Multi-market Value Creation and Competition*

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ABSTRACT

We analyze multi-market interactions between firms which must invest limited budgets in value (surplus) creation as well as in competitive rent-seeking activities. Firms are horizontally differentiated on a line segment and compete for multiple markets/prizes which differ in the relative effectiveness of each firm's competitive rent-seeking spending. Each firm faces a dual trade-off: First they must choose how much to invest in value creation versus to spend in rent-seeking competition. Second, they must decide on how to allocate resources across the different markets. When the market values are exogenous (and identical across markets) the intensity of competition is highest for the market in the middle, rather than in (advantaged) markets which are close or in (disadvantaged) markets which are closer to the rival. Counter to what one would expect, greater firm differentiation actually intensifies the competition in the middle markets. When firms endogenously invest in value creation, they invest more in value creation in closer markets and the investments decline towards the middle. This results in the most intense competition moving away from the middle to a market in each firm's turf. The analysis also provides a competitive perspective on the home turf bias phenomenon.

1 Introduction

Many important business, economic and political contexts involve rivals who must compete by allocating limited resources simultaneously across multiple markets or prizes. Not only must they decide how much to invest in building different markets, but also the extent to which they must compete in each market. Consider the following examples:

- Drug companies spent an estimated $5.2 billion in 2015 in direct to consumer advertising. For new drugs such advertising may induce patients to ask their doctors about their suitability and thus potentially expand the market. The pharmaceutical industry deploys even larger amounts (e.g., $15 billion in 2012) to conduct detailing and promotional activities to doctors across different geographical markets. Detailing activities by medical representatives not only provides information to doctors about the basic drug action, but also involve efforts to persuade doctors to prescribe the firm's drugs over those of rivals.

- In the mobile phone market leading firms like Samsung and HTC invest in promoting the Android platform to convince consumers to adopt the platform over the iPhone. However they also compete for market share in large Asian markets. HTC's advertising campaign is more effective in its home market in Taiwan than in South Korea, and vice versa for Samsung. Advertising campaigns in this product category can work to increase the generic demand for the product category, or it can persuade consumers about the advantages of a firm's product over its rival(s). Samsung and HTC would have to decide on allocating advertising budgets based on the relative preference for their product in each market.

These examples represent some general aspects of competitive interactions in a variety of contexts: First the players/firms have limited resources (advertising budgets, sales-force size) and they compete in multiple markets. This means that they have to decide on how much of the resource to allocate to each market, resulting in the decisions across markets to be affiliated. Allocating more to one market means less to others. Second, the markets can be differentiated (as in the mobile phone market) with each firm having home markets with relatively higher consumer preference. Should firms deploy more or less resources in markets where they are stronger?

Second, the examples also highlight a basic business strategy trade-off: Firms have to choose between surplus or value creation in each market versus competing for the value. In other words, the pie that firms will fight for is in itself endogenous. Further, a firm's investment in creating value

in a market can be subject to free-riding by the competitor who can deploy competitive resources to win the market. For example, pharmaceutical reps have to decide how much to focus their efforts on providing information about the basic drug action versus on persuading doctors that their drug is relatively superior. Efforts to inform doctors about the basic drug action may also end up benefiting competitors in a category. Similarly, direct to consumer advertising for a new class of drugs by one firm can expand the potential market benefiting all the firms in the category. This feature is also related to the classic guns vs. butter trade-off described in the conflict literature: i.e., in competitive markets economic agents may face a trade-off between investing in producing goods of value versus investing to appropriate the value created by other agents (Hirshleifer 1988). Except that in this paper we examine the trade-off in the context of multi-market interactions between players with limited resources.

We construct a framework to analyze multi-market value creation and the competition for that value. The anatomy of the game is as follows: Two firms (players) with limited resources compete simultaneously for different markets which are located evenly on a unit line. Firms are located at the ends of the line and are differentiated and each firm's relative strength in a market depends upon the distance between the firm's location and that of the market: the further a market is from a firm's location the less effective is the firm's competitive rent-seeking spending. Firms simultaneously choose an allocation of their resource endowment among the markets in order to maximize their expected overall payoffs. In each market they simultaneously choose the investment that will determine the size of the market (value or surplus creation) as well as the competitive outlay to win the market from the rival (competitive rent seeking). Firms' outlays in surplus creation are substitutable and they jointly determine the size of the value pie. This then allows for the possibility that the investments in surplus creation by one firm is subject to free-riding by the other. In each market firms' competitive spending jointly determine the winner through a Tullock contest success function (Tullock, 1980). What would the equilibrium allocations be for the players be in terms of the surplus creation and the rent-seeking allocations across the different markets?

Consider first the case where each market has the same size which is exogenously fixed which implies that firms face only the competitive rent-seeking incentive across the markets. Should a firm defend closer markets in its home turf, or should it spend resources to win markets which are farther away and harder to win? We find that each firm's equilibrium resource distribution has a non-monotonic inverted U-shape profile: Each firm spends relatively less resources both in closer markets and in markets which are closer to its competitor. The firms' outlays peak at a

market in the middle which implies that the competition will be the most intense for the middle market. It is particularly interesting and counter to intuition that greater market differentiation leads to a more concentrated resource distribution profile with even more intense competition in the middle markets. These results are consistent with the empirical studies of electoral competition in U.S. presidential elections (Stromberg 2008, Gordon and Hartmann 2016) which show evidence of greater advertising spending in the most competitive markets.

We then analyze the general case in which market value is endogenously determined. Firms' choose not only the allocation across the markets, but also how to split the spending in each market between investments in market creation and in competitive rent-seeking: The former builds the market value or surplus, while the latter allows a firm to compete for the value that is created. Consider the case where the efficiency of the productive investment of a firm is the same across all markets which implies that firms do not have any home turf advantage in more proximate markets. With substitutable value creation efforts, the equilibrium market values are polarized: The home turfs of both firms, i.e., the markets closest to the firm locations have the highest equilibrium investments and markets which are closer to the middle have lower values. The firms' allocations of competitive rent-seeking outlays are different from the pattern in the exogenous market size case. Firms' competitive spending no longer peak in the middle, but rather in each firm's turf. Firms do not compete most intensely in the most valuable markets, rather the intensity of competition is determined by the trade-off between the equilibrium size of the markets and the ease with which the markets are contestable.

Greater market differentiation leads firms to invest relatively more in their home turfs at the expense of the middle markets and to reduce the overall amount of competitive spending. With asymmetric budgets, the firm with the budget advantage balances its equilibrium actions such that it invests more in value creation and also deploys more in competitive spending. This result contrasts with what might be obtained in a single market value creation and competition model: In that case, it is possible that the firm with the lower budget may deploy greater competitive spending even as the firm with the budget advantage invests more in value creation. Finally, as the budget asymmetry increases, the firm with the advantage invests in more markets closer to the weaker firm.

Our results also provide a competitive perspective on the effects home market advantage, i.e., a systematic preference of consumers to purchase local products. The interesting question is when does this lead to a home turf bias in the equilibrium spending or investments by firms. When the

market values are exogenous, the spending in competitive rent seeking is actually higher in the middle markets rather than in the home turfs of the firms. In contrast, with endogenous market values, the value creation investments of the firms are higher in their home turfs even if all markets have the same productivity/effectiveness in value creation. Thus we have the interesting point that even in the absence of any home turf advantage in value creation, the presence of differentiation in competitive rent-seeking can lead to a home turf bias in the equilibrium value creation. When the firms do have an home turf advantage, then predictably this leads to greater equilibrium investments in closer markets.

2 Related Research

This paper is related to the guns vs. butter literature initiated by Hirshleifer (1988). In that literature rival agents can acquire surplus either by producing goods or by appropriating the output produced by others. Therefore, they must strategically allocate their resource endowment between output creating investments (butter) or appropriative technologies (guns) which helps to seize the output of others or defend one's own output. Our paper examines the resource allocation choice of players between value creating investments and competitive rent-seeking actions. This can be seen as the incentives and the trade-offs faced by firms to collaborate in joint production of value and to compete for that value at the same time. Several papers, e.g., Skaperdas (1992) and Hirshleifer (1994), have been developed along this line to investigate how firms' resource position might affect their allocation strategies (see Garfinkel and Skaperdas (2007) for a review). Our analysis contributes by considering value creation and competition in multi-market interactions and therefore jointly considers dual trade-offs: the allocation of resources between value creation and competition and the allocation of resources to different markets. In marketing, Bass et al. (2005) analyze a form of the guns vs. butter trade-off in single market over time in model of dynamic competition in generic and brand advertising. Amaldoss et al. (2000) present a different type trade-off in the context of R&D alliances: Firms in an alliance jointly invest to develop products of higher values in order to compete against rival alliances.

Our analysis is also related to the contest literature and can be seen as new form of multi-dimensional proportional prize contest in which firms allocate resources over a set of differentiated markets. This enriches and qualitatively generalizes the framework of the Colonel Blotto game of duopoly conflicts in multiple battlefields in which firms allocate their resources among these

battlefields to maximize the sum of rents. The game was first proposed by Borel (1921) and analyzed by Borel and Ville (1938) in a special case of three markets.[1] We highlight the strategic effects of an important aspect that is missing in the literature: The Colonel Blotto game assumes a zero-sum payoff structure. Our analysis obviously considers both the creation of the pie as well as competition for it.[2] To our knowledge, the existing literature on Colonel Blotto games has not considered a game which incorporates the trade-off between market creation and rent-seeking as well as firms' strategic interaction in multiple differentiated markets. Within the standard Colonel Blotto class of games allowing for endogenous market creation as well as allocation across multiple differentiated contests is analytically challenging. The game form developed in the paper contributes by providing a tractable proportional prize setup to analyze the guns vs. butter decisions in multi-market contests.

The paper is also related to a growing literature that applies contest/tournament type models of competition to marketing and I.O. issues, such as R&D and product development, and sale force allocation and incentives. The application of contest-like models in marketing goes back to the attraction models literature originating in Bell et al. (1975) which deals with marketing and promotional effort competition for market shares. Recent work in marketing strategy include Ridlon and Shin (2013) who examine whether a firm should favor weaker employees in an attempt to maximize the total sales effort output in a repeated contest model. Iyer and Katona (2017) analyze competition as a contest for consumer attention in social communication markets, while Katona et al. (2017) model a contest between news providers who can strategically choose news topics. There is also a literature on sales contests which focuses on the optimal design of the prize structure to elicit sales agents efforts (see Kalra and Shi (2001) and Lim, Ahearne, and Ham (2009). Finally, Amaldoss and Staelin (2010), and Chen and Lim (2013) study contests between teams/alliances instead of between individual players.

[1] Gross and Wagner (1950) generalize the analysis to a finite number of markets. A number of studies apply the model to various contexts, including campaign financing allocations (Lake, 1978), advertising (Friedman, 1958), and military defense (Clark and Konrad, 2007, and Kovenock and Roberson, 2009), etc.

[2] Kvasov (2007) and Roberson and Kvasov (2012) relax the usual assumption that resources are forfeited if they were not used for rent-seeking competitions.

3 The Model

Consider two firms/players indexed by $i = 1, 2$, that are located at the two ends of a unit line segment with firm 1 located at zero (left edge) and the other firm at location 1 (right edge). Each firm is endowed with a fixed competitive resource budget m_i. Without loss of generality, assume that $m_1 \geq m_2 > 0$. Suppose that the line segment has a set of $2n+1$ markets (or prizes) which are equally spaced and indexed by $k = 1, \ldots, 2n+1$. Market $k = 1$ is at firm 1's location, while the market $k = 2n+1$ is at firm 2's location. Each of these k's could represent consumer market for a product, or an electoral market in a political contest, or different R&D projects that firms may invest in.

Firms utilize their endowment by simultaneously choosing the amount b_{ik} to invest in surplus creation in market k as well as the amount x_{ik} with which to compete for the market. The decisions b_{ik} can represent investments in awareness advertising or marketing activity to build primary/product category demand. For example in pharmaceutical markets several studies have established the role of direct to consumer advertising (DCTA) in informing the market about the basic drug or about increasing patient visits to doctors thereby expanding primary demand (e.g., Berndt et al. 1995, Iizuka and Jin 2005 or Liu and Gupta 2011). The decisions x_{ik} represent competitive or rent-seeking activities that are directed at winning the market from the rival. These could include comparative advertising or competitive promotional spending to convince consumers to buy from a firm rather than from the rival.

3.1 Exogenous Market Values

We begin with a basic analysis of competition between the firms when each market or prize has a fixed exogenous value $v > 0$. This means that each firm would choose a competitive allocation strategy and firm i's allocation strategy can be represented by a vector $\mathbf{x}_i = (x_{i,1}, \ldots, x_{i,2n+1})$, subject to its budget constraints, i.e., $\sum_{k=1}^{2n+1} x_{i,k} \leq m_i$. The effectiveness of a firm's competitive allocation in a market depends on the distance between its own location and the targeted market. For an arbitrary market k given the firms' choice of competitive outlays $x_{i,k}$ effective outlays $y_{i,k}$ are given by:

$$y_{1,k} = [1 - \frac{t(k-1)}{2n+1}]x^r_{1,k}, \text{ and,} \tag{1}$$

$$y_{2,k} = [1 - \frac{t(2n+1-k)}{2n+1}]x^r_{2,k}. \tag{2}$$

Thus a firm's competitive outlay is relatively more effective in a market that is closer to it than to its rival. The effectiveness of a firm's spending depends upon the distance between the firm and the market, and $t \in (0, 1]$ measures the effectiveness loss caused by distance. In other words, it measures the extent of differentiation between the firms in their ability to compete for the different markets. Therefore in this competition each firm is favored in its own turf because its spending has greater relative effectiveness. Thus markets $\{1, \ldots, n\}$ are the "turf" of firm 1, and markets $\{n+2, \ldots, 2n+1\}$ those of firm 2.

In each market k the outcome of the competition is determined by a Tullock contest success function: Each firm i wins a proportion of the market:

$$p_{i,k} = \frac{y_{i,k}}{y_{1,k} + y_{2,k}}. \tag{3}$$

Note that ties are broken fairly if both firms place zero outlays in any market, i.e., each firm secures half of the market. As in the marketing literature based on contest like models (see, for instance, Bell et al., 1975), the function $p_{i,k}$ can be interpreted as a share of market value firm i secures from market k. The allocation decisions can be seen as determining the market shares in each market in a proportional and smooth manner.[3] We assume that the power term $r \in (0, 1)$ which implies the effective outlays y_i are a concave function which captures the standard idea of decreasing returns to additional marketing investments. Note also that this assumption of $r \leq 1$ ensures a pure strategy equilibrium of the proportional prize contest interpreted as the competition for market share in our model (Gradstein and Konrad 1999, and Konrad 2009).[4]

Each firm chooses its allocation strategy to maximize its aggregate payoff from all the markets $\pi_i(\mathbf{x}_i; \mathbf{x}_j) = \sum_{k=1}^{2n+1} \frac{y_{i,k}}{y_{1,k}+y_{2,k}} v$. For a given allocation strategy $\mathbf{x}_j$ by its rival, a firm i will solve a

[3] Alternatively, the function $p_{i,k}$ can also be interpreted as a firm's probability of winning the entire market k in a winner take all competition, as formulated in Tullock (1980). Konrad (2009) and Sheremeta et. al (2018) provide additional discussion of these two standard ways to interpret the contest model.

