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Research Article

Winner-take-all price competition*

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Summary. We analyze an oligopoly model of homogeneous product price competition that allows for discontinuities in demand and/or costs. Conditions under which only zero profit equilibrium outcomes obtain in such settings are provided. We then illustrate through a series of examples that the conditions provided are "tight" in the sense that their relaxation leads to positive profit outcomes.

Keywords and Phrases: Price competition, Discontinuity, Bertrand, Hotelling.

JEL Classification Numbers: D43, C72.

1 Introduction

This paper examines the competitiveness of winner-take-all price competition in homogeneous product oligopoly environments where underlying buyer demands and/or firms' costs need not be continuous. Our analysis is motivated by the observation that a variety of economic settings have these features. For example, in 1996 an Ivy League university solicited bids from several vendors for its initiative to dramatically expand and standardize desktop computer use throughout the university. The extent of this standardization effort at the staff level depended on the unit price of the lowest bid received. In particular, should the price per unit prove too high, then only faculty and administrators would be included in the initiative. If bids were low enough, the initiative would be expanded to cover

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all staff at the university. Thus, vendors bidding for this contract faced a jump in demand once their price fell below this threshold. More generally, discontinuities in demand may be caused by nonconvex preferences, bandwagon effects, lumpiness in consumption, herd behavior, demand network effects, cascades, and a host of other phenomenon.

The university's IT department recognized that it was much less expensive to support identical machines rather than those of mixed types. Dell and Gateway computers with identical specifications are arguably homogeneous products prior to acquisition, but in maintaining existing systems and ensuring that software upgrades work, it is important to adopt *one* "standard" platform. As a consequence, the contract was awarded entirely on a *winner-take-all basis* to the firm offering the lowest price. As noted by Klemperer (2000), many oligopoly environments have a similar auction-like structure.

The university's preference for standardized machines required the winning vendor to ensure that every machine contained identical components, used identical ports, and identically configured drivers. This created potential problems for vendors due to the rapid pace of technological change as well as temporary shortages of components available from subcontractors. Consequently, vendors also faced jumps in costs once the number of units sold exceeded some quantity threshold. Economic reasons for cost discontinuities have been documented as far back as Brems (1952), and include inflexibility in hiring decisions as a result of collective bargaining agreements, imposition of pollution abatement taxes for production beyond a certain scale, lumpiness in production, and congestion effects; Friedman (1972) provides a technical analysis. Indeed, nonconvexities in production are the essence of Milgrom and Roberts' (1990) study of "modern manufacturing."

How competitive are environments such as these? Harrington (1989) provides sufficient conditions for winner-take-all price competition to result in competitive outcomes when demand is continuous and firms enjoy constant returns. In contrast, Dastidar (1995) shows that when cost functions are continuous but strictly convex and demand is divided equally among competing firms in the event of a tie, multiple equilibria (some with positive profits) arise. Dasgupta and Maskin (1986) and Baye, Chen, and Zhou (1993) provide general conditions for the *existence* of equilibrium in games (including winner-take-all price competition) where the underlying payoff functions are discontinuous. However, it is an open question whether winner-take-all price competition leads to zero profit outcomes when identical firms with demand or cost discontinuities compete in a winner-take-all fashion. This paper addresses this question.

Section 2 presents our general model of winner-take-all price competition. Theorem 1 provides necessary and sufficient conditions for zero profit equilibria to exist in the general model, while Theorem 2 identifies conditions sufficient to guarantee the uniqueness of these outcomes. Section 3 illustrates, through a series of examples, that the conditions provided are "tight" in the sense that their relaxation leads to positive profit outcomes. When monopoly payoffs are bounded, only zero profit equilibrium outcomes obtain with discontinuous (but non-increasing) demand. In contrast, the presence of discontinuous (but non-decreasing) costs can result in anticompetitive pricing. We conclude in Section 4.

