

Online Supplement

Incompatible Risk Choices

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S1 Additional results discussed in the main text

This appendix states and proves additional results discussed in the main text. It uses the utility increments and gap polynomials defined in Appendix D and the polynomial lemmas in Appendix E. For each result, we recall the main-text discussion, then state the result and give its proof.

S1.1 Sharpness of the standard-choice bound

Immediately after Theorem 2, we claim that its bound is tight. The following example shows why: two standard choices can have disjoint α -sets while an increasing utility yields indifference in both over a wealth interval one step shorter than the theorem requires.

Proposition S1 (Sharpness of the standard-choice bound). *For every $k \geq 3$, there are two preferences in length- k binary-lottery-versus-sure-amount choices with disjoint weak α -sets that are jointly rationalizable by a strictly increasing utility over*

$$w, w + 1, \dots, w + k - 3.$$

Thus the sufficient window in Theorem 2 cannot be shortened by one wealth level.

Proof. We construct two decisions with disjoint weak α -sets and a strictly increasing utility that yields indifference in both choices through shift $k - 3$.

Step 1: the decisions. Set $\varepsilon = 1/(4k - 1) \in (0, 1/2)$, and let

$$a = \frac{1 + \varepsilon}{k}, \quad b = \frac{2 - \varepsilon}{k}.$$

The first decision selects \$1 for sure over a gamble paying k with probability a ; the second selects the $0/k$ gamble with probability b over \$2 for sure. Their gap

polynomials are

$$P(z) = a(z + z^2 + \cdots + z^k) - z, \quad (\text{S1})$$

$$Q(z) = (z + z^2) - b(z + z^2 + \cdots + z^k). \quad (\text{S2})$$

Each decision gives up exactly ε in expected value: at $t = 1$ the first gamble is worth $ak = 1 + \varepsilon$ and the second $bk = 2 - \varepsilon$, so rejecting the first demands concavity while accepting the second demands convexity. Both choices are standard because $0 < a, b < 1$ makes the coefficients of P/z change sign once from negative to positive and those of Q/z once from positive to negative. If $t \geq 1$, then $t + \cdots + t^k \geq kt$, so $P(t) \geq (ak - 1)t = \varepsilon t > 0$; if $0 < t \leq 1$, then $t + \cdots + t^k \leq t + (k - 1)t^2$, so

$$Q(t) \geq t[(1 - t)(1 - b) + \varepsilon t] > 0.$$

Thus $\mathcal{T}(P) \subset (0, 1)$ and $\mathcal{T}(Q) \subset (1, \infty)$: the weak α -sets are disjoint, and each is nonempty since P is negative near zero and Q is negative for large t .

Step 2: solve the indifference equations. Write

$$W_i = \sum_{j=1}^k \Delta_{i+j}$$

for the sum of the k utility increments starting at $i + 1$. By the utility-gap identity, exact indifference at shift i reads

$$aW_i = \Delta_{i+1} \quad \text{and} \quad \Delta_{i+1} + \Delta_{i+2} = bW_i.$$

We impose both equations for $i = 0, \dots, k - 3$. Substituting the first equation into the second gives

$$\Delta_{i+2} = (b - a)W_i = \rho \Delta_{i+1}, \quad \rho = \frac{b}{a} - 1 = \frac{1 - 2\varepsilon}{1 + \varepsilon} \in (0, 1),$$

so the first $k - 1$ increments are geometric. Normalizing $\Delta_1 = 1$ gives

$$\Delta_n = \rho^{n-1} \quad (1 \leq n \leq k - 1), \quad W_i = \frac{\Delta_{i+1}}{a} = C\rho^i, \quad C = \frac{1}{a}.$$

The equation $W_0 = C$ determines Δ_k :

$$\Delta_k = C - \sum_{n=0}^{k-2} \rho^n.$$

The identity $W_{i+1} = W_i - \Delta_{i+1} + \Delta_{i+k+1}$ determines the remaining increments:

$$\Delta_{i+k+1} = \Delta_{i+1} + C\rho^i(\rho - 1) = \tau\rho^i \quad (0 \leq i \leq k-4), \quad \tau = 1 + C(\rho - 1).$$

This defines $\Delta_1, \dots, \Delta_{2k-3}$. Set $\Delta_n = 1$ outside this range. By construction, the utility yields exact indifference in both choices at shifts $0, \dots, k-3$.

Step 3: check that the increments are positive. Our choice of ε gives

$$a = \frac{4}{4k-1}, \quad C = k - \frac{1}{4}, \quad \rho = \frac{4k-3}{4k}, \quad \tau = \frac{4k+3}{16k}.$$

Since $0 < \rho < 1$ and $\tau > 0$, the geometric increments before and after Δ_k are positive. Also,

$$\Delta_k = C - \sum_{n=0}^{k-2} \rho^n \geq C - (k-1) = \frac{3}{4} > 0.$$

Summing the increments produces a strictly increasing utility that yields exact indifference in both choices at every shift $0, \dots, k-3$. Translating the indices gives the claimed interval from any base wealth w , even though the two α -sets are disjoint. $\square$

S1.2 Why complex choices have no length-only bound

The Introduction notes that choice lengths alone cannot bound the wealth interval needed for complex choices. We show this by keeping both choices at length four while making the required interval arbitrarily long.