[4] More generally a unique pure-strategy equilibrium exists in a N player contest when $r \leq \frac{N}{N-1}$, while a unique mixed-strategy equilibrium with continuously distributed spending/bids exists when $r \to \infty$ resulting in an all-pay auction like outcome, in which case an infinitesimal amount of over-bidding by a player leads to a sure win. There is a limited characterization of the equilibrium behavior when r takes intermediate values: For $0 < r \leq 2$ with two firms there is a pure strategy Nash equilibrium, but for $r > 2$ Baye et al. (1994) show the existence of only a mixed-strategy equilibrium that fully dissipates the rent. There are also no general results if we change the contest structure. For example, Fu and Lu (2012) show that the pure strategy equilibrium breaks down when $r > 1$ in a multi-stage sequential elimination contest.

constrained maximization problem given by

$$\max_{\mathbf{x}_i} \pi_i(\mathbf{x}_i; \mathbf{x}_j)$$
$$s.t. \sum_{k=1}^{2n+1} x_{i,k} \leq m_i$$
$$x_{i,k} \geq 0, \forall k \in \{1, \ldots, 2n+1\}.$$

The firm's allocation problem to maximize payoffs embeds some important trade-offs: First given that the resource endowment is limited, investing more in any given market necessarily means reducing the allocation for one or more of the other markets. Second, the market differentiation represented by t implies that each firm has to decide how much to invest in its home turf versus attack its rival's turf. We establish the unique pure strategy equilibrium of this multi-market game. Because unused budgets do not have any outside option value each firm will exhaust its budget and $\sum_{k=1}^{2n+1} x_{i,k} = m_i$. In section 3.2.1 we will consider the role of outside options. The first step in identifying the equilibrium is the following Lemma:

Lemma 1 *There exists no equilibrium in which a firm places zero outlay in any market, i.e.,* $x_{i,k} > 0$, $\forall k \in \{1, \ldots, 2n+1\}$.

Suppose there were to exist a market where both firms did not allocate any resources, then it will be optimal for one of the firms to shift an infinitesimal amount of resource from elsewhere to this market. Doing so would provide the firm with an incremental payoff of v while having a negligible effect on the payoff of the firm from the alternative market. Similarly, if only one firm were to not allocate any resources in a given market, the other firm would want to reduce its outlay to be negligibly small. Thus in equilibrium both firms compete for each one of the available markets by deploying positive resource allocations.

We now proceed to describe the interior equilibrium of the game in which $x_{i,k} > 0$, $\forall k \in \{1, \ldots, 2n+1\}$. Define $\lambda = \frac{m_2}{m_1}$. The following proposition characterizes the equilibrium:

Proposition 1 *There exists a unique pure strategy Nash equilibrium in the game with fixed market values. In each market k, firms' allocate $x_{1,k} = \frac{m_1 \phi(k)}{\sum_{k=1}^{2n+1} \phi(k)}$ and $x_{2,k} = \frac{m_2 \phi(k)}{\sum_{k=1}^{2n+1} \phi(k)}$, where $\phi(k)$ is given by*

$$\phi(k) = \frac{\left\{ r[1 - \frac{t(k-1)}{2n+1}][1 - \frac{t(2n+1-k)}{2n+1}]\lambda^r \right\}}{\left\{ [1 - \frac{t(k-1)}{2n+1}] + [1 - \frac{t(2n+1-k)}{2n+1}]\lambda^r \right\}^2}.$$

Firms spend the same proportion $\phi(k) \big/ \sum_{k=1}^{2n+1} \phi(k)$, of their resource budget in each market. Hence, the ratio between firms' bids remains constant across markets, i.e., $\frac{x_{2,k}}{x_{1,k}} = \lambda = \frac{m_2}{m_1}$. One might think that a firm can have lower incentives to spend in a market that is farther away and in the rival's home turf. But then it is also the case that it is easier for the rival to defend a closer market because its spending in such a market is more efficient. The rival strategically lowers its spending in its home turf. These opposing incentives cancel out leading firms to choose a constant ratio of resource allocation across markets.[5]

The main point of interest in this proposition is the manner in which the firms split their resource budgets among the $2n+1$ markets. Define $\tilde{\phi}(k) \triangleq \phi(k|\,t) \big/ \sum_{k=1}^{2n+1} \phi(k|\,t)$, which is the portion of resource each firm allocates in equilibrium to a market k. In the following proposition, we explore the properties of the function $\tilde{\phi}(k)$ with respect to k.

Proposition 2 *i) The function $\tilde{\phi}(k)$ first increases with k and peaks at a cutoff $k^* \geq n+1$. It then decreases if $k^* < 2n+1$. ii) When firms are symmetric ($m_1 = m_2$), the peak k^* is located at the middle market, i.e., $k^* = n+1$; when firms are asymmetric, i.e., $m_1 > m_2$, the peak k^* is located right to the mid-point, i.e., $k^* > n+1$.*

The resource allocation function $\tilde{\phi}(k)$ is non-monotonic in k. Consider the case of symmetric firms ($m_1 = m_2$). Each firm's allocation increases as it moves towards the center on its home turf peaking at the market $n+1$ in the middle and so the firms compete most intensely in the middle of the market. In markets close to its location a firm strategically withdraws and allocates less because its spending is more effective. Whereas it also allocates less in markets that are farther away and in the competitor's turf precisely because its spending is relatively less effective. This leads to an inverted U-shaped equilibrium resource allocation profile with the maximum allocation by both firms in the middle.

The above result can be related to empirical studies in political markets. In U.S. presidential elections rival candidates must decide how to allocate their advertising budgets and their campaigning time across electoral markets which are differentiated according to Democratic and Republican preferences. Stromberg (2008) shows that presidential candidates in the 2000 and 2004 elections do not allocate their visits to states simply based on the size of the state, but rather based on the

[5] The equilibrium specified in Proposition 1 remains the same even in a sequential move game in which one firm first commits to its spending allocation and then the rival firm chooses its allocation strategy upon observing the first mover's strategy. In other words, the possibility of commitment by one of the firms does not change the equilibrium incentives. See the Appendix for details.

relative degree of competitiveness (closeness) of the race, with closer races getting greater allocations. More recently, Gordon and Hartmann (2016) analyze multi-market advertising competition in the 2000 and 2004 presidential elections to show evidence of greater concentration of spending in the most competitive/battleground markets relative to the non-battleground markets both under the electoral college rules and in a proportional direct vote counterfactual case.

When the firms are asymmetric and $m_1 > m_2$, the spending peak, k^*, shifts to firm 2's turf. The most intense competition takes place in a market in the weaker firm 2's turf and the greater the asymmetry between the firms the closer is the peak to firm 2's location. The general result in the contest literature is that a more balanced playing field leads to more competition. In any given market k, the balance in the playing field depends on a) the effectiveness of the firms' spending which is determined by the distance between the market k and each firm, and b) the firms' budgets. Symmetric firms have an advantage in their own turfs, and therefore the highest spending and competition occurs at the mid point. When they are asymmetric, the most intense competition occurs in a market which is in firm 2's turf, where the closer distance of the market to firm 2 can counteract the disadvantage of its smaller budget.

Effect of Market Differentiation and Other Comparative Statics

How does the extent of differentiation of the market affect the equilibrium strategies? We turn our attention to this question in the next proposition:

Proposition 3 *i) For symmetric budgets, the ratio* $\frac{\tilde{\phi}(\{k+1\}|t)}{\tilde{\phi}(\{k\}|t)}$ *strictly increases with t for* $k < n+1$*, while it strictly decreases for* $k \geq n+1$*, i.e., competitive rent-seeking resources are increasingly spent in the markets closer to the middle. ii) For asymmetric budgets,* $\frac{\partial k^*}{\partial t} \leq 0$*, the spending peak* k^* *shifts towards the left side of the line.*

Note that $\frac{\tilde{\phi}(\{k+1\}|t)}{\tilde{\phi}(\{k\}|t)}$ is the ratio of spending on market $k+1$ as compared to market k and this ratio strictly increases with market differentiation for $k < n+1$, while it strictly decreases for $k \geq n+1$. As t increases, distance of a market from a firm causes a greater attenuation of the firm's spending effectiveness and so each firm gets a greater advantage in markets which are in its own turf. This reduces a firm's spending in more remote markets and therefore provides incentives for the rival to also spend less in protecting closer markets. Thus we get the unexpected result that greater t leads to increasingly intense competition in marginal markets which are in the middle. This result is noteworthy precisely because an increase in t is equivalent to the market being

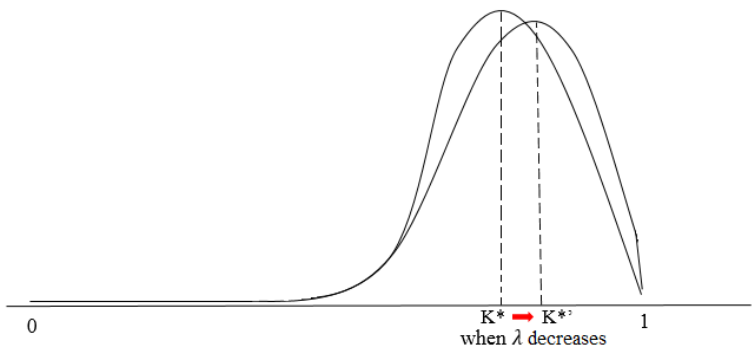

Figure 1: Distribution of Competitive Spendings

more differentiated between the firms. And greater differentiation leads firms to compete more in the middle, rather than in closer markets where they have a natural advantage. The direct vote counterfactual analysis in Gordon and Hartmann (2016) indicate that in the 2000 presidential election the overall national vote margin between the candidates was much smaller when compared to that in the 2004 election suggesting that the election was more competitive (less differentiated) in 2000. And the relative spending by the candidates in the non-battleground states was higher in 2000 and this can be seen as being consistent with the result in Proposition 3 that greater differentiation leads to more concentration of spending in the middle markets.

When firms have asymmetric budgets the location of the spending peak k^* depends on several factors, such as the extent of market differentiation, the asymmetry between their budgets (λ), and the competitive technology. It is interesting to note that $\frac{\partial k^*}{\partial t} \leq 0$: Increasing differentiation leads firms to spend more on the markets on the left: Firm 1 "retreats" as t increases and concentrates more spending in closer markets. Therefore, the market with the most intense competition moves closer to Firm 1 which has the budget advantage. Greater differentiation reduces Firm 1's spending effectiveness in markets which are farther away.

We can also show that $\frac{\partial k^*}{\partial \lambda} \leq 0$. In other words as the asymmetry between the firms increases (i.e., $\lambda = \frac{m_2}{m_1}$ decreases) the spending peak k^* is pushed rightward into Firm 2's turf and firms increasingly concentrate their resources to compete in markets which are on the right. Intuitively, the weaker firm (Firm 2) is induced to focus its limited budget on closer markets. This allows

Firm 1 to divert its spending from its home turf and move them to markets closer to Firm 2. Thus Firm 2 substitutes for its lack of resources by the effectiveness of its spending in closer markets. Figure 1 illustrates the comparative statics. Finally, increases of r leads k^* to move rightward: As r increases the competition for the markets becomes more discriminatory and this magnifies Firm 2's disadvantage forcing it to focus more on closer markets.

3.2 Endogenous Budget Choices

We now extend the basic model by allowing firms to endogenously choose their budgets and therefore the amount of spending they would want to deploy in the market competition. The incentive of firms to choose budgets can be represented in two important ways: First, firms might have an alternative use or outside option for their budget endowments m_i. In this case firms must decide how much of their budget endowment to deploy in the market versus on the outside opportunity. Second, firms may choose the amount of budget resources to deploy given that they have increasing costs. Both these possibilities are examined below.

3.2.1 Budget Choice with Outside Options

Suppose that each firm has a potential alternative use for its budget: Specifically, each firm i can invest its endowment in a numeraire good $x_{i,0}$, which has unit marginal utility. As before firm 2 has a (weakly) larger endowment, $m_2 \geq m_1$. Each firm's strategy is given by a vector $\mathbf{x}_i = (x_{i,0}, x_{i,1}, \ldots, x_{i,2n+1})$ and the following maximization problem:

$$\max_{\mathbf{x}_i} u_i(\mathbf{x}_i; \mathbf{x}_j) = \pi_i(\mathbf{x}_i; \mathbf{x}_j) + x_{i,0}$$
$$s.t. \sum_{k=0}^{2n+1} x_{ik} \leq m_i$$
$$x_{i,k} \geq 0, \forall k \in \{1, \ldots, 2n+1\},$$

with $\pi_i(\mathbf{x}_i; \mathbf{x}_j) = \sum_{k=1}^{2n+1} \frac{y_{i,k}}{y_{1,k}+y_{2,k}} v$.