2 General model and results

A set $N = \{1, 2, ...n\}$ of $n > 1$ identical, risk-neutral firms compete to supply some homogeneous product to a buyer. Let $\pi(p)$ denote the operating profits (that is, profits gross of any unavoidable sunk costs) that a monopolist charging a price $p \in \mathscr{P} \subseteq [0, \infty)$ would earn in this market. At this point we do not place any *a priori* restrictions on demand or costs. Since the game is symmetric, it is natural to restrict attention to the case where all firms have an identical strategy space, $\mathscr{P}$, which we assume is connected. Thus, $\mathscr{P}^n$ denotes the strategy space of the game.

Each firm simultaneously chooses a price, $p_i \in \mathscr{P}$, with the firm charging the lowest price winning a "contract" from the buyer at that price. In the event of a tie for low price, the buyer awards the contract to one of the firms at random.[1] This is in contrast to Dastidar (1995), who studies a class of homogeneous product pricing games where split contracts are awarded in the event of a tie.[2] Thus, if $(p_1, p_2, ..., p_n) \in \mathscr{P}^n$ are the prices chosen by the n firms, the profits of firm i are given by:

$$\pi_i(p_1, p_2, ..., p_n) = \begin{cases} \pi(p_i) & \text{if} \quad p_i < p_j \ \forall j \neq i \\ \frac{1}{m}\pi(p_i) & \text{if} \quad i \text{ ties } m-1 \text{ other firms for low price} \\ 0 & \qquad \text{otherwise} \end{cases} .$$

If we let $\Pi = (\pi_1, \pi_2, ...\pi_n)$ denote the vector of these payoff functions, then a *winner-take-all pricing game* is given by $\Gamma \langle N, \mathscr{P}^n, \Pi \rangle$.

We let Φ be the set of all cumulative distribution functions on $\mathscr{P}$, and Φ^n denote the set of all $n-$*tuples* of such functions. For $F \in \Phi$, let $\mathscr{S}_F$ denote the support of the density associated with F. Thus, $\left(F_1^*, F_2^*, ..., F_n^*\right) \in \Phi^n$ is a mixed strategy Nash equilibrium of Γ if, given the vector of mixed strategies of opponents, F_{-i}^*, firm i's expected profits are no less under F_i^* than under any other strategy $F_i' \in \Phi$; that is,

$$E\pi_i\left(F_i', F_{-i}^*\right) \leq E\pi_i\left(F_i^*, F_{-i}^*\right) \forall i \in N .$$

Notice that if $\mathscr{S}_{F_i^*}$ is a singleton for all $i \in N$, then $\left(F_1^*, F_2^*, ..., F_n^*\right)$ comprises a pure strategy Nash equilibrium.

We now define the analog of Bertrand's paradox for this class of games.

[1] We assume that each firm tied for the lowest price has an equal chance of receiving the contract; however our Theorems 1 and 2 below hold for *all* tie-breaking rules where a single firm receives the entire contract.

[2] When firms do not enjoy constant returns technologies, these differing assumptions give rise to different payoff functions.

Definition 1 *Let Γ be a winner-take-all pricing game.* $\left(F_1^*, F_2^*, ...F_n^*\right) \in \Phi^n$ *is a* zero operating profit equilibrium *if*

a) $\left(F_1^*, F_2^*, ...F_n^*\right)$ *is a Nash equilibrium of Γ, and*
b) $E\pi_i(F_1^*, F_2^*, ...F_n^*) = 0$ *for all* $i \in N$.

This definition generalizes the idea underlying Bertrand's paradox to winner-take-all environments where technologies need not exhibit constant returns. Notice that the original two-firm Bertrand paradox is merely the special case where all probability mass is concentrated at the (constant) marginal cost, c, $\left(\text{i.e. } \mathscr{S}_{F_1^*} = \mathscr{S}_{F_2^*} = c\right)$.