Proposition S2 (Fixed length does not bound the incompatibility window). *For every integer $N \geq 0$, there are two weak preferences in complex choices of length four, supported on the common grid $\{0, 1, 2, 3, 4\}$, whose weak α -sets are nonempty and disjoint but which are jointly rationalizable by a strictly increasing utility at every wealth level*

$$w, w+1, \dots, w+N.$$

To construct the example, we choose gap polynomials and turn them into lottery choices. The following lemma lets us do that.

Lemma S1 (Realizing a gap polynomial as lotteries). *Let $C(z) = \sum_{j=1}^k c_j z^j$ be any nonzero real polynomial with zero constant coefficient and degree at most k . After multiplication by a strictly positive constant, C is the gap polynomial of an observed weak preference between two lotteries supported on $\{0, 1, \dots, k\}$.*

Proof. Set

$$b_0 = -c_1, \quad b_i = c_i - c_{i+1} \quad (1 \leq i < k), \quad b_k = c_k.$$

These numbers sum to zero and satisfy

$$\sum_{i=j}^k b_i = c_j.$$

Let $b_i^+ = \max\{b_i, 0\}$, $b_i^- = \max\{-b_i, 0\}$, and $s = \sum_i b_i^+ = \sum_i b_i^- > 0$. Choose $0 < \lambda \leq 1/s$ and any probability vector q . Then

$$p_i^A = \lambda b_i^+ + (1 - \lambda s)q_i, \quad p_i^B = \lambda b_i^- + (1 - \lambda s)q_i$$

are probability vectors and $p_i^A - p_i^B = \lambda b_i$. Their gap polynomial is λC . Positive scaling changes neither its sign set nor any of the associated preference inequalities. $\square$

Proof of Proposition S2. Set $m = N + 4$, $\theta = \pi/m$, and

$$R_\theta(z) = z^2 - 2 \cos(\theta)z + 1.$$

For every real t ,

$$R_\theta(t) = (t - \cos \theta)^2 + \sin^2 \theta > 0,$$

because $0 < \theta < \pi$. Its roots are $e^{\pm i\theta}$. Define

$$P_\theta(z) = z(z-1)R_\theta(z), \tag{S3}$$

$$Q_\theta(z) = z(2-z)R_\theta(z). \tag{S4}$$

Since $R_\theta(t) > 0$ on the positive line,

$$\mathcal{T}(P_\theta) = (0, 1], \quad \mathcal{T}(Q_\theta) = [2, \infty).$$

The two sets are nonempty and disjoint. Since $m \geq 4$, we have $\cos \theta > 0$, and their coefficient signs alternate three times. These are complex choices. By Lemma S1, positive multiples of the two polynomials are gap polynomials of preferences in length-four lottery choices on the common five-point support.

We now construct an increasing utility that rationalizes both decisions through shift N . On the relevant block, set

$$\Delta_n = \sin \left(\left(n - \frac{1}{2} \right) \theta \right), \quad n = 1, \dots, m,$$

and set $\Delta_n = 1$ outside that block. Every increment is strictly positive because the displayed angles lie strictly between 0 and π . The sine sequence therefore gives $m = \pi/\theta$ consecutive positive increments. The elementary sine identity gives

$$\Delta_n - 2 \cos(\theta) \Delta_{n+1} + \Delta_{n+2} = 0, \quad n = 1, \dots, m-2. \tag{S5}$$

Let $T_\theta(z) = zR_\theta(z)$. Equation (S5) says

$$[T_\theta(S)\Delta]_j = 0 \quad \text{for } j = 0, \dots, m-3.$$

Because

$$P_\theta = (z-1)T_\theta, \quad Q_\theta = (2-z)T_\theta,$$

both $P_\theta(S)\Delta$ and $Q_\theta(S)\Delta$ equal zero at shifts $0, \dots, m-4 = N$. Summing the positive increments produces a strictly increasing utility that yields indifference in both choices throughout the claimed window. Translating the construction gives the same result from any base wealth w .

For this example, we can also bound the required wealth span from above. Define

$$A_m(z) = \sum_{j=0}^{m-2} \frac{\sin((j+1)\theta)}{\sin \theta} z^j.$$

All its coefficients are positive, and the same sine recurrence gives

$$A_m(z)R_\theta(z) = 1 + z^m.$$

Since $P_\theta + Q_\theta = zR_\theta$,

$$A_m P_\theta + A_m Q_\theta = z(1 + z^m).$$

This is a positive certificate with multiplier degree $m - 2$. Thus the same pair that remains rationalizable through shift $m - 4$ cannot be jointly rationalized through shift $m - 2$. Thus, for this family, the smallest span that rules out rationalization is between $m - 3$ and $m - 2$, where $m = \pi/\theta$.