As before in equilibrium each firm will allocate its budget across the markets $k \in \{1, \ldots, 2n+1\}$ such that they yield the same level of marginal utility. Obviously, there exists no equilibrium in which either of the firms invests zero resources in competitive rent-seeking activities. Define $\tilde{m}_i = \sum_{k=1}^{2n+1} x_{i,k}$, i.e., a firm i's endogenously determined total spending on competition across all

markets. Further, define

$$\tau \equiv \tau(\tilde{m}_i|\,\tilde{m}_1,\tilde{m}_2) = \frac{\sum_{k=1}^{2n+1} \frac{\left\{r[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}](\frac{\tilde{m}_2}{\tilde{m}_1})^r\right\}}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}](\frac{\tilde{m}_2}{\tilde{m}_1})^r\right\}^2}}{\tilde{m}_i}.$$

Proposition 4 *The following characterizes the equilibrium of the game with outside options:*

i. (Corner Equilibrium) When $\frac{1}{v} \leq \tau(\tilde{m}_2|\,\tilde{m}_1,\tilde{m}_2)|_{\tilde{m}_1=m_1,\tilde{m}_2=m_2}$*, both firms deploy their entire budget on market competition, i.e.,* $x^*_{i,0} = 0$ *and* $\tilde{m}^*_i = m_i$*.*

ii. (Interior Equilibrium) When the condition $\tau(\tilde{m}_1|\,\tilde{m}_1,\tilde{m}_2)|_{\tilde{m}_1=\tilde{m}_2=m_1} < \frac{1}{v}$*, then neither firm's invests all of its budget in market competition. There exists a unique* $\bar{m}^*$*, which satisfies* $\frac{1}{v} = \frac{\sum_{k=1}^{2n+1} \frac{\left\{r[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}]\right\}}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}^2}}{\bar{m}^*}$*. Each firm spends* $x^*_{i0} = m_i - \bar{m}^*$ *on the numeraire good, and* $\tilde{m}^*_i = \bar{m}^*$ *on market competition.*

iii. (Hybrid Equilibrium) When $m_2 > m_1$ *and the condition* $\tau(\tilde{m}_2|\,\tilde{m}_1,\tilde{m}_2)|_{\tilde{m}_1=m_1,\tilde{m}_2=m_2} < \frac{1}{v} \leq \tau(\tilde{m}_1|\,\tilde{m}_1,\tilde{m}_2)|_{\tilde{m}_1=\tilde{m}_2=m_1}$ *holds, firm 1 exhausts its budget on rent-seeking activities, while firm 2 does not. There exists a unique* $\tilde{m}^*_2$*, which satisfies* $\frac{1}{v} = \frac{\sum_{k=1}^{2n+1} \frac{\left\{r[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}](\frac{\tilde{m}_2}{m_1})^r\right\}}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}](\frac{\tilde{m}_2}{m_1})^r\right\}^2}}{\tilde{m}^*_2}$*. Firm 2 spends in total* $\tilde{m}^*_2$ *on competitive rent-seeking, and* $m_2 - \tilde{m}^*_2$ *on the numeraire good.*

Proof. See Appendix. ■

The endogenous choice of the firms to invest in market competition in the presence of outside options depends upon the value of the market relative to the size of the budget endowments of the firms. When the value of the market (v) is relatively large, while the budgets endowments of both firms are not too large ($\frac{1}{v} \leq \tau(\tilde{m}_2|\,\tilde{m}_1,\tilde{m}_2)|_{\tilde{m}_1=m_1,\tilde{m}_2=m_2}$), we have a corner equilibrium in which both firms exhaust their entire budgets in market competition and thus there is no spending on the outside option. This case reduces to our basic model in the previous section with fixed budgets and the presence of outside options does not affect the results.

In contrast, when ($\frac{1}{v} > \tau(\tilde{m}_1|\,\tilde{m}_1,\tilde{m}_2)|_{\tilde{m}_1=m_1,\tilde{m}_2=m_2}$), then neither firm will exhaust its entire budget because the market value is relatively small while the budget endowments of both firms are sufficiently large. It is interesting to observe that there is ex-post symmetry in the equilibrium, in the sense that firms spend the same amount of resources on competitive rent-seeking despite the asymmetry in their budgets. Thus in this case both firms deploy some part of their budgets on

their outside opportunity. Finally, we have a hybrid equilibrium, in the sense that the firm with the smaller budget, spends all its resources on rent-seeking activities, while the firm with the larger budget divides its budget between rent seeking and the numeraire good. This happens when v falls in an intermediate range or when the differential between m_1 and m_2 is sufficiently large.

It is also useful to explore how the extent of market differentiation (t) affects the amount of the budget allocated to market competition versus the outside opportunity.

Corollary 1 *When t increases, (i) the corner equilibrium is more likely to emerge, in which case both firms spend all their budget resources on competitive rent-seeking; (ii) in the fully interior equilibrium, firms spend more budget resources on competitive rent-seeking as compared to the outside option; (iii) in the hybrid equilibrium, firm 2 spends more of its budget resources on competitive rent seeking.*

We have the interesting finding that greater market differentiation actually leads firms to tilt their resource allocation more towards rent-seeking market competition. This result may be seen as contrary to intuition because one might think that an increase in t should lead to a less effective competitive rent-seeking technology. However, with increasing t a firm's advantage in closer markets increases, while its incentive to invest in remote markets goes down, which leads to a reduction in rent dissipation (wasted spending). Consequently, firms end up with a higher marginal benefit from their competitive rent-seeking activities. In turn, this leads them to divert more resources towards market competition, rather than the outside opportunity.

3.2.2 Costly Budgets

Another natural way to relax the fixed budget assumption is to consider the endogenous choices of firms when the budget decisions are costly (Snyder 1989). Specifically, suppose that a firm's choice of $x_{i,k}$ involves a constant marginal cost c_i. Without loss of generality, let firm 1 have a cost advantage, such that $c_2 \geq c_1 > 0$. Such a cost may represent the firm's cost of raising additional capital or productive resources (e.g., sales force size) that is required for the market. Recall that firms' effective outlays in a k are given by (1) and firm i, at market segment k, wins with a probability given by the contest success function in (3).

Firms simultaneously commit to their distributions of spending outlays in the $2n+1$ segments.

For each segment k, firms' marginal benefits are:

$$\frac{\partial \pi_1}{\partial x_{1,k}} = \frac{\left\{r[1-\frac{(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}]x_{1,k}^{r-1}x_{2,k}^{r-1}\right\}x_{2,k}}{\left\{[1-\frac{t(k-1)}{2n+1}]x_{1,k}^{r}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}^{r}\right\}^2}v;$$

$$\frac{\partial \pi_2}{\partial x_{2,k}} = \frac{\left\{r[1-\frac{(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}]x_{1,k}^{r-1}x_{2,k}^{r-1}\right\}x_{1,k}}{\left\{[1-\frac{t(k-1)}{2n+1}]x_{1,k}^{r}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}^{r}\right\}^2}v.$$

The equilibrium is determined by the following conditions:

$$\frac{\partial \pi_1}{\partial x_{1,k}} = \frac{\partial \pi_1}{\partial x_{1,k'}} = c_1;$$

$$\frac{\partial \pi_2}{\partial x_{2,k}} = \frac{\partial \pi_2}{\partial x_{2,k'}} = c_2.$$

From this we have that in equilibrium $\frac{x_{2,k}}{x_{1,k}} = \frac{c_1}{c_2}$. Define $c = \frac{c_2}{c_1}$. We have the following proposition:

Proposition 5 *In the equilibrium, firm i will deploy a competitive spending allocation given by $x_{i,k} = \frac{r[1-\frac{(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}]c^r}{c_i\left\{[1-\frac{t(k-1)}{2n+1}]c^r+[1-\frac{t(2n+1-k)}{2n+1}]\right\}^2}v$ in each market k, $i \in \{1,2\}, k \in \{1,\ldots,2n+1\}$.*

It can be seen that $x_{i,k}^*$ peaks in a market $k^* \geq n+1$, and each firm has an inverse U-shaped distribution of competitive spending outlays similar to what is described in Proposition 2 for the case of fixed budgets. Thus with endogenous budget choices all the results obtained in section 3 for the case of exogenously fixed budgets are robust with the lower cost firm being analogous to the firm with the higher budget.

4 Endogenous Value Creation: Guns vs. Butter

We now consider the full model in which the firms compete by spending in both value creation as well as rent seeking. Firm i's spending in each market k is a pair $(b_{i,k}, x_{i,k})$: Recall that $b_{i,k}$ is a productive investment that increases the surplus or value created in market k, while $x_{i,k}$ is the firm's competitive rent-seeking spending that helps it to get a larger share of the surplus in the market. Firms simultaneously commit to their strategy $(\mathbf{b}_i, \mathbf{x}_i)$, where $\mathbf{b}_i$ is the vector $(b_{i,1},\ldots,b_{i,2n+1})$, with $\sum_{k=1}^{2n+1} b_{i,k} + \sum_{k=1}^{2n+1} x_{i,k} \leq m_i$. The surplus created in market is given by $v_k = v_k(b_{1,k}, b_{2,k})$ and is increasing and strictly concave in each argument. Further, $v_k(0,0) = 0$ and $v_k(b_{1,k}, b_{2,k}) > 0$ if $\max(b_{1,k}, b_{2,k}) > 0$. We can then show that:

Lemma 2 *There exists no equilibrium in which both firms do not make positive productive investments in a market k, i.e., $v_k > 0$, $\forall k \in \{1,\ldots,2n+1\}$.*

Suppose there were to be a market in which neither firm invests. In that case both firms do not have to deploy competitive rent-seeking spending in that market. This means that one of the firms can gain from decreasing its investment in some other market where it faces competition and shifting it to this market. Thus we have that all markets will have positive surplus, and given this we know that the competitive rent-seeking spending must be positive everywhere as well.

To carry the analysis further we assume that $v_k = \sqrt{b_{1,k} + b_{2,k}}$. This functional form represents the contexts of multi-market competition which motivate our analysis in which the investments in value creation are substitutable and hence subject to free riding. For example, in pharmaceutical markets the firms might invest in DCTA to inform consumers and increase visits to doctors creating primary demand for the product category. Similarly, major cell phone manufacturers like Samsung and HTC have promoted the Android platform to move consumers from the iPhone to Android. The incremental demand for Android phones created by these promotional activities can be appropriated by rivals in a market. As another example, in the early days of the nascent satellite radio market, both the major competitors Sirius and XM invested in advertising which jointly expanded the overall category, in addition to brand-specific advertising (see Bass et al. 2005).

4.1 Symmetric Budgets

This game described above highlights two simultaneous and related trade-offs: How to allocate the limited budget between value (surplus) creation and competition? And how to allocate the budget across the different markets. To explore these trade-offs, consider first the case where firms are ex ante symmetric, with $m_1 = m_2 = m$. For simplicity, we consider the case of $r = 1$. The following proposition characterizes the unique symmetric equilibrium of this game.

Proposition 6 *There exists a unique symmetric equilibrium. In the equilibrium, Firm 1 makes positive investment $b_{1,k}$ in surplus creation only for markets $k \leq n+1$ (Firm 2's strategy is symmetric for markets $k \geq n+1$) given by*

$$b_{1,k} = \begin{cases} \left\{ \frac{[1-\frac{t(k-1)}{2n+1}]}{2\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}\tilde{\mu}} \right\}^2 & \text{if } k \leq n; \\ \left\{ \frac{[1-t\frac{n}{2n+1}]}{2\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}\tilde{\mu}} \right\}^2 / 2 & \text{if } k = n+1; \\ 0 & \text{if } k > n+1, \end{cases} \tag{4}$$

where $\tilde{\mu}$ is the Kuhn-Tucker multiplier as defined in the Appendix.

The equilibrium surplus created in market k is

$$v_k = \begin{cases} \frac{[1-\frac{t(k-1)}{2n+1}]}{2\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}\tilde{\mu}}, & \text{if } k \leq n+1; \\ \frac{[1-\frac{t(2n+1-k)}{2n+1}]}{2\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}\tilde{\mu}} & \text{if } k \geq n+1 \end{cases}.$$

Finally, the two firms choose the same equilibrium competitive rent-seeking spending in each market given by:

$$x_{i,k} = \frac{[1-\frac{t(2n+1-k)}{2n+1}][1-\frac{t(k-1)}{2n+1}]}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}^2\tilde{\mu}} v_k.$$

The proposition identifies the trade-off between value creation and competition. Firms have the incentive to invest in value creation only in markets which are in their own turf and in the middle $(n+1)$ market, but they compete for every market. Specifically, Firm 1's investments in creating value decreases with k for k for $k < n+1$, while Firm 2's investments decreases between from market $2n+1$ to $k > n+1$. Thus firms concentrate more on building value in closer markets and the markets 1 and $2n+1$ end up being the one with the highest value, while the middle market $n+1$, generates the least amount of equilibrium surplus. Figure 2a shows the distribution of surplus across the markets in equilibrium.[6]

This result can also be viewed in the context of the idea of home market advantage; i.e., a systematic preference of the market/consumers to purchase local products. The natural question is when does this lead to a home turf bias in the investments made by firms. With endogenous market values, the value creation investments of the firms are higher in their home turfs even if all markets have the same productivity/effectiveness in value creation. Thus we have the result that even in the absence of any home turf advantage in value creation, the presence of differentiation in competitive rent-seeking can lead to a home turf bias in equilibrium value creation.[7]

[6] This can be compared to the literature on targeted advertising (Iyer et. al 2005) where too firms invest more in advertising in their (closer) high preference markets but relatively less in the more competitive markets, where the latter incentive comes from wanting to reducing price competition.

[7] In a different context of product line expansion Joshi et al. (2016), show an equilibrium in which one firm expands its product line and this allows for better surplus extraction through the basic product from its core consumers.

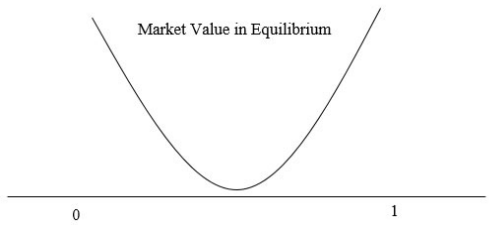

Figure 2a

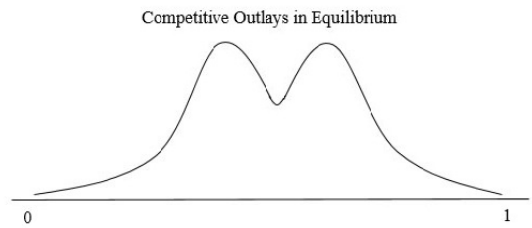

Figure 2b

Figure 2: Distribution of Market Values and Competitive Outlays

Clearly the endogenous market creation profile will affect the competitive rent-seeking spending incentives as the value of each market is no longer the same. Markets which are closer to each firm are larger, creating an incentive for firms to invest more in defending them from the rival's competitive spending. However, there is also the countervailing incentive to compete more intensely for markets which are closer to the middle where each firm's relative dis/advantage in spending effectiveness is not too large. The tension between these two forces determines the equilibrium competitive spending profile $x_{i,k}$.

We can note from the Proposition that the competitive spending (rent-dissipation) rate for market k is $[1-\frac{t(2n+1-k)}{2n+1}][1-\frac{t(k-1)}{2n+1}]/\tilde{\mu}\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}^2$. As in the exogenous market value case, it strictly increases with k for $k < n+1$, and strictly decreases for $k \geq n+1$. But the equilibrium investment in value creation has an opposite profile leading to the highest amount of surplus v_k created at the market coincident with the firms, while the middle market has the lowest surplus size. Thus the incentive to compete most intensely for the middle market is offset by the fact that in equilibrium it will have the smallest size. Hence, the distribution of competitive rent-seeking bids are no longer single-peaked as in the case of fixed market values. Figure 2b. shows the distribution of competitive rent-seeking expenditures.

To examine the distribution of the competitive spending further, note that by symmetry, we will have in equilibrium that $x_{i,k} = x_{i,((2n+1)-(k-1))}$, and so without loss of generality, we focus on the left side of the line for Firm 1, i.e., $k \leq n+1$. Recall $x_{i,k} = \frac{[1-\frac{t(2n+1-k)}{2n+1}][1-\frac{t(k-1)}{2n+1}]}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}^2\tilde{\mu}}$ and $v_k = \frac{[1-\frac{t(k-1)}{2n+1}]}{2\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}\tilde{\mu}}$, which gives

$$x_{i,k} = \frac{[1-\frac{t(2n+1-k)}{2n+1}][1-\frac{t(k-1)}{2n+1}]^2}{2\tilde{\mu}^2\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}^3},$$

We want to evaluate how $x_{i,k}$ changes with k. Because $[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}] = (2-\frac{t\cdot 2n}{2n+1}) > 0$, to evaluate $x_{i,k}$ with respect to k, it is sufficient to examine the numerator and it can be shown

that its sign depends upon $[t(4n+3-3k)-(2n+1)]$.

