Midterm Answer Key

Econ 160, Winter 2002
prepared by David MIller

1.0 Basic concepts

1.1 Proofs

Claim 1: If *G* is a normal form game with more than one Nash equilibrium, then more than one equilibrium must be Pareto optimal.

False. The following game is a counterexample.

Player 2

Player 1		L	R
	T	2,2	0,0
	B	0,0	1,1

Claim 2: In a normal form game, every strategy of each player is either strictly dominated or it is a dominant strategy.

False. Matching pennies is a counterexample.

1.2 Definitions

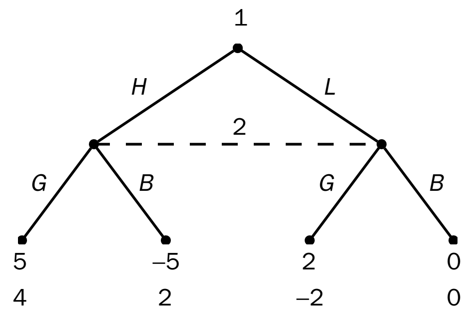
Definition 1: In a normal form game $\langle N, S, u \rangle$, a strategy $s_i' \in S_i$ for player i is a dominant strategy if $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$, and for all $s_i \in S_i \setminus \{s_i'\}$.

Definition 2: In a normal form game $\langle N, S, u \rangle$, a strategy $s_i' \in S_i$ for player i is a strictly (or weakly) dominated strategy if there exists some $s_i'' \in S_i$ such that $u_i(s_i', s_{-i}) < u_i(s_i'', s_{-i})$ for all $s_{-i} \in S_{-i}$ (or $u_i(s_i', s_{-i}) \leq u_i(s_i'', s_{-i})$ for all $s_{-i} \in S_{-i}$ and $u_i(s_i', \hat{s}_{-i}) < u_i(s_i'', \hat{s}_{-i})$ for some $\hat{s}_{-i} \in S_{-i}$).

Definition 3: In a normal form game $\langle N, S, u \rangle$, a strategy $s_i^* \in S_i$ for player i is a best response to a strategy profile $\tilde{s}_{-i} \in S_i$ for the other players if $u_i(s_i^*, \tilde{s}_{-i}) \geq u_i(s_i, \tilde{s}_{-i})$ for all $s_i \in S_i$.

2.0 Technology adoption

2.1 Extensive form



There is only one subgame—the entire game.

2.2 Equilibria

We can use the bimatrix representation to find the pure strategy Nash equilibria.

		Player 2	
		G	B
Player 1	H	5,4	-5,2
	L	2,-2	0,0

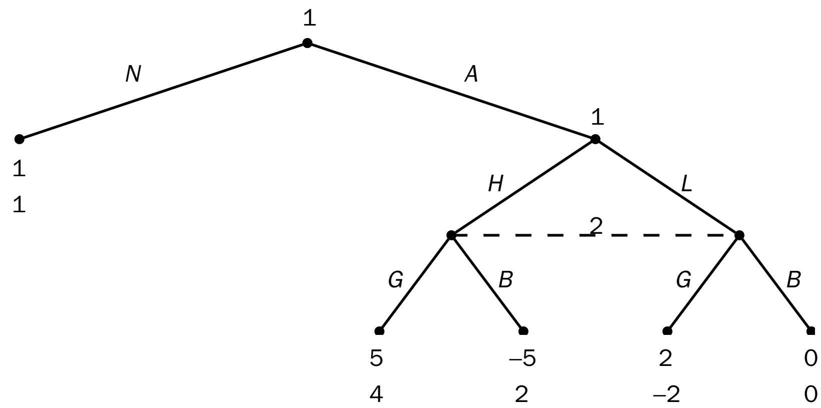
There are no dominated strategies, so we note that H is the best response to G, L is the best response to B, G is the best response to H, and B is the best response to L. Thus (H,G) and (L,B) are the pure strategy Nash equilibria.