The realization lemma may scale the two gap polynomials by different positive constants, say λ_P and λ_Q . Nothing changes: the zero-gap construction is preserved by positive scaling, and the last certificate uses multipliers A_m/λ_P and A_m/λ_Q for the literal lottery pair. $\square$

An explicit lottery example. Here is another example with the lotteries written out. The first decision is to prefer $(\frac{1}{3} : 0, \frac{2}{3} : 2)$ to $(\frac{5}{6} : 1, \frac{1}{6} : 3)$. The second is to prefer B_δ to A_δ , where

$$\begin{aligned} B_\delta &= (1 - \delta)(\tfrac{1}{3} : 0, \tfrac{23}{36} : 2, \tfrac{1}{36} : 4) + \delta(\tfrac{3}{4} : 0, \tfrac{1}{4} : 2), \\ A_\delta &= (1 - \delta)(\tfrac{7}{9} : 1, \tfrac{2}{9} : 3) + \delta[1], \end{aligned}$$

$[1]$ is a sure dollar, and $0 < \delta < 1$. The first choice stays fixed; only the second changes with δ .

Using the convention that a nonpositive gap supports the preferred option, the gap polynomials are

$$P(t) = \frac{t(t-1)(t-2)}{6}, \quad Q_\delta(t) = \frac{t(3-t)}{36} [(1-\delta)(t-2)^2 + 9\delta].$$

For $t > 0$, the first is nonpositive exactly when $1 \leq t \leq 2$, and the second exactly when $t \geq 3$. Since $t = e^{-\alpha}$, the two α -sets are

$$[-\ln 2, 0] \quad \text{and} \quad (-\infty, -\ln 3].$$

They are disjoint and do not change as δ falls. Nevertheless, $P(2) = 0$ and $Q_\delta(2) = \delta/2$: the CARA utility at $\alpha = -\ln 2$ gives indifference in the first choice and comes increasingly close to indifference in the second. The gap in the α -sets does not capture how small this utility difference becomes.

To see how an increasing utility can support both decisions, note that Q_δ has roots $2 \pm i\varepsilon$, where $\varepsilon = 3\sqrt{\delta/(1-\delta)}$. These roots give the utility increments

$$\Delta_n = \rho^n \cos(n\theta), \quad \rho = \sqrt{4 + \varepsilon^2}, \quad \theta = \arctan(\varepsilon/2).$$

As δ falls, ρ approaches 2 and θ approaches zero, so the cosine changes more slowly with wealth. For any desired wealth span N , choose δ small enough that $(N + 4)\theta < \pi/2$. Then all the increments needed for decisions at wealth levels $0, \dots, N$ are positive. Set the increments outside this block to any positive values.

Substitution gives, for $j = 0, \dots, N$,

$$[Q_\delta(S)\Delta]_j = 0, \quad [P(S)\Delta]_j = -\frac{\varepsilon\rho^{j+1}}{6} \left[\varepsilon \cos((j+1)\theta) + \sin((j+1)\theta) \right] < 0.$$

Thus the person is indifferent in the second choice and strictly prefers the observed option in the first. Summing the positive increments gives a strictly increasing utility that rationalizes both decisions throughout the interval.

The interval can therefore be made arbitrarily long while the choice lengths and α -sets stay fixed. The margin defined below approaches zero: at $t = 2$, one gap is zero and the other tends to zero, and the coefficient normalizations have nonzero limits.

S1.3 Long windows require small margins

The discussion following Figure 1 states that, when choice lengths are bounded, arbitrarily long wealth intervals require margins approaching zero. The margin measures how close the best-fitting CARA utility comes to rationalizing both decisions. We define it precisely and prove the claim here.

As in the main proof, write the preferences as $[P(S)\Delta]_w \leq 0$ and $[Q(S)\Delta]_w \leq 0$, where every utility increment is positive. Let $W(P, Q)$ be the smallest N for which no positive increment sequence satisfies both inequalities at every shift $0, \dots, N$. Recall that a certificate $AP + BQ = R \neq 0$, where A, B, R have nonnegative coefficients and $\deg A, \deg B \leq N$, uses only these observed shifts. Adding the inequalities would require a positive weighted sum of positive utility increments to be weakly negative. Thus $W(P, Q) \leq N$: to bound the required interval, it suffices to bound the degrees of the two multipliers.

The margin requires a normalization, since multiplying a gap polynomial by a positive constant changes its size without changing any preference. First divide each nonzero polynomial by its largest absolute coefficient. For $H \in \{P, Q\}$, write $H(t) = \sum_{j=v_H}^{d_H} h_j t^j$, with $h_{v_H}, h_{d_H} \neq 0$, and set

$$p_H(x) = \sum_{j=v_H}^{d_H} h_j x^{j-v_H} (1-x)^{d_H-j}.$$

This is the change of variables $x = t/(1+t)$ used in the positive-translation proof, after removing the lowest power of t . It preserves signs on $(0, 1)$ and puts the extreme-curvature limits at the endpoints. Write $p = p_P$ and $q = p_Q$, and define

$$\mu(P, Q) = \min_{x \in [0,1]} \max\{p(x), q(x)\}. \quad (\text{S6})$$

At each parameter, the maximum is the larger of the two normalized gaps. The minimum finds the parameter at which that larger gap is smallest.