Corollary 2 *(a) Within each half of the line, the distribution of firms' competitive rent-seeking expenditures is in general non-monotonic. The competitive spending $x_{i,k}$ reach their peak symmetrically in two markets $\tilde{k}_1$ and $\tilde{k}_2$, with $\tilde{k}_1 = 2(n+1) - \tilde{k}_2 < n+1$ and correspondingly $\tilde{k}_2 = 2(n+1) - \tilde{k}_1 > n+1$.*

(b) The locations of the peak competitive spending move toward the midpoint as t increases and the firms becomes more differentiated .

The distribution of competitive spending is no longer monotonic within each firm's turf. As already described above the middle market no longer faces the most intense competition. Rather in an interesting contrast to the case of exogenous market values, we get that each firm chooses the highest the competitive spending at two symmetrically located markets on either side of the middle market $k = n+1$. Thus when firms endogenously create the market, the most intense competition shifts to each firm turf, and this reflects the balance between fighting in markets where there is more surplus vs. in markets where competitive spending is more effective in winning the market from the rival.

The particular pattern depends on the size of t. Consider, for instance, the case of $t = 1$. The expression $[t(4n+3-3k)-(2n+1)]$ reduces to $2(n+1)-3k$. The competitive spending $x_{i,k}$ increases and then decreases after reaching an interior peak. In contrast, suppose that t is sufficiently small, i.e., $t \leq \frac{2n+1}{4n}$, the most intense competition simply occur in each firm's home court because in this case, $[t(4n+3-3k)-(2n+1)]_{k=1} \leq 0$. The second part of the corollary shows the interesting effect of firm differentiation on the competitive spending. The markets with the most intense competition are closer to the middle even as differentiation increases.

We now examine how market differentiation affects the extent to which firms invest in value creation versus spending on competitive rent-seeking. This analysis is tractable for the three-market case for which we get:

Proposition 7 *Consider a three-market case with $n = 1$. When t increases, firms invest less in the middle market and invest more on their home markets, i.e., $\frac{dv_2}{dt} < 0$ and $\frac{dv_1}{dt}, \frac{dv_3}{dt} > 0$. They deploy less in competitive rent-seeking activities in all markets, i.e., $\frac{dx_k}{dt} < 0, \forall k \in \{1,2,3\}$.*

The degree of market differentiation affects both i) firms' division of resources between rent-seeking activities and value-creating investments, and ii) their resource allocations across market.

With greater market differentiation both firms invest less in the middle market, and more in their own turfs. A more differentiated market leads to lower effectiveness of a firm's competitive spending in the remote market, and strengthens its advantage at its own turf. This increases the return to the firm's investment in its own turf, as it is more able to protect it from possible predation. Hence, firms shift investments in value creation to closer markets in their own turfs. At the same time, the greater advantage in the home market reduces the rival firm's ability to win those markets, which leads each firm to reduce its competitive expenditure accordingly. Further, because firms invest less in the middle market, the reduced value also elicits less competitive spending in the middle.

4.2 Asymmetric Budgets

Finally, consider the general case that allows firms to be endowed with asymmetric budgets with $m_1 > m_2$. Despite the asymmetry, Lemma 2 continues to hold: There exists no equilibrium in which a market ends up with zero surplus. As a result, both firms will invest in competitive rent-seeking activities in all markets, i.e., $x_{i,k} > 0, \forall i \in \{1, 2\}, k \in \{1, \ldots, 2n + 1\}$. While a closed-form solution to the equilibrium cannot be obtained due to the nonlinearity of the production function and the budget asymmetry, it is still possible for us to characterize important properties of the equilibrium.

Proposition 8 *When $m_1 > m_2$ we have that in equilibrium:*

a. *Firm 1 invests more in value creation and deploys higher competitive rent-seeking spending.*

b. *The equilibrium surplus across markets, v_k, is distributed as a U-shaped curve, strictly decreasing first and then strictly increasing.*

c. *A firm makes more investments in value creation in markets closer to its own location.*

d. *Firm 1 makes investment in value creation in markets 1 to $\bar{k}$, with $\bar{k} \geq n+1$, while only Firm 2 invests in markets $\bar{k}+1$ to $2n+1$. Firm 2 may also invest on $\bar{k}$ (in a knife-edge case), but for markets $\{1, \ldots, 2n+1\} \setminus \{\bar{k}\}$ only one firm invests in equilibrium.*

In the equilibrium, the ratio between firms' competitive rent-seeking expenditures remains constant across all markets (i.e., $\frac{x_{1,k}}{x_{2,k}} = \frac{\mu_2}{\mu_1}$), where μ_1 and μ_2 are the Kuhn-Tucker multipliers for firms 1's and 2's constrained maximization problem. This leads to the result in part (a) of the proposition that the firm with the budget advantage spends more in the rent-seeking competition.

But at the same time it also invests more in value creation. This contrasts with the finding in the literature on single-market guns and butter competition that the firm with the budget advantage may invest more in value creation (butter) while the weaker firm with the smaller budget invests more in guns (see Skaperdas 1992). Because the joint surplus is subject to competition, this leads to the firm with the smaller budget ending up with a larger expected payoffs. In contrast, part (a) of proposition shows that this result does not carry over to value creation competition in multi-market settings. With multiple markets and with market differentiation, firms have to trade-off where to invest as well as how much to invest in value creation and in rent seeking competition. This allows for greater productive investments in closer markets by Firm 1 where Firm 2's competitive spending is less efficient.

As shown in the proposition, in general, in each market only one firm makes productive investments and so the equilibrium value creation profiles of the two firms are mutually exclusive. This can be seen as a strategic attempt by each firm to reduce free-riding of their value creation investments. This is different from the symmetric case, where we see (minimal) overlap: i.e., both firms overlap in a single market in the middle $(n+1)$. The overlap is thus an artifact of symmetry.

In the asymmetric case, Firm 1 invests on strictly more number of markets than Firm 2. The distribution of value creation is similar to that in the symmetric case and each firm is more willing to invest in markets closer to its own position. As a result, the market values are distributed as a U-shape curve: Firm 1's investments strictly decrease toward the other end of the line until it stops investing; in contrast, firm 2's investments pick up in markets closer to the right end of the line. This observation is qualitatively similar to that in the symmetric case. In the symmetric case, equilibrium productive investment is minimized at the middle market, i.e., $n+1$. In the asymmetric case, as expected, it is minimized at a market to the right of the middle point, because of the asymmetry. In fact, if the asymmetry is sufficiently large, Firm 2 may stop investing in value creation, and focus its resource only on competitive rent-seeking activities.

We can also establish some comparative statics pertaining to how the extent of asymmetry in the firms' budgets affect their strategies: Specifically, greater asymmetry in the budget between the firms $(\frac{m_1}{m_2})$ leads to increases in the equilibrium $\bar{k}$: i.e., Firm 1 makes productive investments in more markets while Firm 2 invests in fewer markets. Consistent with this result greater budget asymmetry also leads to increases the ratio of firms' competitive expenditure $(\frac{x_{1,k}}{x_{2,k}})$. Firm 1 therefore deploys relatively higher spending in rent-seeking. Thus the overall message is that in the presence of market differentiation and multi-market interactions greater budget advantage leads a firm to

balance its actions such that it not only invests relatively more in value creation, but also competes with greater resources.

4.3 Efficiency Differences in Productive Investments

Until this point we assumed that the market differentiation pertains to the competitive rent seeking efforts of the firms, and not to the productive investments. This may be seen as consistent with the contexts that motivate the paper: For example, generic advertising which informs new consumers about the objective characteristics of the product category should have similar effects on whether they consider purchasing in the category irrespective of their relative preference for that firm. In this section we consider an extension in which a firm's efficiency in productive investments declines as the distance of the market from the firm increases. Specifically, we assume that by investing an amount $b_{i,k}$, a firm's effective investment in a market segment k is given by

$$\tilde{b}_{1,k} = [1 - \frac{d(k-1)}{2n+1}]b_{1,k}, \text{ and,}$$
$$\tilde{b}_{2,k} = [1 - \frac{d(2n+1-k)}{2n+1}]b_{2,k},$$

with $d \in (0, 1]$. We consider only sufficiently small d to ensure the existence of well-behaved equilibrium. As in the basic model, we derive symmetric equilibrium, with $x_{1,k} = x_{2,k} = x_{1,2n+1-(k-1)} = x_{2,2n+1-(k-1)}$ and $b_{1,k} = b_{2,2n+1-(k-1)}$.

Proposition 9 *In the symmetric equilibrium, each firm deploys competitive rent-seeking spending*

$$x_k^* = \frac{[1 - \frac{t(k-1)}{2n+1}][1 - \frac{t(2n+1-k)}{2n+1}]}{\left\{[1 - \frac{t(k-1)}{2n+1}] + [1 - \frac{t(2n+1-k)}{2n+1}]\right\}^2 \tilde{\mu}^*} v_k^*$$

in market k, where v_k^* is the equilibrium market value and $\tilde{\mu}^*$ is a constant specified in the Appendix. Firm 1 makes positive productive investments in markets $1, \ldots, n+1$, while firm 2 in markets $n+1, \ldots, 2n+1$. Each market has an equilibrium value

$$v_k^* = \begin{cases} \frac{[1-\frac{t(k-1)}{2n+1}][1-\frac{d(k-1)}{2n+1}]}{2\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}\tilde{\mu}^*}, & k \in \{1, \ldots, n+1\} \\ \frac{[1-\frac{t(2n+1-k)}{2n+1}][1-\frac{d(2n+1-k)}{2n+1}]}{2\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}\tilde{\mu}^*}, & k \in \{n+1, \ldots, 2n+1\}. \end{cases}$$

Further, we obtain the following observations in the equilibrium.

Corollary 3 *In the symmetric equilibrium,*

(i) v_k strictly decreases with k for $k \in \{1, \ldots, n+1\}$ and strictly increases for $k \in \{n+1, \ldots, 2n+1\}$.

(ii) The rent-dissipation rate, i.e., x_k^*/v_k^* strictly increases with $k \in \{1, \ldots, n+1\}$ and strictly increases for $k \in \{n+1, \ldots, 2n+1\}$.

Proof. See Appendix. ■

Even if the efficiency of productive investments declines with distance, the equilibrium yields similar implications to those obtained in the baseline model. Firms invest more in segments closer to their own turfs, and market rents bottom out at midpoint. However, firms continue to compete more intensely in the middle. The same trade-off occurs in the extended setting: Competition dissipates a larger portion of rents in the middle even as the value creation is smaller.

Finally, its is useful to consider the limiting case in which $t = 0$. This is the opposite to our baseline model: Firms no longer have home turf advantage in competitive rent seeking in closer markets, but they are more productive in creating value in these closer markets. The analysis reveals the following:

$$x_k = \frac{1}{4\tilde{\mu}^*} v_k, \text{and } v_k = \frac{1 - \frac{d(k-1)}{2n+1}}{4\tilde{\mu}^*}, k \leq n+1.$$

The rent-dissipation rate, $\frac{1}{4\tilde{\mu}^*}$, now becomes uniform across the different markets. However, the equilibrium market values v_k^* are strictly decreasing with k for $k \leq n+1$, and increasing for $k \geq n+1$. As a result, competitive rent-seeking spending x_k also decreases with k for $k \leq n+1$ and then increases.

5 Conclusion

This paper develops a theory of multi-market interactions which brings together two basic trade-offs common to many important economic and business contexts: First, firms must invest resources in value creation which they can profitably extract. But in competitive markets they also have to compete for the market with rivals. This leads to the trade-off of allocating limited resources to creating the market versus competing for it. Second, firms competing in differentiated markets must also decide how to allocate resources between different markets where they have more or less competitive advantage. In this paper we develop and analyze a model of multi-market competition which captures these trade-offs and their effects on firm strategies.

When the market size is fixed then the firms' allocation strategies are only governed by the competitive rent seeking incentives and each firm's equilibrium resource allocation strategies follow an inverted-U profile. A firm spends less in closer markets where its spending is relatively more effective because it is easier to defend these markets from competition. At the same time it also spends relatively less in far away markets precisely because its spending is relatively less effective and it is harder to win these markets from competition. Thus the most intense competition is for a market in the middle. Further, and counter to intuition, the competition for the marginal market in the middle becomes more intense even as the firms become more differentiated.

Next the paper considers the dual trade-off in which firms decide how much to invest in creating value in each market as well as how much to spend in competing for the markets. In equilibrium a firm invests more in closer markets and the investment profile declines monotonically. For symmetric firms this leads to the most intense competition to move away from the middle market to one in each firm's turf. We also find that with asymmetric budgets the firm with the advantage invests more in both value creation as well as in competitive rent-seeking spending. As the budget asymmetry increases the firm with the advantage invests in more markets closer to the weaker firm. Greater market differentiation leads to more value creation by firms in their home turfs and a reduction in the overall amount of competitive spending.

In the standard tradition of contest models, our study assumes that firms' strategic decisions are resource investments rather than prices. Thus our model can be seen as representing market situations in which prices are either non-strategic while firms make advertising or selling allocations (for example in the pharmaceutical and health market), or markets where prices are not relevant (for example political markets). There exist no analytically tractable models in the contest literature with pricing. This is because if firms' ability to price depends on the outcome of the rent-seeking (advertising) competition, this could potentially lead to endogenously determined market rents. A contest model with pricing is definitely worth studying. One possible micro foundation is to consider a logit like consumer choice formulation, but such a setup might require numerical analysis. Suppose we have a model such that a firm is able to increase its product's value to consumers if the firm prevails in advertising competition. Then compared to our analysis with fixed and uniform market values, a firm is more able to protect its existing advantage at closer market segments, which deters the rival firm from aggressive spending. In contrast, in middle markets where neither firm has a clear advantage, a firm should be able to price higher once it prevails in the advertising competition. This would compel both firms to step up spending efforts. We would therefore expect

a qualitatively similar prediction that firms spend less in markets closer to the ends of line, while competing more in markets towards the middle.

An aspect of the problem that we have not explored is the role of potential uncertainty of firms about their rivals. For instance, a firm might be uncertain about the size of its rival's resource budget and the nature of this uncertainty should have a bearing on the extent to which firms invest in value creation versus rent seeking. The analysis of multi-market guns and butter competitions under incomplete information is a challenging problem which may be investigated in future work.