Our first Theorem establishes that a necessary and sufficient condition for the existence of a zero operating profit outcome in winner-take-all pricing games is that there exists a lowest price that a monopolist would have to receive in order to cover its operating costs. We refer to this price as an *initial breakeven price* and formalize the concept in:

Condition B (Initial Breakeven Price) The monopoly profit function has an initial breakeven price; that is, there exists a price $c \in \mathscr{P}$ such that for all $p \in \mathscr{P}$, $p < c$ implies $\pi(p) \leq \pi(c) = 0$.

Theorem 1 *A zero operating profit equilibrium of Γ exists if and only if Condition B holds.*

Proof.

($\Leftarrow$) If Condition B holds, then $\mathscr{S}_{F_i^*} = \{c\}$ for all $i \in N$ comprises a zero operating profit equilibrium.

($\Rightarrow$) If $\left(F_1^*, F_2^*, ...F_n^*\right)$ is a zero operating profit equilibrium, a firm charging p earns operating profits of at least $\frac{1}{n}\pi(p)$ when p is the lowest price charged in equilibrium and zero otherwise. By definition, each firm earns zero expected operating profits under $\left(F_1^*, F_2^*, ...F_n^*\right)$. It follows that $\pi(p) = 0$ for almost all $p \in \mathscr{S}_{G_{\min}}$ where $\mathscr{S}_{G_{\min}}$ is the support of the distribution of the lowest price induced by the equilibrium. Hence, there exists a $p' \in \mathscr{S}_{G_{\min}}$ such that $\pi\left(p'\right) = 0$. This and the hypothesis that $\left(F_1^*, F_2^*, ...F_n^*\right)$ comprises a Nash equilibrium implies that for all $p < p', \pi(p) \leq \pi(p') = 0$; otherwise, a firm could re-allocate probability mass to a $p'' < p'$ where $\pi(p'') > 0$ and earn positive expected operating profits. $\square$

Thus, Condition B is necessary and sufficient for Bertrand's "price undercutting" logic to lead to the zero profit outcome as a Nash equilibrium to a winner-take-all pricing game. The intuition is straightforward: For sufficiency, notice that if the monopoly operating profit function has an initial breakeven price, then there are no gains to undercutting a rival who prices at that level; hence all firms charging the initial breakeven price comprises a Nash equilibrium. For necessity, simply observe that if for a given zero profit equilibrium price, there did exist a lower price at which a monopolist could earn positive operating profits, then a firm could profitably deviate by charging that price. This contradicts the hypothesis that the original prices comprised an equilibrium in the first place.

Thus, a zero profit outcome exists in winner-take-all pricing games in fairly general environments; neither continuity (of demand or costs) nor constant returns to scale are required. When each firm prices at c, where Condition B holds, no firm can gain by undercutting. Finally, note that Theorem 1 also implies that any winner-take-all pricing game having zero profit equilibria must have at least one equilibrium in pure strategies.

In light of Theorem 1, it remains to identify conditions under which *every* Nash equilibrium to a given winner-take-all pricing game is, in fact, a zero profit equilibrium when there are discontinuities in demand and/or cost. Before we present these conditions, we need the following definition.

Definition 2 *A function* $\pi : \mathscr{P} \to \Re$ *is* left lower semicontinuous *if, for all* $p^* \in \mathscr{P}$, $\liminf_{p \uparrow p^*} \pi(p) \geq \pi(p^*)$.

It is a simple matter to show that either of the following is sufficient for $\pi(p)$ to be left lower semicontinuous: (1) $\pi(p)$ is lower semi-continuous,[3] or (2) $\pi(p)$ is continuous but for downward jumps.[4] For instance, the function $F(x)$ in Figure 1 is left lower semicontinuous but not lower semicontinuous.

We are now in a position to state our main result.