Proposition S3 (Long windows require small margins). *For nonzero gap polynomials normalized as above, and every maximal degree k and $\mu_0 > 0$, there is a finite bound $F(k, \mu_0)$ such that*

$$\deg P, \deg Q \leq k, \quad \mu(P, Q) \geq \mu_0 \quad \implies \quad W(P, Q) \leq F(k, \mu_0).$$

Consequently, any sequence of incompatible pairs with bounded degrees and incompatibility windows tending to infinity must have margins tending to zero.

Proof. We use the certificate construction from Appendix E. The additional point is that the degrees can be chosen using only k and μ_0 .

At each parameter, at least one gap is at least μ_0 . Write $p_+ = \max\{p, 0\}$ and similarly for q_+ . Using the positive gaps as weights gives

$$p_+p + q_+q \geq \mu_0^2.$$

This uses only the inequalities that are violated at a given parameter. We approximate the weights by polynomials so that they can be used in a certificate.

Normalization and the degree bound control the sizes and Lipschitz constants of p, q , and hence of their positive parts, using k alone. Bernstein approximation therefore gives polynomials a, b with nonnegative Bernstein coefficients such that $f = ap + bq \geq \mu_0^2/2$ on $[0, 1]$. The approximation error for a function g is at most $\text{Lip}(g)/(2\sqrt{n})$, so one degree n works for every pair under consideration.

The degrees and coefficients of these polynomials f are now bounded in terms of k, μ_0 , and each f stays at least $\mu_0^2/2$ above zero. The coefficient estimate in the proof of Lemma A6 therefore gives one common degree at which all their Bernstein coefficients are positive. Choose this degree also to be at least $n + D$, where $D = \max\{d_P - v_P, d_Q - v_Q\} \leq k$ is the common compactification degree. Returning to t gives certificates $AP + BQ = R$, where A, B, R have nonnegative coefficients and $R \neq 0$. The additional degree needed to restore the removed terms and match degrees is at most $2k$. Thus the multiplier degrees are bounded using only k, μ_0 . The certificate argument above then gives $F(k, \mu_0)$.

For incompatible pairs, $\max\{p, q\} > 0$ on $(0, 1)$ and is nonnegative at the endpoints by continuity, so $\mu \geq 0$. If the windows grew without bound but the margins did not approach zero, infinitely many pairs would have margins above some fixed $\mu_0 > 0$. The uniform bound just proved rules this out. $\square$

Small margins need not produce long windows: Theorem 2 bounds the required interval for standard choices even when their margins are arbitrarily small.

S1.4 Finite exact indifference

At the end of the main-text discussion of wealth spans, we note that the short bound from Theorem 2 extends to exact indifference in arbitrary choices, subject to a restriction on common roots. We define the reduced polynomials used in that restriction and then state and prove the result.

For a choice D of length k_D , let

$$P_D(z) = \sum_{j=1}^{k_D} c_j^D z^j = z R_D(z), \quad R_D(z) = \sum_{j=1}^{k_D} c_j^D z^{j-1}.$$

We call R_D the *reduced recurrence polynomial*. Trimming common tails gives

$$c_1^D = -b_0^D \neq 0, \quad c_{k_D}^D = b_{k_D}^D \neq 0,$$

and hence

$$\deg R_D = k_D - 1. \tag{S7}$$

For exact CARA indifference,

$$\alpha \in \alpha(D_A \sim D_B) \iff R_D(e^{-\alpha}) = 0.$$

This includes risk neutrality: $\alpha = 0$ corresponds to the root $z = 1$.

Proposition S4 (Finite wealth bound: exact indifference). *Consider two choices $L = (L_A, L_B)$ and $G = (G_A, G_B)$, each comparing distinct lotteries, with lengths $k_L, k_G \geq 2$ and the same lowest differing outcome. Suppose their reduced polynomials R_L, R_G have no common negative or nonreal root. For any base wealth w , the following are equivalent:*

- (i) *There is a strictly increasing utility function such that $L_A \sim L_B$ at every wealth level*

$$w, w + 1, \dots, w + k_G - 2$$

and $G_A \sim G_B$ at every wealth level

$$w, w + 1, \dots, w + k_L - 2.$$

- (ii) *The exact CARA-indifference sets intersect:*

$$\alpha(L_A \sim L_B) \cap \alpha(G_A \sim G_B) \neq \emptyset.$$

Proof. An indifference is an equality, so we can combine its wealth-shifted copies with weights of either sign. We use the Bézout identity: two polynomials with no common root can be combined, using polynomial weights, to give the constant 1. The degrees of those weights determine how many shifts of each indifference we need.

If the exact CARA sets intersect, a common α gives a CARA utility that yields indifference in both choices at every wealth level, hence on the two blocks.