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Appendix

Proof of Lemma 1

The proof is by contradiction. Suppose there exists a market $k' \in \{1, \ldots, 2n+1\}$ such that $x_{i,k'} = x_{j,k'} = 0$. Then let firm i deviate by finding an infinitesimal ε, such that it places an outlay of ε in market k', but reduces its bid in some other market k'' by ε. In this case, it will gain v at market k' with probability one, but its probability of winning v in market k'' decreases negligibly. By continuity, the firm must get strictly better with such a deviation, which establishes the contradiction.

Suppose now that $x_{j,k'} > 0$ and $x_{i,k'} = 0$. Now firm j can always gain by reducing $x_{j,k'}$ to an infinitesimally small ε and reallocating to other markets. This means that firm i has the incentive to deviate from $x_{i,k'} = 0$, which establishes a contradiction.

Proof of Proposition 1

Evaluating π_i with respect to an arbitrary $x_{i,k}$ yields

$$\frac{\partial \pi_1}{\partial x_{1,k}} = \frac{\left\{ r[1 - \frac{(k-1)}{2n+1}] \frac{k}{2n+1} x_{1,k}^{r-1} x_{2,k}^{r-1} \right\} x_{2k}}{\left\{ [1 - \frac{(k-1)}{2n+1}] x_{1,k}^{r} + \frac{k}{2n+1} x_{2,k}^{r} \right\}^2} v;$$

$$\frac{\partial \pi_2}{\partial x_{2,k}} = \frac{\left\{ r[1 - \frac{(k-1)}{2n+1}] \frac{k}{2n+1} x_{1,k}^{r-1} x_{2,k}^{r-1} \right\} x_{1k}}{\left\{ [1 - \frac{(k-1)}{2n+1}] x_{1,k}^{r} + \frac{k}{2n+1} x_{2,k}^{r} \right\}^2} v.$$

An interior optimum must satisfy $\frac{\partial \pi_1}{\partial x_{1,k}} = \frac{\partial \pi_{1k'}}{\partial x_{1,k'}} = \mu_1$, and $\frac{\partial \pi}{\partial x_{2,k}} = \frac{\partial \pi_2}{\partial x_{2,k'}} = \mu_2, \forall k, k' \in \{1, \ldots, 2n+1\}, k \neq k'$. Hence, we must have $\frac{x_{2,k}}{x_{1,k}} = \frac{x_{2,k'}}{x_{1,k'}} = \frac{\mu_1}{\mu_2}, \forall k, k' \in \{1, \ldots, 2n+1\}, k \neq k'$.

Define $\lambda \equiv \frac{\mu_1}{\mu_2}$. We then have $x_{2,k} = \lambda x_{1,k}$ for all k. We can then rewrite $\frac{\partial \pi_i}{\partial x_{i,k}}$ as

$$\frac{\partial \pi_i}{\partial x_{i,k}} = \frac{\left\{ r[1 - \frac{(k-1)}{2n+1}] \frac{k}{2n+1} \lambda^r \right\}}{\left\{ [1 - \frac{(k-1)}{2n+1}] + \frac{k}{2n+1} \lambda^r \right\}^2} \frac{v}{x_{i,k}},$$

Define $\phi(k) \equiv \frac{\left\{ r[1 - \frac{(k-1)}{2n+1}] \frac{k}{2n+1} \lambda^r \right\}}{\left\{ [1 - \frac{(k-1)}{2n+1}] + \frac{k}{2n+1} \lambda^r \right\}^2}$. We then have $x_{i,k} = \frac{\phi(k)}{\phi(k')} x_{i,k'}$ and we must recursively obtain $x_{i,k} = \frac{\phi(k)}{\phi(1)} x_{i,1}$. Further, the resource (budget) constraints can be rewritten as

$$\sum\nolimits_{k=1}^{2n+1} x_{1,k} = [\frac{\sum_{k=1}^{2n+1} \phi(k)}{\phi(1)}] x_{1,1} = m_1;$$

$$\sum\nolimits_{k=1}^{2n+1} x_{2,k} = \lambda \sum\nolimits_{k=1}^{2n+1} x_{1,k} = \lambda [\frac{\sum_{k=1}^{2n+1} \phi(k)}{\phi(1)}] x_{1,1} = m_2.$$

As a result, we have $\lambda = \frac{m_2}{m_1}$ and $x_{1,1} = \frac{m_1 \phi(1)}{\sum_{k=1}^{2n+1} \phi(k)}$, and therefore $x_{1,k} = \frac{m_1 \phi(k)}{\sum_{k=1}^{2n+1} \phi(k)}$, and $x_{2k} = \frac{m_2 \phi(k)}{\sum_{k=1}^{2n+1} \phi(k)}$. This proves the Proposition.

Proof of Proposition 2

Apparently, $\tilde{\phi}(\,k|\,t)$ continues to be single-peaked, as the sign is determined by the term

$$\left\{\begin{array}{c}\left[(2n+1)^2-t(2n+1)(n-1)-t^2n\right]\\ -(2n+1)\lambda^r\left[(2n+1)-t(3n+1)+t^2n\right]\\ -t(1+\lambda^r)\left[(2n+1)-tn\right]k\end{array}\right\}.$$

Define $\hat{k}=\dfrac{\left\{\begin{array}{c}\left[(2n+1)^2-t(2n+1)(n-1)-t^2n\right]\\ -(2n+1)\lambda^r\left[(2n+1)-t(3n+1)+t^2n\right]\end{array}\right\}}{t(1+\lambda^r)[(2n+1)-tn]}$. If k is treated as a continuous varaible, the function $\tilde{\phi}(\,k|\,t)$ is then maximized at $k=\hat{k}$. Hence, $\tilde{\phi}(\,k|\,t)$ reaches its peak at

$$k^*=\left\{\begin{array}{ccc}0 & & \text{if } \hat{k}\leq 0\\ 2n+1 & & \text{if } \hat{k}\geq 2n+1\\ \arg\max\limits_{(int(\hat{k}),int(\hat{k})+1)} & \tilde{\phi}(k). & \text{if otherwise}\end{array}\right..$$

Obviously, $\hat{k}$ decreases with λ. It has a value of precisely $\dfrac{\left\{\begin{array}{c}\left[(2n+1)^2-t(2n+1)(n-1)-t^2n\right]\\ -(2n+1)\left[(2n+1)-t(3n+1)+t^2n\right]\end{array}\right\}}{2t[(2n+1)-tn]}=$ $\dfrac{\left\{\begin{array}{c}[-(2n+1)(n-1)-tn]\\ -(2n+1)\left[-(3n+1)+tn\right]\end{array}\right\}}{2[(2n+1)-tn]}=\frac{4n^2+6n+2-2tn^2-2tn}{2[(2n+1)-tn]}=\frac{\left[2n^2+3n+1-tn^2-tn\right]}{[(2n+1)-tn]}=n+1$. This implies that in the asymmetric case, the peak appears to the right of $n+1$.

Because $\lambda<1$, λ^r decreases with r. As a result, $\hat{k}$ increases with r.

Proof of Proposition 3

Evaluating $\hat{k}$ with respect to t leads to

$$\frac{\partial\hat{k}}{\partial t}=\frac{1}{\{t(1+\lambda^r)\left[(2n+1)-tn\right]\}^2}\times$$
$$\left\{\begin{array}{c}\left[\begin{array}{c}-(2n+1)(n-1)-2tn\\ +(2n+1)(3n+1)\lambda^r-2tn(2n+1)\end{array}\right]\{t(1+\lambda^r)\left[(2n+1)-tn\right]\}\\ -\left\{\begin{array}{c}\left[(2n+1)^2-t(2n+1)(n-1)-t^2n\right]\\ -(2n+1)\lambda^r\left[(2n+1)-t(3n+1)+t^2n\right]\end{array}\right\}(1+\lambda^r)\left[(2n+1)-tn\right]\\ +\left\{\begin{array}{c}\left[(2n+1)^2-t(2n+1)(n-1)-t^2n\right]\\ -(2n+1)\lambda^r\left[(2n+1)-t(3n+1)+t^2n\right]\end{array}\right\}tn(1+\lambda^r)\end{array}\right\}.$$

Let $\varphi(t)$ denote the numerator. We have $\varphi(t)$ rewritten as

$$\varphi(t) = \left\{ \begin{array}{c} \left[\begin{array}{c} -t(2n+1)(n-1) - 2t^2 n \\ +t(2n+1)(3n+1)\lambda^r - 2t^2 n(2n+1)\lambda^r \end{array} \right] \{(1+\lambda^r)\left[(2n+1) - tn\right]\} \\ - \left[\begin{array}{c} -t(2n+1)(n-1) - t^2 n \\ +t(2n+1)(3n+1)\lambda^r - t^2 n(2n+1)(2n+1)\lambda^r \end{array} \right] (1+\lambda^r)\left[(2n+1) - tn\right] \\ -\left[(2n+1)^2 - (2n+1)^2\lambda^r\right](1+\lambda^r)\left[(2n+1) - tn\right] \\ + \left\{ \begin{array}{c} \left[(2n+1)^2 - t(2n+1)(n-1) - t^2 n\right] \\ -(2n+1)\lambda^r\left[(2n+1) - t(3n+1) + t^2 n\right] \end{array} \right\} tn(1+\lambda^r) \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \left[(2n+1)^2\lambda^r - (2n+1)^2 - t^2 n - t^2 n(2n+1)\lambda^r\right]\{(1+\lambda^r)\left[(2n+1) - tn\right]\} \\ + \left\{ \begin{array}{c} \left[(2n+1)^2 - t(2n+1)(n-1) - t^2 n\right] \\ -(2n+1)\lambda^r\left[(2n+1) - t(3n+1) + t^2 n\right] \end{array} \right\} tn(1+\lambda^r) \end{array} \right\}$$

This is further rewritten as

$$= (1+\lambda^r) \left\{ \begin{array}{c} \left[(2n+1)^2\lambda^r - (2n+1)^2 - t^2 n - t^2 n(2n+1)\lambda^r\right](2n+1) \\ + \left\{ \begin{array}{c} \left[2(2n+1)^2 - t(2n+1)(n-1)\right] \\ -(2n+1)\lambda^r\left[2(2n+1) - t(3n+1)\right] \end{array} \right\} tn \end{array} \right\}$$

$$= (1+\lambda^r) \left\{ \begin{array}{c} (2n+1)^2(\lambda^r - 1)[(2n+1) - 2tn] \\ +t^2 n^2(2n+1)(\lambda^r - 1) \end{array} \right\}.$$

It is negative because $\lambda < 1$.

Proof of Proposition 4

Proof. An optimum requires

$$\frac{\partial \pi_{1,k}}{\partial x_{1,k}} = \frac{\partial \pi_{1,k'}}{\partial x_{1,k'}} = \lambda_1;$$
$$\frac{\partial \pi_{2,k}}{\partial x_{2,k}} = \frac{\partial \pi_{2,k'}}{\partial x_{2,k'}} = \lambda_2,$$

$\forall k, k' \in \{1, \ldots, 2n+1\}, k \neq k'$. Hence, we continue to have $\frac{x_{2,k}}{x_{1,k}} = \frac{x_{2,k'}}{x_{1,k'}} = \frac{\lambda_1}{\lambda_2}, \forall k, k' \in \{1, \ldots, 2n+1\}, k \neq k'$. Analogous to the baseline setting, we must obtain recursively $x_{i,k} = \frac{\phi(k)}{\phi(1)} x_{i,1}$, with $\phi(k) \equiv \frac{\left\{r[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}](\frac{\tilde{m}_2}{\tilde{m}_1})^r\right\}}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}](\frac{\tilde{m}_2}{\tilde{m}_1})^r\right\}^2}$.

A firm i's marginal utility from its advertising in market segment 1 is given by $\left.\frac{\partial \pi_i}{\partial x_{i1}}\right|_{x_{i1}=x_{i1}^*} = \frac{\phi(1)v}{x_{i1}}$, which further leads to $\frac{\left.\frac{\partial \pi_i}{\partial x_{i1}}\right|_{x_{i1}=x_{i1}^*}}{v} = \frac{\phi(1)}{x_{i1}}$. Note $\frac{\phi(1)}{x_{i,1}} = \frac{\sum_{k=1}^{2n+1}\phi(k)}{\tilde{m}_i}$. We then rewrite the condition to obtain a firm's marginal utility from rent-seeking activities:

$$\left.\frac{\partial \pi_i}{\partial x_{i,1}}\right|_{x_{i,1}=x_{i,1}^*} = \frac{\sum_{k=1}^{2n+1}\phi(k)}{\tilde{m}_i} v = \tau(\tilde{m}_i | \tilde{m}_1, \tilde{m}_2) v.$$

Before we proceed, we first verify the following.

Claim 1 *The function τ strictly decreases with $\tilde{m}_i$.*

Rewrite τ as

$$\tau \equiv \tau(\tilde{m}_i|\,\tilde{m}_1,\tilde{m}_2) = \sum_{k=1}^{2n+1}\left\{\frac{\left\{r[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}](\frac{\tilde{m}_2}{\tilde{m}_1})^r\right\}}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}](\frac{\tilde{m}_2}{\tilde{m}_1})^r\right\}^2}/\tilde{m}_i\right\}.$$

We first verify the negative effect of $\tilde{m}_2$. Consider each item in the sum:

$$\frac{\left\{r[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}](\frac{\tilde{m}_2}{\tilde{m}_1})^r\right\}}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}](\frac{\tilde{m}_2}{\tilde{m}_1})^r\right\}^2}/\tilde{m}_2 = \frac{\left\{r[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}](\frac{1}{\tilde{m}_1})^r\tilde{m}_2^{r-1}\right\}}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}](\frac{\tilde{m}_2}{\tilde{m}_1})^r\right\}^2}.$$

Because $r \leq 1$, the numerator is nonincreasing with $\tilde{m}_2$; the denominator strictly increases with it. Hence, the sum must strictly decreases with $\tilde{m}_2$.

We then consider $\tilde{m}_1$. Consider the inverse of each item in the sum:

$$\frac{\tilde{m}_1\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}](\frac{\tilde{m}_2}{\tilde{m}_1})^r\right\}^2}{\left\{r[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}](\frac{\tilde{m}_2}{\tilde{m}_1})^r\right\}} = \frac{\left\{[1-\frac{(k-1)}{2n+1}]\tilde{m}_1^{\frac{1+r}{2}}+\frac{k}{2n+1}\tilde{m}_2^r m_1^{\frac{1-r}{2}}\right\}^2}{\left\{r[1-\frac{(k-1)}{2n+1}]\frac{k}{2n+1}\tilde{m}_2^r\right\}}.$$

The numerator strictly increases with $\tilde{m}_1$, while the demoninator is independent of it. Hence, the item in the sum must be strictly decreasing with it, so is the sum.