To search for a mixed strategy Nash equilibrium, let $p=\sigma_1(H)$, $1-p=\sigma_1(L)$, $q=\sigma_2(G)$, and $1-q=\sigma_2(B)$, where $\sigma_i(s_i)=\Pr(s_i|\sigma_i)$. Then we write the indifference conditions for the two players and solve.

$$\begin{array}{c} u_1(H)=5q-5(1-q)=u_1(L)=2q+0(1-q) \\ u_2(G)=4p-2(1-p)=u_2(B)=2p+0(1-p) \end{array} \Rightarrow (p,q)=\left(\tfrac{1}{2},\tfrac{5}{8}\right) \qquad \text{(EQ 1)}$$

So there are three Nash equilibria: (H,G), (L,B), and $\left(\tfrac{1}{2},\tfrac{5}{8}\right)$. Since there is only one subgame, these are also the subgame perfect equilibria.

2.3 Extensive form with adoption decision



There are two subgames: one in which the technology has been adopted and the players simultaneously make the decisions modeled in the first part of the question, and one which is the whole game.

2.4 Pure strategy SPEs

We find the SPEs by backward induction, using each pure strategy equilibrium in the last subgame in turn.

Suppose (H, G) is played in the last subgame. Then the CEO's payoff to adopting the technology is $5 > 1$, so the best response at the root node is to play A. Thus (AH, G) is a SPE.

Suppose (L, B) is played in the last subgame. Then the CEO's payoff to adopting the technology is $0 < 1$, so the best response at the root node is to play N. Thus (NL, B) is a SPE.

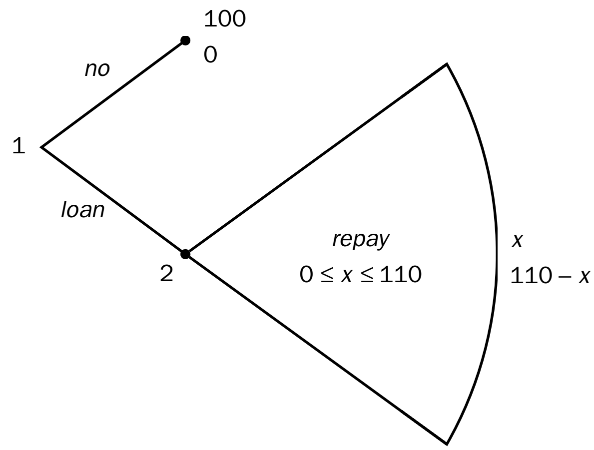
2.5 Mixed strategy SPEs

There is also an SPE in which the $\left(\frac{1}{2}, \frac{5}{8}\right)$ equilibrium is played in the last subgame. By backward induction, there exists a best response for the CEO at the root node. (We were not asked to actually calculate this SPE.)

3.0 Debt and repayment

3.1 No legal system

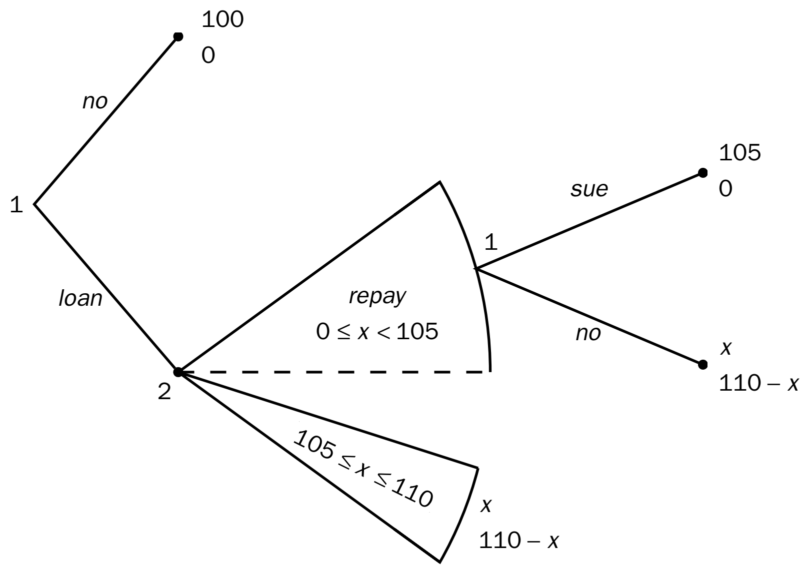
Since the strategy space for the borrower is continuous, we cannot draw a proper game tree. However, we can draw a diagram that fairly represents the extensive form.