Conversely, suppose an increasing utility satisfies the two blocks and, toward a contradiction, that the exact CARA sets are disjoint. Disjointness rules out a common positive real root. The assumption rules out common negative and nonreal roots, and both constant coefficients are nonzero. Thus R_L and R_G are coprime. By the extended Euclidean algorithm and (S7), there are real polynomials A, B with

$$AR_L + BR_G = 1, \quad \deg A \leq k_G - 2, \quad \deg B \leq k_L - 2.$$

Translate the base wealth to zero and write $\Delta_n = u(w + n) - u(w + n - 1) > 0$. Since $P_L = zR_L$, exact indifference in L at the observed shifts reads

$$[R_L(S)\Delta]_{1+j} = 0, \quad j = 0, \dots, k_G - 2, \tag{S8}$$

and similarly

$$[R_G(S)\Delta]_{1+j} = 0, \quad j = 0, \dots, k_L - 2. \tag{S9}$$

The degree bounds ensure that $A(S)$ and $B(S)$ use only the observed shifts. Hence

$$[A(S)R_L(S)\Delta]_1 = [B(S)R_G(S)\Delta]_1 = 0.$$

Applying the Bézout identity gives $\Delta_1 = 0$, contradicting strict monotonicity. The exact CARA sets must intersect. $\square$

For equal-length choices $k_L = k_G = k$, both indifferences need only be observed at

$$w, w + 1, \dots, w + k - 2.$$

This is $k - 1$ consecutive wealth observations, or a wealth span of $k - 2$.

S1.5 Scale consistency for arbitrary choices

The Introduction states that two CRRA-incompatible decisions cannot be rationalized over a sufficiently long scale interval. That claim also covers complex choices. Corollary 4 gives an explicit bound for standard choices with the same first differing outcome. The following result allows arbitrary finite choices with positive outcomes, but only establishes that some finite scale bound exists.

Write sL for the lottery obtained by multiplying every outcome in L by $s > 0$.

Corollary S1 (Scale consistency and a finite scale bound). *Let $L = (L_A, L_B)$ and $G = (G_A, G_B)$ each compare distinct finite lotteries with strictly positive outcomes. There is a finite ratio $R > 1$, depending only on the two choices and the designated preferences, such that the following are equivalent:*

- (i) *the sets $\eta(L_A \succeq L_B)$ and $\eta(G_A \succeq G_B)$ intersect;*
- (ii) *one strictly increasing utility function on $(0, \infty)$ rationalizes $sL_A \succeq sL_B$ and $sG_A \succeq sG_B$ at every scale $s > 0$;*
- (iii) *for every base scale $s_0 > 0$, there exists a strictly increasing utility function on $(0, \infty)$, possibly depending on s_0 , that rationalizes both preferences at every scale $s \in [s_0, Rs_0]$.*

We first establish the approximation result needed for the proof.

As in the proof of Corollary 4, set $u(z) = V(e^z)$, so $V(sy) = u(\ln s + \ln y)$. We use the original positive outcomes, without subtracting the first differing outcome.

For the reference utilities, the same change of variables gives

$$v(e^z; \eta) = r(z; \eta - 1), \tag{S10}$$

using the usual positive-affine normalization of CRRA and the CARA reference utility $r(z; \alpha) = (1 - e^{-\alpha z})/\alpha$, with $r(z; 0) = z$. Thus η belongs to the CRRA set of a decision if and only if $\alpha = \eta - 1$ belongs to the CARA set of its log version. Write $\mathcal{A}(D)$ for the CARA set of the observed preference in a choice D over real outcomes.

The remaining issue is that the log outcomes need not lie on a common lattice. We cannot simply round to the nearest lattice point, since this might reverse a weak preference. Instead, we round in the direction that makes the observed preference easier to satisfy: preferred outcomes go up and alternative outcomes go down. We

then show that sufficiently small changes of this kind preserve disjointness of the CARA sets.

Within each choice, first cancel probability mass common to the two options at the same outcome. These terms contribute equally to both expected utilities. Write the remaining distinct outcomes as $z_0 < \dots < z_m$ and their probability differences as b_i . The expected-utility gap is

$$\text{Gap}_D(u, w) = \sum_{i=0}^m b_i u(w + z_i), \quad b_i \neq 0, \quad \sum_{i=0}^m b_i = 0. \quad (\text{S11})$$

Here b_i is the preferred option's probability minus the alternative's, so the observed preference is $\text{Gap}_D(u, w) \geq 0$. Canceling common mass does not change this inequality. The remaining masses on the two sides can also be divided by their common positive total to obtain two lotteries without changing any ranking.

For an integer $N > 0$, replace each outcome z_i by

$$q_N(z_i, b_i) = \begin{cases} N^{-1} \lceil Nz_i \rceil, & b_i > 0, \\ N^{-1} \lfloor Nz_i \rfloor, & b_i < 0. \end{cases} \quad (\text{S12})$$

This puts every outcome on $N^{-1}\mathbb{Z}$, rounding upward in the preferred option and downward in the alternative. Denote the resulting choice by $D^{(N)}$, retaining the observed preference.