Our main claim can then be verified. There are altogether three cases. Recall $\left.\frac{\partial \pi_i}{\partial x_{i1}}\right|_{x_{i1}=x_{i1}^*} = \tau(\tilde{m}_i|\,\tilde{m}_1,\tilde{m}_2)v$, which indicates a firm's marginal utility obtained from rent-seeking activities.

Case 1: $\frac{1}{v} \leq \tau(\tilde{m}_2|\,\tilde{m}_1,\tilde{m}_2)|_{\tilde{m}_1=m_1,\tilde{m}_2=m_2}$. In this case, $\tau(\tilde{m}_1|\,\tilde{m}_1,\tilde{m}_2)|_{\tilde{m}_1=m_1,\tilde{m}_2=m_2}\,v \geq \tau(\tilde{m}_2|\,\tilde{m}_1,\tilde{m}_2)|_{\tilde{m}_1=m_1,\tilde{m}_2}$ 1, which implies that both firms obtain higher marginal utilities from rent-seeking activities even if they spend all their resources on those. Then the numerie goods will be ignored entirely.

Case 2: $\tau(\tilde{m}_2|\,\tilde{m}_1,\tilde{m}_2)|_{\tilde{m}_1=m_1,\tilde{m}_2=m_2}\,v \leq \tau(\tilde{m}_1|\,\tilde{m}_1,\tilde{m}_2)|_{\tilde{m}_1=\tilde{m}_2=m_1}\,v < 1$. In this case, both firms end up with lower marginal utilities from rent-seeking activities than the numerie good when they allocate all resources to rent seeking. Then investment on the numerie good cannot be zero and interior equilibrium emerges. The interior equilibrium requires

$$\frac{\sum_{k=1}^{2n+1}\frac{\left\{r[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}](\frac{\tilde{m}_2}{\tilde{m}_1})^r\right\}}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}](\frac{\tilde{m}_2}{\tilde{m}_1})^r\right\}^2}}{\tilde{m}_1} = \frac{\sum_{k=1}^{2n+1}\frac{\left\{r[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}](\frac{\tilde{m}_2}{\tilde{m}_1})^r\right\}}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}](\frac{\tilde{m}_2}{\tilde{m}_1})^r\right\}^2}}{\tilde{m}_2} = 1,$$

which thus implies $\tilde{m}_1 = \tilde{m}_2 = \bar{m}$. Then we have

$$\tau(\tilde{m}_1|\,\tilde{m}_1,\tilde{m}_2) = \tau(\tilde{m}_2|\,\tilde{m}_1,\tilde{m}_2) = \frac{\sum_{k=1}^{2n+1}\frac{\left\{r[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}]\right\}}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}^2}}{\bar{m}}.$$

It strictly decreases with $\bar{m}$. There must exist a unique interior solution $\bar{m}^*$, which satisifies

$$\frac{\sum_{k=1}^{2n+1} \frac{\left\{r[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}]\right\}}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}^2}}{\bar{m}^*} v = 1.$$

Case 3: $\tau(\tilde{m}_2|\,\tilde{m}_1,\tilde{m}_2)|_{\tilde{m}_1=m_1,\tilde{m}_2=m_2}\, v < 1 \leq \tau(\tilde{m}_1|\,\tilde{m}_1,\tilde{m}_2)|_{\tilde{m}_1=\tilde{m}_2=m_1}\, v$. In this case, firm 2 obtains a lower marginal utility from rent seeking when all resources go to such activities than the numerie good. So firm 2 must divide its resources between rent seeking and the numerie good. Firm 1, however, prefers to spend all its resources on rent-seeking activities. ■

Proof of Corollary 1

Proof. We first take first order derivative of the term $\frac{\left\{r[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}]\lambda^r\right\}}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\lambda^r\right\}^2}$ with respect to t for an arbitrary k. The sign of this derivative is the same as that of $\frac{\left\{[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}]\right\}}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\lambda^r\right\}^2}$. We have

$$\frac{d\frac{\left\{[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}]\right\}}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\lambda^r\right\}^2}}{dt}$$

$$= \frac{\left\{\begin{array}{c} -\left[\frac{(k-1)}{2n+1}\frac{t(2n+1-k)}{2n+1} + \frac{(2n+1-k)}{2n+1}\frac{t(k-1)}{2n+1}\right]\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\lambda^r\right\} \\ +2[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}]\left[\frac{(k-1)}{2n+1}+\frac{(2n+1-k)}{2n+1}\lambda^r\right] \end{array}\right\}}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\lambda^r\right\}^3}$$

$$= \frac{\left\{\begin{array}{c} -\left[\frac{2t(k-1)(2n+1-k)}{(2n+1)^2}\right]\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\lambda^r\right\} \\ +2[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}]\left[\frac{(k-1)}{2n+1}+\frac{(2n+1-k)}{2n+1}\lambda^r\right] \end{array}\right\}}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\lambda^r\right\}^3}.$$

We only need to look at the sign of the numerator. Rewrite it as

$$\left\{\begin{array}{c} -\left[\frac{2t(k-1)(2n+1-k)}{(2n+1)^2}\right]\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\lambda^r\right\} \\ +2[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}]\left[\frac{(k-1)}{2n+1}+\frac{(2n+1-k)}{2n+1}\lambda^r\right] \end{array}\right\}$$

$$= -2\frac{2t(k-1)(2n+1-k)}{(2n+1)^2} + 2\frac{[(2n+1)-t\cdot 2n][(k-1)+(2n+1-k)\lambda^r]}{(2n+1)^2}$$

$$+ 2\frac{2t^2(k-1)(2n+1-k)[(k-1)+(2n+1-k)\lambda^r]}{(2n+1)^3}.$$

The sum is positive when $t = 0$. Evaluating it with respect to t yields

$$
\begin{aligned}
&-2\frac{2(k-1)(2n+1-k)}{(2n+1)^2} - 2\frac{2n[(k-1)+(2n+1-k)\lambda^r]}{(2n+1)^2} \\
&+2\frac{4t(k-1)(2n+1-k)[(k-1)+(2n+1-k)\lambda^r]}{(2n+1)^3} \\
&= -\frac{2}{(2n+1)^2}\left\{\begin{array}{c} 2(k-1)(2n+1-k)+2n[(k-1)+(2n+1-k)\lambda^r] \\ -\frac{4t(k-1)(2n+1-k)[(k-1)+(2n+1-k)\lambda^r]}{(2n+1)} \end{array}\right\}.
\end{aligned}
$$

This is negative because

$$
\begin{aligned}
&2n[(k-1)+(2n+1-k)\lambda^r] - \frac{4t(k-1)(2n+1-k)[(k-1)+(2n+1-k)\lambda^r]}{(2n+1)} \\
&= \frac{2[(k-1)+(2n+1-k)\lambda^r]}{(2n+1)}\left[n(2n+1)-2t(k-1)(2n+1-k)\right] > 0.
\end{aligned}
$$

To see that, note $(k-1)(2n+1-k) \leq n^2$.

We now consider the situation of $t = 1$. In this case, the sum is rewritten as

$$
\begin{aligned}
&= -2\frac{2(k-1)(2n+1-k)}{(2n+1)^2} + 2\frac{[(k-1)+(2n+1-k)\lambda^r]}{(2n+1)^2} \\
&+2\frac{2(k-1)(2n+1-k)[(k-1)+(2n+1-k)\lambda^r]}{(2n+1)^3} \\
&= \frac{2}{(2n+1)^3}\left\{\begin{array}{c} -2(k-1)(2n+1-k)(2n+1)+[(k-1)+(2n+1-k)\lambda^r](2n+1) \\ +2(k-1)(2n+1-k)[(k-1)+(2n+1-k)\lambda^r] \end{array}\right\}.
\end{aligned}
$$

It increases with λ. Let $\lambda = 1$. In this case, the terms in the bracket can be written as

$$
\begin{aligned}
&-2(k-1)(2n+1-k)(2n+1)+2n(2n+1)+2(k-1)(2n+1-k)2n \\
&= -2(k-1)(2n+1-k)+2n(2n+1) > 0,
\end{aligned}
$$

which verifies the claim. ■

Proof of Lemma 2

Suppose otherwise. Then neither firm would exert competitive rent-seeking effort in a market with no surplus. Hence, a firm can strictly increase its payoff by decreasing its investment from other markets where its rival also invests, but increase its investment in this market where it is the sole claimant of the market surplus.

Proof of Proposition 6

We first demonstrate that in any symmetric equilibrium, firms exert the same amount of rent-seeking effort on every market, i.e., $x_{1,k} = x_{2,k}, \forall k \in \{1, \ldots, 2n+1\}$.

Symmetric equilibrium, with $(b_{1,k}, x_{1,k}) = (b_{2,2n+1-(k-1)}, x_{2,2n+1-(k-1)})$, leads to $\mu_1 = \mu_2$. Hence,

$$[1 - \frac{t(k-1)}{2n+1}][1 - \frac{t(2n+1-k)}{2n+1}]x_{2,k} / \left\{[1 - \frac{t(k-1)}{2n+1}]x_{1,k} + [1 - \frac{t(2n+1-k)}{2n+1}]x_{2,k}\right\}^2$$
$$= [1 - \frac{t(k-1)}{2n+1}][1 - \frac{t(2n+1-k)}{2n+1}]x_{1,k} / \left\{[1 - \frac{t(k-1)}{2n+1}]x_{1,k} + [1 - \frac{t(2n+1-k)}{2n+1}]x_{2,k}\right\}^2,$$

which gives $x_{1,k} = x_{2,k}$.

Define $\tilde{\mu} = \mu_1 = \mu_2$ and $x_k = x_{1,k} = x_{2,k}$. The Kuhn-Tucker conditions can be rewritten as

$$\frac{\partial \pi_1}{\partial x_{1,k}} = \frac{\partial \pi_2}{\partial x_{2,k}} = \frac{[1 - \frac{t(k-1)}{2n+1}][1 - \frac{t(2n+1-k)}{2n+1}]}{\left\{[1 - \frac{t(k-1)}{2n+1}] + [1 - \frac{t(2n+1-k)}{2n+1}]\right\}^2 x_k} \sqrt{b_{1,k} + b_{2,k}} = \tilde{\mu};$$
$$\frac{\partial \pi_1}{\partial b_{1,k}} = \frac{[1 - \frac{t(k-1)}{2n+1}]}{[1 - \frac{t(k-1)}{2n+1}] + [1 - \frac{t(2n+1-k)}{2n+1}]} \frac{1}{2\sqrt{b_{1,k} + b_{2,k}}} \leq \tilde{\mu};$$
$$\frac{\partial \pi_1}{\partial b_{2,k}} = \frac{[1 - \frac{t(2n+1-k)}{2n+1}]}{[1 - \frac{t(k-1)}{2n+1}] + [1 - \frac{t(2n+1-k)}{2n+1}]} \frac{1}{2\sqrt{b_{1,k} + b_{2,k}}} \leq \tilde{\mu}.$$

It is impossible to have $\frac{\partial \pi_1}{\partial b_{1,k}} = \frac{\partial \pi_1}{\partial b_{2,k}} = \tilde{\mu}$ for $k \neq n+1$. Hence, except for the mid-point, no market has both firms make positive productive investment. In any symmetric equilibrium, $v_k = v_{2n+1-(k-1)}$. We must have $\frac{\partial \pi_1}{\partial b_{1,k}} > \frac{\partial \pi_1}{\partial b_{1,2n+1-(k-1)}}$ and $\frac{\partial \pi_1}{\partial b_{2,k}} < \frac{\partial \pi_1}{\partial b_{1,2n+1-(k-1)}}$. That is, for $k < (>)\ n+1$, only firm 1(2) invests.

We then consider the mid-point, i.e., $k = n+1$. Firms must both invest positively here, because $\frac{\partial \pi_1}{\partial b_{1,k}} = \frac{\partial \pi_1}{\partial b_{2,k}} = \frac{1}{2} \cdot \frac{1}{2v_k}$.

Hence, we rewrite the Kuhn-Tucker conditions as

$$x_k = \frac{[1 - \frac{t(k-1)}{2n+1}][1 - \frac{t(2n+1-k)}{2n+1}]}{\left\{[1 - \frac{t(k-1)}{2n+1}] + [1 - \frac{t(2n+1-k)}{2n+1}]\right\}^2 \tilde{\mu}} v_k,$$
$$\text{and } v_k = \frac{1}{2} \frac{[1 - \frac{t(k-1)}{2n+1}]}{\left\{[1 - \frac{t(k-1)}{2n+1}] + [1 - \frac{t(2n+1-k)}{2n+1}]\right\} \tilde{\mu}}, k \leq n+1.$$

Recall $\sum_{k=1}^{2n+1} v_k^2 + 2\sum_{k=1}^{2n+1} x_k = 2m$. LHS can further be written as

$$\sum_{k=1}^{2n+1} v_k^2 + 2\sum_{k=1}^{2n+1} x_k$$
$$= 2\sum_{k=1}^{n} v_k^2 + 4\sum_{k=1}^{n} x_k + v_{n+1}^2 + 2x_{n+1}.$$

We have

$$\sum_{k=1}^{n} v_k^2 = \sum_{k=1}^{n} \frac{1}{4} \frac{[1-\frac{t(k-1)}{2n+1}]^2}{\tilde{\mu}^2 \left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}^2},$$

$$\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} \frac{[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}]}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}^2 \tilde{\mu}} v_k$$

$$= \sum_{k=1}^{n} \frac{1}{2} \frac{[1-\frac{t(k-1)}{2n+1}]^2[1-\frac{t(2n+1-k)}{2n+1}]}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}^3 \tilde{\mu}^2}.$$

Hence, $2\sum_{k=1}^{n} v_k^2 + 4\sum_{k=1}^{n} x_k = 2\sum_{k=1}^{n} \frac{[2n+1-t(k-1)]^2}{4\tilde{\mu}^2[(2n+1)-tn]^2}\left\{\frac{1}{4}+2\frac{[2n+1-t(2n+1-k)]}{2[(2n+1)-tn]}\right\}$.Further, $v_{n+1}^2 = \frac{1}{16\tilde{\mu}^2}$ and $2x_{n+1} = 2(\frac{1}{4\tilde{\mu}}\frac{1}{4\tilde{\mu}}) = \frac{1}{8\tilde{\mu}^2}$. We then have

$$2\sum_{k=1}^{n} v_k^2 + 4\sum_{k=1}^{n} x_k + v_{n+1}^2 + 2x_{n+1}.$$
$$= \frac{1}{\tilde{\mu}^2}\left\{\sum_{k=1}^{n} \frac{[2n+1-t(k-1)]^2}{[(2n+1)-tn]^2}\left\{\frac{1}{2}+2\frac{[2n+1-t(2n+1-k)]}{[(2n+1)-tn]}\right\}+\frac{3}{16}\right\}.$$

Hence,

$$\tilde{\mu} = \sqrt{\frac{\left\{\sum_{k=1}^{n} \frac{[2n+1-t(k-1)]^2}{[(2n+1)-tn]^2}\left\{\frac{1}{4}+\frac{[2n+1-t(2n+1-k)]}{[(2n+1)-tn]}\right\}+\frac{3}{32}\right\}}{m}}$$

Proof of Proposition 7

Consider the special case of $n = 1$. We have

$$\tilde{\mu} = \sqrt{\frac{\left\{\sum_{k=1}^{n} \frac{[2n+1-t(k-1)]^2}{[(2n+1)-tn]^2}\left\{\frac{1}{4}+\frac{[2n+1-t(2n+1-k)]}{[(2n+1)-tn]}\right\}+\frac{3}{32}\right\}}{m}}$$
$$= \sqrt{\frac{\frac{9}{(3-t)^2}\left(\frac{1}{4}+\frac{9}{3-t}\right)+\frac{3}{32}}{m}}.$$

We can then calculate v_k by $v_k = \frac{1}{2}\frac{[1-\frac{t(k-1)}{2n+1}]}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}\tilde{\mu}}, k \leq n+1$. We have

$$v_1 = \frac{1}{2}\frac{[1-\frac{t(k-1)}{2n+1}]}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}\tilde{\mu}} = \frac{3}{4}\frac{1}{\sqrt{\frac{\frac{9(39-t)}{4}+\frac{3(3-t)}{32}}{m}}},$$

which is increasing in t. So is v_3.