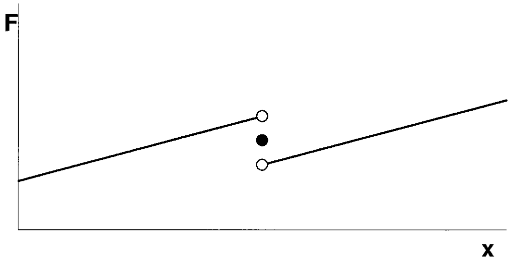

Figure 1. A left lower semicontinuous function

Theorem 2 *Let Γ be a winner-take-all pricing game in which*

a) Condition B holds;

b) $\pi(p)$ is bounded from above; and

c) $\pi(p)$ is left lower semicontinuous.

Then every Nash equilibrium of Γ is a zero operating profit equilibrium.

Proof. By Theorem 1, a zero operating profit equilibrium exists. The following argument establishes (by exhaustion) that every equilibrium is a zero operating profit equilibrium.

Case 1: By way of contradiction, suppose there exists an equilibrium $\left(F_1^*, F_2^*, \ldots, F_n^*\right)$ such that for some firm i

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[3] A function $\pi : \mathscr{P} \to \Re$ is **lower semicontinuous** if, for all $p^* \in \mathscr{P}$, $\liminf_{p \to p^*} \pi(p) \geq \pi(p^*)$.

[4] A function $\pi : \mathscr{P} \to \Re$ is *continuous but for downward jumps* if, for all $p^* \in \mathscr{P}$, $\liminf_{p \uparrow p^*} \pi(p) \geq \pi(p^*) \geq \limsup_{p \downarrow p^*} \pi(p)$; see Milgrom and Roberts (1994).

$$E\pi_i\left(F_i^*, F_{-i}^*\right) = \pi' < 0$$

Then, by Condition B, there exists a price c such that $\pi(c) = 0$. By re-allocating all probability mass to c, firm i earns $E\pi_i(c, F_{-i}^*) = 0$. This contradicts the hypothesis that $\left(F_1^*, F_2^*, ..., F_n^*\right)$ is a Nash equilibrium.

Case 2: By way of contradiction, suppose that there exists an equilibrium $\left(F_1^*, F_2^*, ..., F_n^*\right)$ such that for some firm i

$$E\pi_i\left(F_i^*, F_{-i}^*\right) = \pi' > 0$$

Define:

$$\begin{aligned} \underline{p_i} &= \inf\left\{p_i \in \mathscr{S}_{F_i^*}\right\} \\ \overline{p_i} &= \sup\left\{p_i \in \mathscr{S}_{F_i^*}\right\} \end{aligned}$$

Subcase A: Suppose that $\left(F_1^*, F_2^*, ..., F_n^*\right)$ entails a positive probability that i ties for the lowest price at $\underline{p_i}$. Then $E\pi_i(\underline{p_i}, F_{-i}^*) = \pi'$, and by left lower semi-continuity and Condition B, there exists $p_i' \in \mathscr{P}$ where $p_i' < \underline{p_i}$ such that $E\pi_i\left(p_i', F_{-i}^*\right) > \pi'$. This contradicts the hypothesis that $\left(F_1^*, F_2^*, ..., F_n^*\right)$ is a Nash equilibrium.

Subcase B: Suppose that $\left(F_1^*, F_2^*, ..., F_n^*\right)$ entails a zero probability that i ties for the lowest price at $\underline{p_i}$. Then,

$$\lim_{p \uparrow \overline{p_i}} E\pi_i(p, F_{-i}^*) \equiv \lim_{p \uparrow \overline{p_i}} \prod_{j \neq i} \left(1 - F_j^*(p)\right) \pi(p) = \pi'.$$

Letting $\overline{\pi}$ denote the upper bound on monopoly profits, it follows that

$$\pi' \leq \lim_{p \uparrow \overline{p_i}} \prod_{j \neq i} \left(1 - F_j^*(p)\right) \overline{\pi}.$$