Lemma S2 (Favorable lattice approximation). *Let D and G be two reduced choices between distinct finite lotteries on arbitrary real outcomes, each with its observed weak preference.*

(i) *For every increasing u and every w ,*

$$\text{Gap}_D(u, w) \leq \text{Gap}_{D^{(N)}}(u, w).$$

Consequently, every increasing utility that satisfies the observed weak preference in D also satisfies it in the favorable approximation $D^{(N)}$.

(ii) *For all sufficiently large N , the approximated outcomes remain strictly ordered and distinct.*

(iii) If

$$\mathcal{A}(D) \cap \mathcal{A}(G) = \emptyset,$$

then, for all sufficiently large N ,

$$\mathcal{A}(D^{(N)}) \cap \mathcal{A}(G^{(N)}) = \emptyset.$$

Proof. Part (i) follows from the direction of the rounding. If $b_i > 0$, we raise z_i , so $b_i u(w + z_i)$ weakly increases. If $b_i < 0$, we lower z_i ; utility weakly falls, but multiplying by the negative coefficient reverses the inequality. In both cases,

$$b_i u(w + q_N(z_i, b_i)) \geq b_i u(w + z_i).$$

Adding across outcomes proves the claim. This uses only monotonicity, not continuity of the unknown utility.

For part (ii), each outcome moves by less than $1/N$. The original outcomes are finite in number and distinct, so their minimum separation is positive. Once $2/N$ is smaller than that separation, the rounded outcomes remain in the same order.

For part (iii), let $C_D(\alpha) = \sum_i b_i r(z_i; \alpha)$ be the CARA utility gap. CARA utility is jointly continuous in the outcome and the parameter, including at zero; for example, $r(z; \alpha) = \int_0^z e^{-\alpha s} ds$. Thus, if the rounded outcomes and the parameters both converge, their CARA gaps converge as well. The issue is whether a common parameter for the approximations could become arbitrarily large in either direction.

For very large positive α , the lowest differing outcome determines the ranking. Since $\sum_i b_i = 0$, for $\alpha \neq 0$,

$$C_D(\alpha) = -\frac{1}{\alpha} \sum_i b_i e^{-\alpha z_i}. \tag{S13}$$

Factoring out the lowest-outcome term gives

$$C_D(\alpha) = -\frac{e^{-\alpha z_0}}{\alpha} \left[b_0 + \sum_{i>0} b_i e^{-\alpha(z_i - z_0)} \right].$$

The bracket converges to $b_0 \neq 0$, so for all sufficiently large positive α ,

$$\text{sign } C_D(\alpha) = -\text{sign } b_0. \quad (\text{S14})$$

For very negative α , the highest differing outcome instead determines the ranking. Writing $\alpha = -a$, with $a > 0$,

$$C_D(-a) = \frac{e^{az_m}}{a} \left[b_m + \sum_{i < m} b_i e^{-a(z_m - z_i)} \right],$$

so for all sufficiently negative α ,

$$\text{sign } C_D(\alpha) = \text{sign } b_m. \quad (\text{S15})$$

These conclusions hold with the same curvature cutoffs for every sufficiently fine approximation. The coefficients do not change, and the distances between distinct rounded outcomes remain bounded below by a positive number. The exponential remainder terms in the two displays therefore converge to zero uniformly over those approximations.

Now suppose the original CARA sets are disjoint but arbitrarily fine approximations have a common parameter. Choose $N_j \rightarrow \infty$ and such parameters α_j , so that

$$C_{D^{(N_j)}}(\alpha_j) \geq 0, \quad C_{G^{(N_j)}}(\alpha_j) \geq 0.$$

At least one original decision must disagree with all sufficiently large positive α ; otherwise both sets would contain that entire range and would intersect. The same is true for sufficiently negative α . Since the two extreme rankings are unchanged by fine rounding, the common parameters α_j must stay in a bounded interval, after discarding finitely many terms.

Take a convergent subsequence, $\alpha_j \rightarrow \alpha^*$. The outcomes also converge, so joint continuity gives

$$C_{D^{(N_j)}}(\alpha_j) \rightarrow C_D(\alpha^*), \quad C_{G^{(N_j)}}(\alpha_j) \rightarrow C_G(\alpha^*).$$

Both limits are weakly nonnegative. Thus α^* rationalizes both original decisions,

contradicting disjointness. □

The same argument works on a lattice $c + N^{-1}\mathbb{Z}$ with any fixed starting point c . Each outcome still moves by less than $1/N$ in the favorable direction.

Proof of Corollary S1. If the two η -sets intersect, choose a common CRRA parameter η^* . For $\eta^* \neq 1$, scaling every outcome by s multiplies the expected-utility gap by the positive factor $s^{1-\eta^*}$. For $\eta^* = 1$, it adds the same constant $\ln s$ to both options. Thus the same utility rationalizes both decisions at every scale, and therefore on every finite scale interval. This proves (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii); when (i) holds, any $R > 1$ will do.

It remains to show that disjoint η -sets give some finite ratio R for which (iii) is impossible. Take logs and cancel common probability mass, writing D, G for the resulting choices with their observed preferences. Their CARA sets are disjoint by (S10). Lemma S2 lets us choose N large enough that $D^{(N)}, G^{(N)}$ also have disjoint CARA sets. Any increasing utility rationalizing the original decisions still rationalizes these approximations at the same translations.