For v_2, we have

$$v_2 = \frac{1}{2}\frac{[1-\frac{t(k-1)}{2n+1}]}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}\tilde{\mu}} = \frac{1}{4\tilde{\mu}},$$

which is decreasing in t, because $\tilde{\mu}$ increases with it.

Proof of Proposition 8

Note that the Kuhn-Tucker conditions laid out above continue to hold in the asymmetric case:

$$\frac{\partial \pi_1}{\partial x_{1,k}} = \frac{[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}{\left\{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}\right\}^2}\sqrt{b_{1,k}+b_{2,k}} = \mu_1;$$

$$\frac{\partial \pi_1}{\partial b_{1,k}} = \frac{[1-\frac{t(k-1)}{2n+1}]x_{1,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}\frac{1}{2\sqrt{b_{1,k}+b_{2,k}}} \leq \mu_1;$$

$$\frac{\partial \pi_2}{\partial x_{2,k}} = \frac{[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}]x_{1,k}}{\left\{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}\right\}^2}\sqrt{b_{1,k}+b_{2,k}} = \mu_2;$$

$$\frac{\partial \pi_1}{\partial b_{2,k}} = \frac{[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}\frac{1}{2\sqrt{b_{1,k}+b_{2,k}}} \leq \mu_2.$$

By the argument laid out above, the ratio between the firms' competitive rent-seeking outlays is constant across all markets, $\frac{x_{1,k}}{x_{2,k}} = \frac{\mu_2}{\mu_1}$. As a result, in each market k, firm 1 wins with a probability

$$p_{1,k} = \frac{[1-\frac{t(k-1)}{2n+1}]\mu_2}{[1-\frac{t(k-1)}{2n+1}]\mu_2+[1-\frac{t(2n+1-k)}{2n+1}]\mu_1},$$

and firm 2 wins with the complementary probability. Obviously, as k increases, i.e., on a market further away from the left end, $p_{1,k}$ strictly decreases and $p_{2,k}$ strictly increases.

We now lay out the following arguments successively, which build the proof for the proposition.

Claim 1 There exists at most one market in which both firms make positive productive investments.

Assume otherwise that there are markets $k, k' \in \{1, \ldots, 2n+1\}$, such that $b_{i,k}, b_{i,k'} > 0$, $\forall i \in \{1,2\}$. This implies

$$\frac{[1-\frac{t(k-1)}{2n+1}]x_{1,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}\frac{1}{2\sqrt{b_{1,k}+b_{2,k}}}$$
$$= \frac{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'}}{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'}+[1-\frac{t(2n+1-k')}{2n+1}]x_{2,k'}}\frac{1}{2\sqrt{b_{1,k'}+b_{2,k'}}} = \mu_1;$$

$$\frac{[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}\frac{1}{2\sqrt{b_{1,k}+b_{2,k}}}$$
$$= \frac{[1-\frac{t(2n+1-k')}{2n+1}]x_{2,k'}}{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'}+[1-\frac{t(2n+1-k')}{2n+1}]x_{2,k'}}\frac{1}{2\sqrt{b_{1,k'}+b_{2,k'}}} = \mu_2;$$

This further leads to

$$\frac{[1-\frac{t(k-1)}{2n+1}]x_{1,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}\frac{1}{2\sqrt{b_{1,k}+b_{2,k}}}$$
$$+\frac{[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}\frac{1}{2\sqrt{b_{1,k}+b_{2,k}}}$$
$$=\frac{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'}}{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'}+[1-\frac{t(2n+1-k')}{2n+1}]x_{2,k'}}\frac{1}{2\sqrt{b_{1,k'}+b_{2,k'}}}$$
$$+\frac{[1-\frac{t(2n+1-k')}{2n+1}]x_{2,k'}}{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'}+[1-\frac{t(2n+1-k')}{2n+1}]x_{2,k'}}\frac{1}{2\sqrt{b_{1,k'}+b_{2,k'}}},$$

which gives $b_{1,k}+b_{2,k}=b_{1,k'}+b_{2,k'}$. However,

$$\frac{[1-\frac{t(k-1)}{2n+1}]x_{1,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}\frac{1}{2\sqrt{b_{1,k}+b_{2,k}}}$$
$$=\frac{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'}}{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'}+[1-\frac{t(2n+1-k')}{2n+1}]x_{2,k'}}\frac{1}{2\sqrt{b_{1,k'}+b_{2,k'}}}$$

implies $b_{1,k}+b_{2,k}\neq b_{1,k'}+b_{2,k'}$, because $\frac{[1-\frac{t(k-1)}{2n+1}]x_{1,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}\neq\frac{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'}}{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'}+[1-\frac{t(2n+1-k')}{2n+1}]x_{2,k'}}$ by the fact $\frac{x_{1,k}}{x_{2,k}}=\frac{x'_{1,k}}{x'_{2,k}}=\frac{\mu_2}{\mu_1}$. Contradiction.

Claim 2 In the equilibrium, market rents are distributed as a U-shaped curve: It strictly decreases with k until a point $\bar{k}$, and then strictly increases.

Consider three arbitrary adjacent markets, $k-1, k$ and $k+1$. Assume that $b_{1,k-1}+b_{2,k-1}, b_{1,k+1}+b_{2,k+1}\leq b_{1,k}+b_{2,k}$. Recall the Kuhn-Tucker conditions. Because $p_{1,k}$ strictly decreases with k, and $p_{2,k}$ strictly increases with k, we must have the following: If firm 1 makes productive investment on $k-1$, then it cannot invest on k; if firm 2 makes productive investment on $k+1$, then it cannot invest on k.

Suppose $b_{1,k-1}>0$, $b_{2,k+1}>0$, then $b_{1,k}+b_{2,k}=0$. Contradiction.

Suppose $b_{1,k-1}=0$, and $b_{2,k+1}>0$. Then only firm 1 has productive investment on k. This implies $b_{2,k-1}>0$. By Kuhn-Tucker condition, it also implies $b_{2,k-1}>b_{1,k}$, which leads to contradiction. By the same logic, we conclude that it is impossible to have $b_{2,k+1}=0$, and $b_{1,k-1}>0$.

Suppose $b_{1,k-1}=b_{2,k+1}=0$. This implies $b_{2,k-1}, b_{1,k+1}>0$. Because firm 1 invests positively on $k+1$ but not $k-1$, we must have $b_{1,k+1}<b_{2,k-1}$. Because firm 2 invests positively on $k-1$ but not $k+1$, we must have $b_{1,k+1}>b_{2,k-1}$. Contradiction.

Claim 3 Suppose that there exists a k_0, with $b_{1,k_0}, b_{2,k_0}>0$, then we must have (1) for all $k<k_0$, $b_{1,k}>0$ and $b_{2,k}=0$, and (2) for all $k>k_0$, $b_{2,k}>0$ and $b_{1,k}=0$.

Suppose otherwise that there is a $k<k_0$, with $b_{2,k}>0$. Then we must have $b_{1,k}=0$ by Claim 1.

Because firm 2 has positive productive investment at both k and k_0, we must have

$$\frac{[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}\frac{1}{2\sqrt{b_{2,k}}}$$
$$=\frac{[1-\frac{t(2n+1-k_0)}{2n+1}]x_{2,k_0}}{[1-\frac{t(k_0-1)}{2n+1}]x_{1,k_0}+[1-\frac{t(2n+1-k_0)}{2n+1}]x_{2,k_0}}\frac{1}{2\sqrt{b_{1,k_0}+b_{2,k_0}}}=\mu_2,$$

which implies $b_{2,k}<b_{1,k_0}+b_{2,k_0}$, because $\frac{[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}<\frac{[1-\frac{t(2n+1-k_0)}{2n+1}]x_{2,k_0}}{[1-\frac{t(k_0-1)}{2n+1}]x_{1,k_0}+[1-\frac{t(2n+1-k_0)}{2n+1}]x_{2,k_0}}$. However, because firm 1 invests zero on k, we have

$$\frac{[1-\frac{t(k-1)}{2n+1}]x_{1,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}\frac{1}{2\sqrt{b_{2,k}}}$$
$$<\frac{[1-\frac{t(k_0-1)}{2n+1}]x_{1,k_0}}{[1-\frac{t(k_0-1)}{2n+1}]x_{1,k_0}+[1-\frac{t(2n+1-k_0)}{2n+1}]x_{2,k_0}}\frac{1}{2\sqrt{b_{1,k_0}+b_{2,k_0}}}=\mu_1,$$

which implies $b_{2,k}>b_{1,k_0}+b_{2,k_0}$, because $\frac{[1-\frac{t(k-1)}{2n+1}]x_{1,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}>\frac{[1-\frac{t(k_0-1)}{2n+1}]x_{1,k_0}}{[1-\frac{t(k_0-1)}{2n+1}]x_{1,k_0}+[1-\frac{t(2n+1-k_0)}{2n+1}]x_{2,k_0}}$. Contradiction.

The same argument applies to the second part of the claim.

Claim 4 Suppose there does not exist a k_0 as described in Claim 3. Consider two arbitrary markets $k,k'\in\{1,\ldots,2n+1\}$, $k<k'$. If $b_{1,k}=0$, then $b_{1,k'}=0$; similarly, if $b_{2,k'}=0$, then $b_{2,k}=0$.

Suppose $b_{1,k}=0$, then $b_{1,k'}>0$. Then we must have $b_{2,k}>0$. The Khun-Tucker condition requires

$$\frac{[1-\frac{t(k-1)}{2n+1}]x_{1,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}\frac{1}{2\sqrt{b_{2,k}}}$$
$$<\frac{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'}}{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'}+[1-\frac{t(2n+1-k')}{2n+1}]x_{2,k'}}\frac{1}{2\sqrt{b_{1,k'}+b_{2,k'}}}=\mu_1.$$

This implies $b_{2,k}>b_{1,k'}+b_{2,k'}$ because $\frac{[1-\frac{t(k-1)}{2n+1}]x_{1,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}>\frac{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'}}{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'}+[1-\frac{t(2n+1-k')}{2n+1}]x_{2,k'}}$. However, $b_{2,k}>0$ implies

$$\mu_2=\frac{[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}\frac{1}{2\sqrt{b_{2,k}}}$$
$$\geq\frac{[1-\frac{t(2n+1-k')}{2n+1}]x_{2,k'}}{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'}+[1-\frac{t(2n+1-k')}{2n+1}]x_{2,k'}}\frac{1}{2\sqrt{b_{1,k'}+b_{2,k'}}},$$

which implies $b_{2,k}<b_{1,k'}+b_{2,k'}$ because $\frac{[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}<\frac{[1-\frac{t(2n+1-k')}{2n+1}]x_{2,k'}}{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'}+[1-\frac{t(2n+1-k')}{2n+1}]x_{2,k'}}$. Contradiction.

The same argument applies to the second part of the claim.

Claim 5 Suppose there exist two markets $k,k'\in\{1,\ldots,2n+1\}$, $k<k'$, with $b_{1,k},b_{1,k'}>0$, then $b_{1,k}>b_{1,k'}$.

Because $p_{1,k}$ strictly decreases, by Kuhn Tucker condition, we must have $b_{1,k}+b_{2,k}>b_{1,k'}+b_{2,k'}$. By Claims 1, 3 and 4, we have $b_{1,k}>b_{1,k'}+b_{2,k'}$.

Claim 6 Suppose there exist two markets $k, k' \in \{1, \ldots, 2n+1\}$, $k < k'$, with $b_{2,k}, b_{2,k'} > 0$, then $b_{2,k} < b_{2,k'}$.

This is implied by the proof of Claim 5.

Define $k_1 = \max(k|\, b_{1,k} > 0, b_{2,k} = 0)$ and $k_2 = \max(k|\, b_{1,k} = 0, b_{2,k} > 0)$. If k_0 exists, by definition, $k_0 = k_1 + 1 = k_2 - 1$; if k_0 does not exist, $k_2 = k_1 + 1$.

Claim 7 $\mu_1 < \mu_2$.

Suppose $\mu_1 \geq \mu_2$. This implies that firm 1's spending on rent seeking is no more than firm 2's on every market. As a result, $\frac{\partial \pi_1}{\partial b_{1,k}} \leq \frac{\partial \pi_1}{\partial b_{1,2n+1-(k-1)}}$. This implies that $k_1 \leq 2n+1$, i.e., firm 1 makes productive investments on a smaller number of markets than firm 2. At the same time, $v_{1,k} \leq v_{2,2n+1-(k-1)}$ must hold to make sure that firm 1 invests for $k \leq k_1$.

These facts imply that firm 1 spend less than firm 2 on both rent seeking and productive investment, which contradict the fact that firm 1 has a bigger budget, since a firm in this game has no reason to leave resource unused.

Claim 8 $k_1 \geq 2n+1$

Given $\mu_1 < \mu_2$, the claim is self-evident by the same argument that proves Claim 7.