We solve by backward induction. In the last subgame, player 2 wants to minimize x, so $x = 0$. Then at the root node player 1 plays "no" in order to get a payoff of 100 rather than 0. Thus $(no, 0)$ is a subgame perfect equilibrium. Furthermore, since backward induction identified a unique best response at every information set, it is unique.

We can find another Nash equilibrium that is not subgame perfect: the debtor chooses $x = 50$ and the lender chooses *no*.

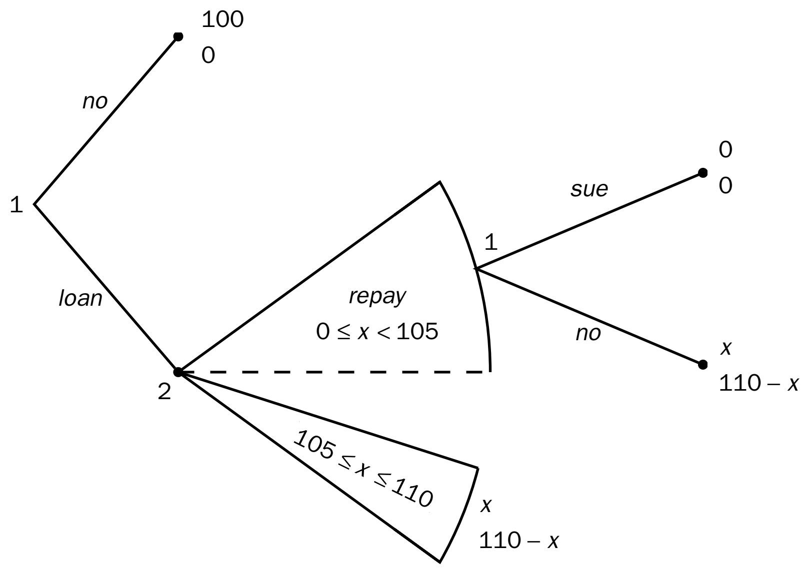
3.2 Free legal system



Again, we solve by backward induction. In the last subgame, player 1 has a strict best response of "sue" (regardless of x) in order to get a payoff of 105 rather than $x < 105$. Then in the middle subgame, player 2 has a strict best response of $x = 105$, which yields a payoff of 5. (This is better than $x < 105$, which yields a payoff of 0, and better than $x > 105$, which yields a payoff of $110 - x < 5$.) Then at the root node player 1 has a strict best response of "loan" to get a payoff of 105 rather than 100. Thus $\left(\left(\mathit{loan}, \mathit{sue}\ \forall\ x\right), 105\right)$ is the unique subgame perfect equilibrium.

There are no other Nash equilibria because the normal form best response for the lender to any $x < 105$ is to loan and sue for that x, while the normal form best response for the debtor to any $x < 105$ for which the lender would not sue is to choose that x. Thus the lender must sue for all x in Nash equilibrium and the debtor's unique best response is $x = 105$.