Multiply every approximating outcome by N , turning outcomes m/N into integers m . This changes the units, but not whether a common CARA parameter exists, because

$$r(m; \alpha/N) = Nr(m/N; \alpha).$$

In particular, a common integer-grid parameter β would give a common real-unit parameter $\alpha = N\beta$. Corollary A1 therefore supplies an $M \geq 0$ such that the two integer decisions cannot be rationalized at all shifts $0, 1, \dots, M$. If needed, translate all four lotteries by one common integer to make their outcomes nonnegative; this only changes the base wealth in that corollary.

Set $R = \exp((M+1)/N) > 1$. Suppose some strictly increasing V rationalizes both original decisions throughout $[s_0, Rs_0]$. Let $u(z) = V(e^z)$ and $w_0 = \ln s_0$. The log preferences then hold throughout $[w_0, w_0 + (M+1)/N]$, and in particular at $w_0 + j/N$ for $j = 0, \dots, M$. Favorable rounding preserves all of these inequalities.

Define $U_n = u(w_0 + n/N)$ for $n \in \mathbb{Z}$. This is a strictly increasing utility sequence, and an approximating outcome m/N at translation $w_0 + j/N$ is evaluated as

$$u\left(w_0 + \frac{j+m}{N}\right) = U_{j+m}.$$

Thus U rationalizes the two integer decisions at every shift $0, \dots, M$, a contradiction. The R we chose depends only on the decisions and works for every $s_0 > 0$. This proves (iii) $\Rightarrow$ (i). $\square$

S1.6 Any finite number of decisions

Following Theorem 1, we note that the same compatibility test applies to any finite number of decisions. The following corollary states this extension: all the decisions must share a single CARA parameter.

Corollary S2 (Finite-family compatibility). *Any finite family of weak preferences, each between finite-support lotteries on the common monetary lattice, is jointly rationalized at every lattice wealth level by one strictly increasing utility function if and only if all of the weak α -sets have a common point.*

The proof uses the following extension of Lemma A2 to finitely many polynomials.

Proposition S5 (Finite-family positive-translation alternative). *Let $m \geq 1$ and let $P_1, \dots, P_m$ be real polynomials. Exactly one of the following holds:*

(i) *There is a $t > 0$ such that*

$$P_i(t) \leq 0 \quad \text{for every } i = 1, \dots, m.$$

(ii) *There are $A_1, \dots, A_m, R \in \mathbb{R}_+[z]$, with $R \neq 0$, such that*

$$\sum_{i=1}^m A_i(z)P_i(z) = R(z).$$

Proof. The two alternatives cannot both hold, for the same sign reason as in the two-polynomial case. Suppose there is no common weakly nonpositive point. Discard any identically zero polynomials for now; their multipliers will be zero. Normalize the remaining polynomials to common lowest and highest powers as in the proof of Lemma A2, writing $\hat{P}_i = M_i P_i$, and compactify $t > 0$ to $x \in [0, 1]$. This gives ordinary polynomials $p_1, \dots, p_m$ satisfying

$$\max_i p_i(x) > 0 \quad \text{for every } x \in [0, 1].$$

Define the continuous nonnegative weights

$$a_i(x) = \max\{p_i(x), 0\}.$$

At least one $p_i(x)$ is positive at every x , so

$$f(x) = \sum_{i=1}^m a_i(x)p_i(x) = \sum_{i:p_i(x)>0} p_i(x)^2 > 0 \quad (\text{S16})$$

throughout the compact interval. Bernstein-approximate all the a_i 's at one sufficiently large common degree n . Uniform approximation, the strictly positive minimum of f , and finiteness of the family imply

$$\sum_{i=1}^m (\mathcal{B}_n a_i)(x)p_i(x) > 0 \quad \text{on } [0, 1].$$

Each Bernstein approximation has coefficients $a_i(r/n)\binom{n}{r} \geq 0$. Converting back to t , just as in the two-polynomial proof, therefore produces nonnegative-coefficient polynomials $A_{0,i}$ for which

$$H(t) = \sum_i A_{0,i}(t)\hat{P}_i(t) > 0 \quad \text{for every } t > 0.$$

Apply the one-polynomial certificate to H , and absorb both that multiplier and the earlier sign-preserving normalization multipliers into the $A_{0,i}$'s. The result is

$$\sum_i A_i P_i = R,$$

where every A_i and R has nonnegative coefficients and $R \neq 0$. Restore the discarded zero polynomials with multiplier zero. $\square$

Proof of Corollary S2. A common point supplies one CARA utility, so sufficiency is immediate. For necessity, orient every gap polynomial so that its observed preference says $P_i(S)\Delta \leq 0$. If the α -sets had no common point, Proposition S5 would give $\sum_i A_i P_i = R \neq 0$. Nonnegative combinations of the shifted observed inequalities would then imply $R(S)\Delta \leq 0$, while the nonzero nonnegative coefficients of R and the strictly positive increments make $R(S)\Delta > 0$. This is the same contradiction as before. $\square$

In particular, exact indifference at every wealth level is covered without any root restriction: treat each equality as the two opposing weak preferences. The restriction on common negative and nonreal roots in Proposition S4 is needed only to obtain its explicit short finite window.