Assume that k_0 does not exist. We have for $k \leq k_1$,

$$\frac{[1 - \frac{t(2n+1-k)}{2n+1}]\mu_1}{\left\{[1 - \frac{t(k-1)}{2n+1}]\mu_2 + [1 - \frac{t(2n+1-k)}{2n+1}]\mu_1\right\}} 2b_{1,k} = x_{1,k} = \frac{\mu_2}{\mu_1} x_{2,k};$$

$$\frac{\left\{[1 - \frac{t(k-1)}{2n+1}]\mu_2\right\}^2}{4\left\{[1 - \frac{t(k-1)}{2n+1}]\mu_2 + [1 - \frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^2} = b_{1,k}\mu_1.$$

Hence, for $k \leq k_1$,

$$b_{1,k} = \frac{\left\{[1 - \frac{t(k-1)}{2n+1}]\mu_2\right\}^2}{4\mu_1\left\{[1 - \frac{t(k-1)}{2n+1}]\mu_2 + [1 - \frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^2},$$

$$x_{1,k} = \frac{\left\{[1 - \frac{t(k-1)}{2n+1}]\mu_2\right\}^2 \left\{[1 - \frac{t(2n+1-k)}{2n+1}]\mu_1\right\}}{2\mu_1\left\{[1 - \frac{t(k-1)}{2n+1}]\mu_2 + [1 - \frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^3},$$

$$x_{2,k} = \frac{\left\{[1 - \frac{t(k-1)}{2n+1}]\mu_2\right\}^2 \left\{[1 - \frac{t(2n+1-k)}{2n+1}]\mu_1\right\}}{2\mu_1\mu_2\left\{[1 - \frac{t(k-1)}{2n+1}]\mu_2 + [1 - \frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^3}$$

For $k \geq k_2$,

$$\frac{[1-\frac{t(k-1)}{2n+1}]\mu_2}{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2+[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}}2b_{2,k}=x_{2,k};$$

$$\frac{\left\{[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^2}{4\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2+[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^2}=b_{2,k}\mu_2.$$

Hence, for $k \geq k_2$,

$$b_{2,k}=\frac{\left\{[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^2}{4\mu_2\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2+[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^2},$$

$$x_{2,k}=\frac{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2\right\}\left\{[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^2}{2\mu_2\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2+[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^3},$$

$$x_{1,k}=\frac{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2\right\}\left\{[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^2}{2\mu_1\mu_2\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2+[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^3}$$

The equilibrium is determined by the following equations:

$$
\begin{aligned}
&\textstyle\sum_{k=1}^{k_1} b_{1,k} + \sum_{k=1}^{k_1} x_{1,k} + \sum_{k=k_2}^{2n+1} x_{1,k} \\
&= \frac{1}{4\mu_1} \textstyle\sum_{k=1}^{k_1} \frac{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2\right\}^2}{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2 + [1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^2} \\
&+ \frac{1}{2\mu_1} \textstyle\sum_{k=1}^{k_1} \frac{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2\right\}^2 \left\{[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}}{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2 + [1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^3} \\
&+ \frac{1}{2\mu_1\mu_2} \textstyle\sum_{k=k_2}^{2n+1} \frac{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2\right\} \left\{[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^2}{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2 + [1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^3} \\
&= m_1; \\
&\textstyle\sum_{k=k_2}^{2n+1} b_{2,k} + \sum_{k=1}^{k_1} x_{2,k} + \sum_{k=k_2}^{2n+1} x_{2,k} \\
&= \frac{1}{4\mu_2} \textstyle\sum_{k=k_2}^{2n+1} \frac{\left\{[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^2}{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2 + [1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^2} \\
&+ \frac{1}{2\mu_2} \textstyle\sum_{k=k_2}^{2n+1} \frac{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2\right\} \left\{[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^2}{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2 + [1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^3} \\
&+ \frac{1}{2\mu_1\mu_2} \textstyle\sum_{k=1}^{k_1} \frac{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2\right\}^2 \left\{[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}}{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2 + [1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^3} \\
&= m_2.
\end{aligned}
$$

Proofs of Proposition 9 and Corollary 3

The Kuhn-Tucker conditions are

$$
\begin{aligned}
\frac{\partial \pi_1}{\partial x_{1,k}} &= \frac{\partial \pi_2}{\partial x_{2,k}} = \frac{[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}]}{\left\{[1-\frac{t(k-1)}{2n+1}] + [1-\frac{t(2n+1-k)}{2n+1}]\right\}^2 x_k} \sqrt{\tilde{b}_{1,k} + \tilde{b}_{2,k}} = \tilde{\mu}; \\
\frac{\partial \pi_1}{\partial b_{1,k}} &= \frac{[1-\frac{t(k-1)}{2n+1}]}{[1-\frac{t(k-1)}{2n+1}] + [1-\frac{t(2n+1-k)}{2n+1}]} \frac{[1-\frac{d(k-1)}{2n+1}]}{2\sqrt{\tilde{b}_{1,k} + \tilde{b}_{2,k}}} \leq \tilde{\mu}; \\
\frac{\partial \pi_1}{\partial b_{2,k}} &= \frac{[1-\frac{t(2n+1-k)}{2n+1}]}{[1-\frac{t(k-1)}{2n+1}] + [1-\frac{t(2n+1-k)}{2n+1}]} \frac{[1-\frac{d(2n+1-k)}{2n+1}}{2\sqrt{\tilde{b}_{1,k} + \tilde{b}_{2,k}}} \leq \tilde{\mu}.
\end{aligned}
$$

Brcause of symmetry, the proof focuses on $k \leq n+1$. In the equilibrium, a firm's rent-seeking expenditures are positive everywhere. Firms' rent-seeking expenditures are the same on each segment, which we

denote by x_k. As in the baseline model, only one firm invests in each segment except the midpoint $n+1$. Let $v_k = \sqrt{\tilde{b}_{1,k} + \tilde{b}_{2,k}}$, which indicates the rent on a segment k. The conditions lead to

$$x_k = \frac{[1 - \frac{t(k-1)}{2n+1}][1 - \frac{t(2n+1-k)}{2n+1}]}{\left\{[1 - \frac{t(k-1)}{2n+1}] + [1 - \frac{t(2n+1-k)}{2n+1}]\right\}^2 \tilde{\mu}} v_k,$$

$$\text{and } v_k = \frac{[1 - \frac{t(k-1)}{2n+1}][1 - \frac{d(k-1)}{2n+1}]}{2\left\{[1 - \frac{t(k-1)}{2n+1}] + [1 - \frac{t(2n+1-k)}{2n+1}]\right\} \tilde{\mu}}, k \leq n+1.$$

The rent-dissipation rate $x_k/v_k = \frac{[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}]}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}^2 \tilde{\mu}}$. Because $[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]=2-\frac{2tn}{2n+1}$, the denominator is indepent of k. It is straightforward to verify that the numerator $[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}]$ strictly decreases with k for $k \leq n+1$, and increases for $k \geq n+1$.

Clearly, v_k must strictly decrease with k for $k \leq n+1$, and strictly increase for $k \geq n+1$. To see that,

$$\frac{[1 - \frac{t(k-1)}{2n+1}][1 - \frac{d(k-1)}{2n+1}]}{2\left\{[1 - \frac{t(k-1)}{2n+1}] + [1 - \frac{t(2n+1-k)}{2n+1}]\right\} \tilde{\mu}} = \frac{[1 - \frac{t(k-1)}{2n+1}][1 - \frac{d(k-1)}{2n+1}]}{2\frac{2(2n+1)-2tn}{2n+1} \tilde{\mu}}$$

$$= \frac{[1 - \frac{t(k-1)}{2n+1}][1 - \frac{d(k-1)}{2n+1}]}{4\frac{(2n+1)-tn}{2n+1} \tilde{\mu}},$$

which strictly decreases with t. However, as in the baseline model, rent-dissipation rate strictly increases with k for $k \leq n+1$, and strictly decreases for $k \geq n+1$.

We have yet to verify that $b_{i,k}$ strictly decreases with k for $k \leq n+1$. We consider two arbitrary consecutive segments, $k, k+1$ with $k < n$. The equilibrium requires

$$v_k = \frac{[1 - \frac{t(k-1)}{2n+1}][1 - \frac{d(k-1)}{2n+1}]}{2\left\{[1 - \frac{t(k-1)}{2n+1}] + [1 - \frac{t(2n+1-k)}{2n+1}]\right\} \tilde{\mu}};$$

$$v_{k+1} = \frac{[1 - \frac{tk}{2n+1}][1 - \frac{dk}{2n+1}]}{2\left\{[1 - \frac{tk}{2n+1}] + [1 - \frac{t(2n-k)}{2n+1}]\right\} \tilde{\mu}}.$$

These can be rewritten as

$$\sqrt{[1 - \frac{d(k-1)}{2n+1}]b_{1,k}} = \frac{[1 - \frac{t(k-1)}{2n+1}][1 - \frac{d(k-1)}{2n+1}]}{2\left\{[1 - \frac{t(k-1)}{2n+1}] + [1 - \frac{t(2n+1-k)}{2n+1}]\right\} \tilde{\mu}};$$

$$\sqrt{[1 - \frac{dk}{2n+1}]b_{1,k+1}} = \frac{[1 - \frac{tk}{2n+1}][1 - \frac{dk}{2n+1}]}{2\left\{[1 - \frac{tk}{2n+1}] + [1 - \frac{t(2n-k)}{2n+1}]\right\} \tilde{\mu}}.$$

Thus,

$$b_{1,k} = [1 - \frac{d(k-1)}{2n+1}] \left\{ \frac{[1 - \frac{t(k-1)}{2n+1}]}{2\left\{[1 - \frac{t(k-1)}{2n+1}] + [1 - \frac{t(2n+1-k)}{2n+1}]\right\}\tilde{\mu}} \right\}^2;$$

$$b_{1,k+1} = [1 - \frac{dk}{2n+1}] \left\{ \frac{[1 - \frac{tk}{2n+1}]}{2\left\{[1 - \frac{tk}{2n+1}] + [1 - \frac{t(2n-k)}{2n+1}]\right\}\tilde{\mu}} \right\}^2.$$

Because $\left\{[1 - \frac{t(k-1)}{2n+1}] + [1 - \frac{t(2n+1-k)}{2n+1}]\right\} = \left\{[1 - \frac{tk}{2n+1}] + [1 - \frac{t(2n-k)}{2n+1}]\right\}$, and $[1 - \frac{t(k-1)}{2n+1}] > [1 - \frac{tk}{2n+1}]$, $b_{1,k} < b_{1,k+1}$.

We then consider the case $k = n$. Consider the two segments $n, n+1$. Because at midpoint, $b_{1,n+1} = b_{2,n+1}$, we have

$$\sqrt{[1 - \frac{d(n-1)}{2n+1}]b_{1,n}} = \frac{[1 - \frac{t(n-1)}{2n+1}][1 - \frac{d(n-1)}{2n+1}]}{2\left\{[1 - \frac{t(n-1)}{2n+1}] + [1 - \frac{t(n+1)}{2n+1}]\right\}\tilde{\mu}};$$

$$\sqrt{2[1 - \frac{dn}{2n+1}]b_{1,n+1}} = \frac{[1 - \frac{tn}{2n+1}][1 - \frac{dn}{2n+1}]}{2\left\{[1 - \frac{tn}{2n+1}] + [1 - \frac{tn}{2n+1}]\right\}\tilde{\mu}}.$$

These lead to

$$b_{1,n} = [1 - \frac{d(n-1)}{2n+1}] \left\{ \frac{[1 - \frac{t(n-1)}{2n+1}]}{2\left\{[1 - \frac{t(n-1)}{2n+1}] + [1 - \frac{t(n+1)}{2n+1}]\right\}\tilde{\mu}} \right\}^2;$$

$$2b_{1,n+1} = [1 - \frac{dn}{2n+1}] \left\{ \frac{[1 - \frac{tn}{2n+1}]}{2\left\{[1 - \frac{tn}{2n+1}] + [1 - \frac{tn}{2n+1}]\right\}\tilde{\mu}} \right\}^2.$$

It is obvious to see $b_{1,n} > 2b_{1,n+1}$.

We now calculate the sum of firms's rent-seeking expenditures. By symmetry, the sum can be given by

$$2\sum_{k=1}^{2n+1} x_k = 4\sum_{k=1}^{n} x_k + 2x_{n+1}$$

For market segment 1 to n,

$$\begin{aligned}\sum_{k=1}^{n} x_k &= \sum_{k=1}^{n} \frac{[1 - \frac{t(k-1)}{2n+1}][1 - \frac{t(2n+1-k)}{2n+1}]}{\left\{[1 - \frac{t(k-1)}{2n+1}] + [1 - \frac{t(2n+1-k)}{2n+1}]\right\}^2 \tilde{\mu}} v_k \\ &= \sum_{k=1}^{n} \frac{[1 - \frac{t(k-1)}{2n+1}]^2[1 - \frac{t(2n+1-k)}{2n+1}][1 - \frac{d(k-1)}{2n+1}]}{2\left\{[1 - \frac{t(k-1)}{2n+1}] + [1 - \frac{t(2n+1-k)}{2n+1}]\right\}^3 \tilde{\mu}^2}.\end{aligned}$$

For market segment $n+1$,

$$x_{n+1} = \frac{[1-\frac{tn}{2n+1}][1-\frac{tn}{2n+1}]}{\left\{[1-\frac{tn}{2n+1}]+[1-\frac{tn}{2n+1}]\right\}^2 \tilde{\mu}} \frac{[1-\frac{tn}{2n+1}][1-\frac{dn}{2n+1}]}{2\left\{[1-\frac{tn}{2n+1}]+[1-\frac{tn}{2n+1}]\right\}\tilde{\mu}}$$
$$= \frac{[1-\frac{dn}{2n+1}]}{16\tilde{\mu}^2}.$$

We then consider firms' productive investments:

$$2\sum_{k=1}^{n} b_{1,k} + 2b_{1,n+1} = 2\sum_{k=1}^{n}[1-\frac{d(k-1)}{2n+1}]\left\{\frac{[1-\frac{t(k-1)}{2n+1}]}{2\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}\tilde{\mu}}\right\}^2$$
$$+\frac{1}{16}[1-\frac{dn}{2n+1}]\frac{1}{\tilde{\mu}^2}.$$

We then have

$$4\sum_{k=1}^{n} x_k + 2x_{n+1} + 2\sum_{k=1}^{n} b_{1,k} + 2b_{1,n+1}$$
$$= 2\sum_{k=1}^{n}\frac{[1-\frac{t(k-1)}{2n+1}]^2[1-\frac{t(2n+1-k)}{2n+1}][1-\frac{d(k-1)}{2n+1}]}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}^3\tilde{\mu}^2}$$
$$+2\sum_{k=1}^{n}[1-\frac{d(k-1)}{2n+1}]\left\{\frac{[1-\frac{t(k-1)}{2n+1}]}{2\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}\tilde{\mu}}\right\}^2 + \frac{[1-\frac{dn}{2n+1}]}{8\tilde{\mu}^2}$$
$$= 2m.$$

Hence, the constant $\tilde{\mu}$ can be solved as

$$\tilde{\mu} = \sqrt{\frac{\sum_{k=1}^{n}[1-\frac{d(k-1)}{2n+1}]\frac{[1-\frac{t(k-1)}{2n+1}]^2}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}}\left\{\frac{[1-\frac{t(2n+1-k)}{2n+1}]}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}^2}+1\right\}+\frac{[1-\frac{dn}{2n+1}]}{16}}{m}}.$$