Interpretation of $\beta$ in log-linear models

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1 Model

Our econometric specification for the relationship between $x$ and $y$ is

$$\log(y) = x\beta + \varepsilon$$

We are interested in the interpretation of $\beta$, specifically, when does $\beta$ mean that a one unit change in $x$ is associated with a 100\% change in $y$?

2 Approximate $\% \Delta y$

As $x_0 \to x_1$, what happens to $y$ in percentage terms? In other words, what can we say about

$$\% \Delta y \equiv \frac{y_1 - y_0}{y_0}$$

in relation to

$$\Delta x \equiv x_1 - x_0$$

(where conventionally we think about $\Delta x = 1$)

Well, let’s start with what we know about $\log y$. We know that for a change $\Delta x$, the corresponding change in $\log y$ is

$$\Delta \log y = \Delta x\beta$$

$$= \log y_1 - \log y_0$$

$$= \log \left( \frac{y_1}{y_0} \right)$$

$$= \log \left( \frac{\Delta y + y_0}{y_0} \right)$$

$$= \log \left( \frac{\Delta y}{y_0} + 1 \right)$$

$$\approx \frac{\Delta y}{y_0} \equiv \% \Delta y$$

Where the approximation (from a Taylor Series expansion around $z = 0$) that $\log(1 + z) \approx z$ for small $z$ was used in the last step.

Thus, a change $\Delta x$ is associated with approximately a 100\% $\Delta x \beta$ percent change in $y$.

3 Exact $\% \Delta y$

First, let’s write down the exact quantity that we want to examine for a given change in $x$
\[ \% \Delta y = \frac{y_1 - y_0}{y_0} \]
\[ = \frac{y_1}{y_0} - 1 \]
\[ = \exp(\log(\frac{y_1}{y_0})) - 1 \]
\[ = \exp(x_1\beta + \varepsilon_1 - x_0\beta - \varepsilon_0) - 1 \]
\[ = \exp(\Delta x\beta + \Delta \varepsilon) - 1 \]

So, if we are interested in the percentage change in \( y \) for a \( \Delta x \) change (e.g., \( \Delta x = 1 \)) in \( x \) (ceteris paribus, holding \( \Delta \varepsilon = 0 \)), then the exact percentage change in \( y \) implied by our log-linear model is

\[ \% \Delta y = \exp(\Delta x\beta) - 1 \]

4 Comparison of log points and percentage points

The approximation in Section 2 used the fact that \( \Delta y/y_0 \) was small, which is likely to be the case for a small quantity \( \Delta x\beta \). However, using the log point change in \( y \) implied by \( \beta \) as the approximation to the percentage point change in \( y \) always gives a biased downward estimate of the exact percentage change in \( y \) associated with \( \Delta x \).

For example, if \( \hat{\beta} = .3 \), then, while the approximation is that a one-unit change in \( x \) is associated with a 30% increase in \( y \), if we actually convert 30 log points to percentage points, the percent change in \( y \)

\[ \% \Delta y = \exp(\hat{\beta}) - 1 = .35 \]

So instead of a 30% increase as suggested by our approximation, the exact percentage increase implied by our estimate is 35%. The approximation is a lower bound.

If \( \hat{\beta} = -.3 \), then a one-unit change in \( x \) is associated with a \( \exp(-.3) - 1 \approx .26 \) or 26% decrease. Again, the approximation was biased downward relative to the exact implied percentage change in \( y \).