sqrt{\frac{\lambda}{2(V_I + E_I^2)\rho}} + 1 > n^*$. For example, one sufficient condition is $n^* = 1$. $\hat{\rho^*}$ is decreasing in λ when n^* is sufficiently large: $n^* > 2\sqrt{\frac{2\lambda}{(V_I + E_I^2)\rho}}$.

I Proof of Proposition 4

With subscription fee, under noisy push with noise level k ($k \leq k^*$), the profit is given by the following:

$$
\begin{aligned}
\pi_{noisy} &= E[U(n,k)] + \frac{(k+1)\pi_c\lambda}{n} \\
&= \lambda E_I - \frac{(k+1)\lambda}{n} - \frac{\rho(n-1)}{2(k+1)}\left(V_I + E_I^2\frac{k}{1+k}\right) + \frac{(k+1)\pi_c\lambda}{n}
\end{aligned}
$$

When $k = 0$ we have the profit under genuine push strategy.

Taking the derivative with respect to k, we have that

$$\frac{\partial \pi_{noisy}}{\partial k} = \frac{(\pi_c - 1)\lambda}{n} + \frac{\rho(n-1)}{2(k+1)^2}(V_I - E_I^2 \frac{k-1}{k+1})$$

Clearly $V_I + E_I^2 \frac{k-1}{k+1} > 0$ because $V_I > E_I^2$. If $\pi_c - 1 > 0$, then $\frac{\partial \pi_{noisy}}{\partial k} > 0$. The firm would prefer to increase k as long as $k \leq k^*$. If $\pi_c - 1 < 0$, there may exist an interior $k \leq k^*$ such that $\frac{\partial \pi_{noisy}}{\partial k} = 0$.

Notice that at $k = 0$, we have

$$\begin{aligned}
\frac{\partial \pi_{noisy}}{\partial k}|_{k=0} =& \frac{(\pi_c - 1)\lambda}{n} + \frac{\rho(n-1)}{2}(V_I - E_I^2) \\
>& \frac{(\pi_c - 1)}{n} \frac{\rho V_I n(n-1)}{2} + \frac{\rho(n-1)}{2}(V_I - E_I^2) \\
=& \frac{\rho(n-1)}{2}[\pi_c V_I - E_I^2]
\end{aligned}$$

and as long as $\pi_c > \frac{E_I^2}{V_I}$, $\frac{\partial \pi_{noisy}}{\partial k}|_{k=0} > 0$, implying the optimal level of noise is strictly positive.

The special case is when consumer checks every notification ($n = 1$). We have $\frac{\partial \pi_{noisy}}{\partial k} = (\pi_c - 1)$. Therefore, the optimal level of noise is either no-push ($k_p^* = 0$) if $\pi_c < 1$ and optimal noisy push strategy (i.e. $k_p^* = k^*$) if $\pi_c > 1$.