Hence, the hypothesis that $\pi' > 0$, implies that $\lim_{p \uparrow \overline{p_i}} F_j^*(p) < 1$ for all $j \neq i$: The implied equilibrium mixed strategies employed by firms other than i are such that all firms $j \neq i$ allocate positive probability to prices which either have no possibility of winning or, at best, offer some chance of a tie for lowest price at $\overline{p_i}$. Since $\pi(p)$ is left lower semicontinuous, some firm $j \neq i$ can profitably deviate by re-allocating the probability mass from prices at or above $\overline{p_i}$ to prices just below $\overline{p_i}$. This contradicts the hypothesis that $\left(F_1^*, F_2^*, ..., F_n^*\right)$ is a Nash equilibrium. □

Obviously, Condition B is necessary for the existence of a zero operating profit equilibrium (see Theorem 1). The following section presents examples which highlight the role of the other assumptions in Theorem 2.

3 Examples

Our first two examples are based on an environment where two identical, risk-neutral, price-setting firms compete for the demand function, $q = D(p)$, and have cost functions, $C(q)$. The firm setting the lowest price captures the entire market and earns the monopoly profits corresponding to that price:

$$\pi(p) = D(p)\,p - C(D(p)).$$

In the event of a tie, each firm has an equal probability of being awarded the entire contract.

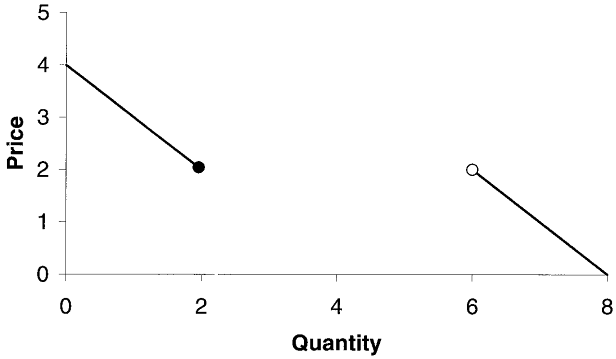

Figure 2. Discontinuous demand

Example 1 (Discontinuous Demand) Two identical firms produce at zero costs and compete for a buyer whose demand function (sketched in Figure 2) is given by

$$D(p) = \begin{cases} 8 - p & \text{if} \quad 0 \le p < 2 \\ 4 - p & \text{if} \quad 2 \le p \le 4 \end{cases}.$$

Each firm's strategy consists of a price $p_i \in [0, 4]$. In this case, the monopoly profit function is given by

$$\pi(p) = \begin{cases} p(8 - p) & \text{if} \quad 0 \le p < 2 \\ p(4 - p) & \text{if} \quad 2 \le p \le 4 \end{cases}.$$

Notice that, as a result of the discontinuity in demand, $\pi(p)$ is discontinuous, as shown in Figure 3. Nonetheless, the conditions of Theorem 2 are satisfied. To see this, note first that Condition B holds (letting $c = 0$, $\pi(c) = 0$; hence the initial breakeven price is zero). Thus, by Theorem 1, we know that a zero operating profit equilibrium exists. Furthermore, note in Figure 3 that $\pi(p)$ is bounded and left lower semicontinuous. By Theorem 2, we can conclude that every Nash equilibrium to the game in Example 1 is, in fact, a zero profit equilibrium. The discontinuous demand in Example 1 does not undermine the traditional price undercutting logic.