S2 Experiment Instructions

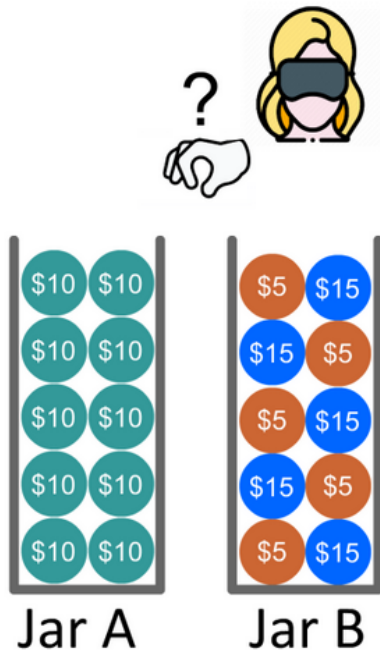
Experiment Instructions (Page 1)

In this study you will be asked to make monetary decisions. These are **real decisions and will determine bonus payments**.

The goal of the study is to determine how you make monetary choices when uncertainty is involved. In every decision:

- You will choose between two Jars: "Jar A" and "Jar B."
- Each Jar will contain balls that give you money.

For example, you could be shown the following two Jars.



Jar A has 10 balls that are all worth \$10. *If you drew a ball out of Jar A at random you would always get \$10.*

Jar B also has 10 total balls, but 5 balls are worth \$5 and the other 5 balls are worth \$15. *If you drew a ball out of Jar B at random you would always get at least \$5, but you would also have a 50% chance of getting \$15.*

In this study, you will have to choose between Jars like the ones above. In particular, you will have to choose which jar you would rather RANDOMLY pick one ball from.

However, the Jars will be shown to you as tables rather than pictures. For example, instead of the picture above you would be shown:

Jar A
<u>100% of Balls</u>
<u>\$10</u>

Jar B	
<u>50% of Balls</u>	<u>50% of Balls</u>
<u>\$5</u>	<u>\$15</u>

The table and picture above say the **exact same thing**. So please pay careful attention to the percentages and amounts of money involved when making your decisions.

Experiment Instructions (Page 2)

Then you will have to decide how much *more* you value your preferred Jar. That is, you will decide what amount of money would need to be taken away from your preferred Jar for you to like it exactly as much as the Jar you liked less.

For example, suppose you said that you would rather draw a ball from Jar A (shown below).

Jar A	
<u>50% of Balls</u>	<u>50% of Balls</u>
<u>\$15</u>	<u>\$25</u>

Jar B	
<u>50% of Balls</u>	<u>50% of Balls</u>
<u>\$10</u>	<u>\$30</u>

Then you would decide how much **money would have to be taken away from the balls in Jar A** for you to like Jar A exactly as much as you like Jar B.

If you said \$2.50 would have to be **SUBTRACTED** from Jar A for you to like it exactly as much as Jar B, you would be saying that you are **equally happy** to draw a ball at random from either of the following Jars:

Jar A - \$2.50	
50% of Balls	50% of Balls
\$12.50	\$22.50

Jar B	
50% of Balls	50% of Balls
\$10	\$30

Please note, that some answers do NOT make sense. For example, if you said that \$5.00 would have to be subtracted from Jar A for you to like it exactly as much as Jar B, you would be saying you are **equally happy** to draw a ball from either of the following Jars:

Jar A - \$5.00	
50% of Balls	50% of Balls
\$10	\$20

Jar B	
50% of Balls	50% of Balls
\$10	\$30

But Jar B gives you \$10 whenever "Jar A - \$5.00" gives you \$10, and Jar B gives you \$30 when "Jar A - \$5.00" gives you \$20. So you should like Jar B better than "Jar A - \$5.00".

Experiment Instructions (Page 3)

We will ask you several questions like the examples on the previous pages. **To make sure the decision is meaningful, there is a chance each choice will determine your bonus.**

You will be paid according to a special set of rules (detailed below) that ensure it is **always in your best interest to answer accurately and honestly.**

It is also in your best interest to tell us an precise amounts: that is, if you would be equally happy with \$4.40, please say \$4.40 rather than \$4.

***Details** (Not tested in comprehension questions): For every 2-part decision you make, there will be a 50% chance that you will be given a random draw from your preferred Jar with no money subtracted (a computer will randomly determine the outcome of a draw). There will also be a 50% chance that the computer will randomly choose a "staying price" for your decision.*

- If the staying price is more than the amount you said would need to be subtracted from your preferred Jar for you to be equally happy to draw from your non-preferred Jar, your potential bonus will be the outcome of a random, computerized draw from your non-preferred Jar.*
- If the staying price is less than the amount you said would need to be subtracted from your preferred Jar to be equally happy to draw from your non-preferred Jar, your potential bonus will be equal to the outcome of a draw from your preferred Jar minus the staying price.*