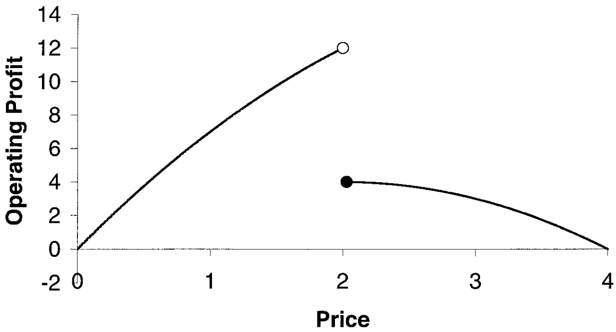

Figure 3. Discontinuous monopoly operating profits due to discontinuous demand

Example 2 (Discontinuous Costs) Two identical firms face a market demand of $D(p) = 4 - p$ and have cost functions given by

$$C(q) = \begin{cases} q & \text{if} \quad 0 \le q \le 2 \\ q + 1.5 & \text{if} \quad 2 < q \le 4 \end{cases}.$$

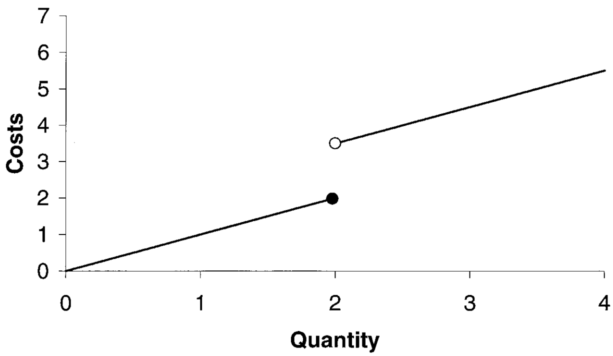

Figure 4. Discontinuous costs

This cost function is illustrated in Figure 4. Each firm's strategy consists of a price $p_i \in [0, 4]$, with the firm charging the lowest price earning

$$\pi(p) = \begin{cases} (p - 1)(4 - p) - 1.5 & \text{if} \quad 0 \le p < 2 \\ (p - 1)(4 - p) & \text{if} \quad 2 \le p \le 4 \end{cases}.$$

Figure 5 illustrates the discontinuity in $\pi(p)$ caused by the discontinuity in costs. It is clear that the monopoly operating profit function satisfies Condition B. Thus, by Theorem 1, we conclude that a zero operating profit equilibrium exists.[5] However, note that the conditions of Theorem 2 fail, as $\pi(p)$ is not left lower semicontinuous. This leaves open the possibility that, in addition to the

[5] In particular, setting $\pi(c) = 0$ and solving yields $c = \frac{5-\sqrt{3}}{2}$. Since $\pi(p) < 0$ for all $p < c$, it follows that $p_1 = p_2 = \frac{5-\sqrt{3}}{2}$ comprises a symmetric zero profit equilibrium.

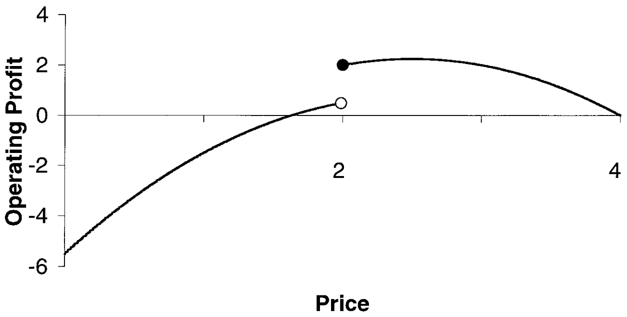

Figure 5. Discontinuous monopoly operating profits due to discontinuous costs

zero profit equilibrium, there also exists a positive profit equilibrium. Indeed, this is the case; one may easily verify that $p_1 = p_2 = 2$ constitutes a Nash equilibrium in which each firm earns expected profits of $\pi_i = 1$.[6] Here, the jump in costs associated with market expansion beyond 2 units undermines the usual undercutting argument.

Examples 1 and 2 suggest that the impact of discontinuities in $\pi(p)$ on winner-take-all price competition critically depends on whether the discontinuities are caused by discontinuities in demand or costs. These inferences are, in fact quite general, as the following remarks reveal.

Remark 1 *Over the range where demand is non-increasing and price exceeds marginal cost, demand discontinuities cause $\pi(p)$ to jump down when the price rises above the point of discontinuity in demand. Thus, discontinuities in demand tend to give rise to a monopoly operating profit function that is left lower semi-continuous. Provided there are no other discontinuities and the profit function is otherwise "well-behaved," the conditions of Theorem 2 will hold. In short, with discontinuous but downward sloping demand, winner-take-all price competition tends to result in zero profit outcomes.*

Remark 2 *If the production possibilities set is closed and monotonic, then the corresponding cost function is lower semi-continuous and non-decreasing in q (see Nadiri, 1982). Provided there are no other discontinuities, it follows that cost discontinuities lead to the* **failure** *of $\pi(p)$ to be left lower semicontinuous. The general conclusion is that zero profit outcomes are by no means assured in the presence of discontinuous costs.*

Thus, with discontinuities in demand, $\pi(p)$ tends to be left lower semicontinuous. In contrast, discontinuities in cost invariably lead to the failure of left lower semicontinuity. Whether such a failure leads to positive profit equilibria depends on the size of the upward jump in costs. In Example 2, for instance, the jump in costs of 1.5 is sufficiently severe to lead to positive profit equilibria. If the jump in costs had been smaller (< 1), positive profit equilibria would not arise. For a

[6] For other winner-take-all tie-breaking rules, one can construct similar examples where the failure of left lower semicontinuity results in positive profit equilibria.

larger (≥ 1) jump in costs, equilibria in which firms earn positive operating profits emerge. Conclusions regarding the competitiveness of winner-take-all pricing games in settings with discontinuous costs are, inevitably, model-specific.

Our final example illustrates the importance of bounded monopoly profits.

Example 3 (Hotelling Models) Consider the Hotelling model presented by Gabszewicz and Thisse (1992) in the *Handbook of Game Theory*. Two firms produce a homogeneous product at zero cost and are located at distances a_i from the endpoints of a line of unit length ($a_1 + a_2 \leq 1$; $a_i \geq 0$). Customers are uniformly distributed over this line and have transportation costs, $T(x) = tx$ to visit a store that is distance x away. Consumers have a perfectly inelastic demand for one unit and buy from the firm offering the product at the lowest overall cost (price + transportation costs). When the two firms set prices $p_i \in [0, \infty)$, their payoffs are:

$$\pi_i(p_i, p_j) = \begin{cases} \left(\frac{1-a_j+a_i}{2}\right) p_i + \frac{1}{2t}\left(p_i p_j - p_i^2\right) & \text{if} \quad |p_i - p_j| \leq t(1 - a_1 - a_2) \\ p_i & \text{if} \quad p_i < p_j - t(1 - a_1 - a_2) \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Proposition 1 in Gabszewicz and Thisse asserts that for $a_1 + a_2 = 1$, "the unique price equilibrium is given by $p_1^* = p_2^* = 0$." (p. 286). It is not readily apparent that our theorems shed light on this issue. Note, however, that when $a_1 + a_2 = 1$, equation (1) simplifies to

$$\pi_i(p_i, p_j) = \begin{cases} a_i p_i & \text{if} \quad p_i = p_j \\ p_i & \text{if} \quad p_i < p_j \\ 0 & \text{otherwise} \end{cases}. \tag{2}$$

These payoff functions are isomorphic to those that arise in a winner-take-all pricing game where $C(q) = 0$, $D(p) = 1$, and the monopoly profit function is $\pi(p) = p$ for all $p \in [0, \infty)$.[7] Since monopoly profits are unbounded, the conditions of our Theorem 2 are not satisfied. Thus, our theorem leaves open the possibility that there exist positive profit equilibria in the Hotelling model. In fact, contrary to the claim by Gabszewicz and Thisse, positive profit equilibria do exist, as the following theorem demonstrates.

Theorem 3 *When $a_1 + a_2 = 1$, the Hotelling model has a continuum of positive profit Nash equilibria.*

Proof. Fix some $k \in (0, \infty)$, and suppose that both firms price according to the cumulative distribution function

[7] While the tie-breaking rule we have assumed in our model gives equal weight to each firm being the winning firm, Gabszewicz and Thisse' model assumes that firm i has probability a_i of being the winner. Since firms have zero costs, one can show that the results of our theorems extend to this case.

$$F(p) = \begin{cases} 0 & \text{if} \quad p \le k \\ 1 - \frac{k}{p} & \text{if} \quad p > k \end{cases}$$

Notice this is a well defined, atomless probability distribution on $[k, \infty)$, as $F(k) = 0$, $F(\infty) = 1$, and $F'(p) > 0$ for all $p \in [k, \infty)$.

By symmetry, it is sufficient to show that if firm j adopts F as its strategy, firm i cannot gain by choosing a strategy different from F. Suppose firm j chooses a price according to F. If firm i charges p_i, then with probability $F(p_i)$ firm j's realized price is less than p_i. By equation (1) firm i earns zero profits in this case. With probability $[1 - F(p_i)]$ firm j's price exceeds p_i, and in this event firm i earns profits of p_i. Thus, the expected profits that firm i earns by charging p_i when the rival prices according to F is $E\pi_i(p_i) = [1 - F(p_i)]p_i$. (Since F is atomless, we can ignore ties; hence the tie-breaking rule is irrelevant here.) Using the definition of F, $E\pi_i(p_i) = k$ for $p_i \in [k, \infty)$. This means that firm i earns the same expected profits, namely k, by pricing at or above k. If firm i sets a price that is strictly less than k, it is certain to win the entire market, but the corresponding profits are strictly less than k. Since firm i's profits are constant and equal to k for each $p_i \in [k, \infty)$, and strictly less than k for $p_i < k$, any $p_i \in [k, \infty)$ is a best response by firm i to firm j's strategy, F. Since F allocates all probability in the interval $[k, \infty)$, firm i can do no better than to price according to F. Thus, F constitutes a symmetric Nash equilibrium in which each firm earns positive expected profits of k. Since this construction holds for all $k \in (0, \infty)$, there exists a continuum of positive profit equilibria. □

Theorem 3 thus establishes that, in addition to the well-known zero-profit equilibrium to the Hotelling model where firms have the same location, there also exists a continuum of positive profit Nash equilibrium payoffs. This game satisfies all of the conditions of our Theorem 2 except for bounded monopoly profits. Absent the assumption of bounded payoffs, one cannot in general rule out positive profit equilibria in winner-take-all pricing games.[8] One can show that the construction given in Theorem 3 is robust to alternative specifications of transportation costs and distributions of consumers.

4 Conclusion

How competitive are winner-take-all pricing games? Our results reveal that the answer is intimately related to the properties of the monopoly profit function. Theorem 1 shows that a zero profit equilibrium exists if and only if the monopoly profit function possesses an initial break-even price. Theorem 2 shows that two additional conditions on monopoly profits are required to guarantee that all equilibria entail zero profits. The examples show that the conditions in Theorem 2 are "tight," in the sense that if monopoly profits not bounded (Example 3) or

[8] Baye and Morgan (1999) set forth general conditions for the existence of a continuum of positive profit equilibria in homogeneous product pricing games.

not left lower semicontinuous (Example 2), positive profit equilibria can arise. Remarks 1 and 2 suggest that cost discontinuities are more likely to lead to non-competitive outcomes than are demand discontinuities. The usual argument used to guarantee the zero profit outcome requires that, in the event of a tie, a firm benefits by undercutting its rivals. The weak form of continuity in Definition 2 is needed to generalize this reasoning to cover cases where demand or costs are discontinuous. Among other things, it guarantees that undercutting the rival's price dominates accepting any tie outcome where firms earn positive operating profits.

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