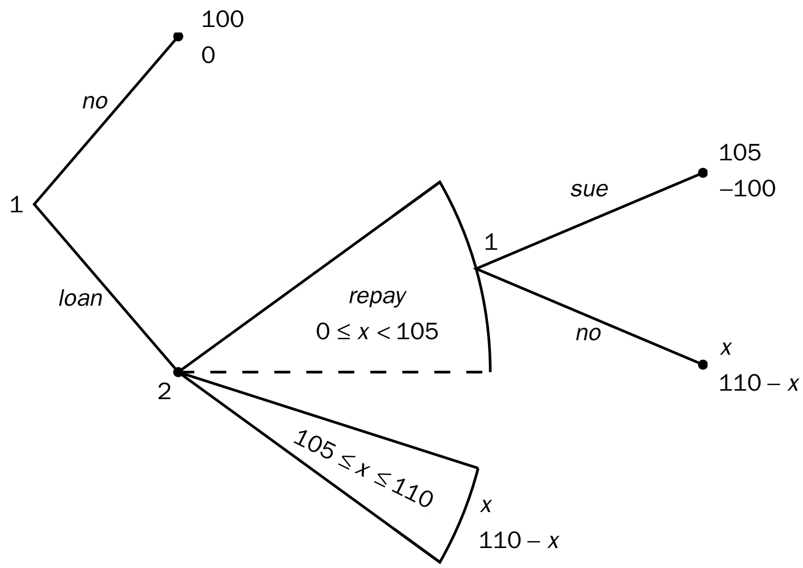
3.3 Costly legal system



In the last subgame, player 1 has a strict best response of "no" if $x > 0$, and is indifferent between "no" and "sue" if $x = 0$. In the middle subgame, player 2 wants to minimize x as long as player 1 will then play "no." Note that $x = 0$ is a strict best response if and only if player 1 will then play "no" with probability 1; otherwise player 2 does not have a well-defined best response. (We cannot say that player 2 plays something very close to zero, because there is always something even closer to zero that is better.) Thus the only Nash equilibrium in the middle subgame is $(0, no\ \forall\ x)$. Then at the root node, player 1 has a strict best response of "no" in order to get a payoff of 100 rather than 0. Thus the unique SPE is $((no, no\ \forall\ x), 0)$.

We can find another Nash equilibrium that is not subgame perfect: the debtor chooses $x = 50$ and the lender chooses $(no, no\ \forall\ x)$.

3.4 "Loser pays" legal system



In the last subgame, player 1 has a strict best response of "sue" (regardless of x) to get a payoff of 105 instead of $x < 105$. Then in the middle subgame, player 2 has a strict best response of $x = 105$, which yields a payoff of 5. (This is better than $x < 105$, which yields a payoff of −100, and better than $x > 105$, which yields a payoff of $110 - x < 5$.) Then at the root node player 1 has a strict best response of "loan" to get a payoff of 105 rather than 100. Thus $((loan, sue\ \forall\ x), 105)$ is the unique subgame perfect equilibrium.

The "loser pays" legal system gives player 1 a payoff of 105, which is 5 higher than her payoff under the costly legal system. Thus she would be willing to pay up to 5 for the change in the law. Furthermore, since the project is implemented under the "loser pays" legal system, yielding payoffs of $(105, 5)$, while it is not implemented under the costly legal system, yielding payoffs of $(100, 0)$, the law induces a Pareto improvement. The social planner should implement it.

There are no other Nash equilibria because the normal form best response for the lender to any $x < 105$ is to loan and sue for that x, while the normal form best response for the debtor to any $x < 105$ for which the lender would not sue is to choose that x. Thus the lender must sue for all x in Nash equilibrium and the debtor's unique best response is $x = 105$.

4.0 Entry deterrence

4.1 Normal form for Cournot

This game has normal form $\langle N, S, u \rangle$, where $N = \{1, 2\}$ is the set of players, $S = S_1 \times S_2$ is the strategy profile space, $S_i = [0, 100]$ is the strategy space for each player, $u = (u_1, u_2)$ is the vector of payoff functions, and $u_i : S \to \mathbb{R}$ is given by

$$u_i(q_i, q_{-i}) = (100 - (q_i + q_{-i}))q_i - k1_{q_i > 0} \qquad \text{(EQ 2)}$$

where 1_x is the indicator function of event *x*.

4.2 Equilibrium

We seek an interior maximum of u_i by taking the first order condition:

$$\frac{\partial}{\partial q_i} u_i(q_i, q_{-i}) = 100 - q_{-i} - 2q_i = 0 \quad \Rightarrow \quad \tilde{q}_i = \frac{100 - q_{-i}}{2} \tag{EQ 3}$$

This is the familiar Cournot best response. However, when gross revenues are below 1000, profits net of fixed cost will be negative, so the best response is not to produce at all, and therefore not incur the fixed production cost. Gross revenues from playing the Cournot best response will be below 1000 when q_{-i} exceeds some $\hat{q}_{-i}(k)$, which is a quantity that we need not calculate.

$$q_i^*(q_{-i}) = \begin{cases} \dfrac{100 - q_{-i}}{2} & \text{if } q_{-i} \le \hat{q}_{-i}(k) \\ 0 & \text{if } q_{-i} \ge \hat{q}_{-i}(k) \end{cases} \tag{EQ 4}$$

We continue by finding the interior solution; we will then check to see whether it provides gross revenues that exceed 1000. The Cournot solution, of course, is

$$\tilde{q}_i = \frac{100 - \tilde{q}_i}{2} = \tilde{q}_{-i} = \frac{100}{3} \tag{EQ 5}$$

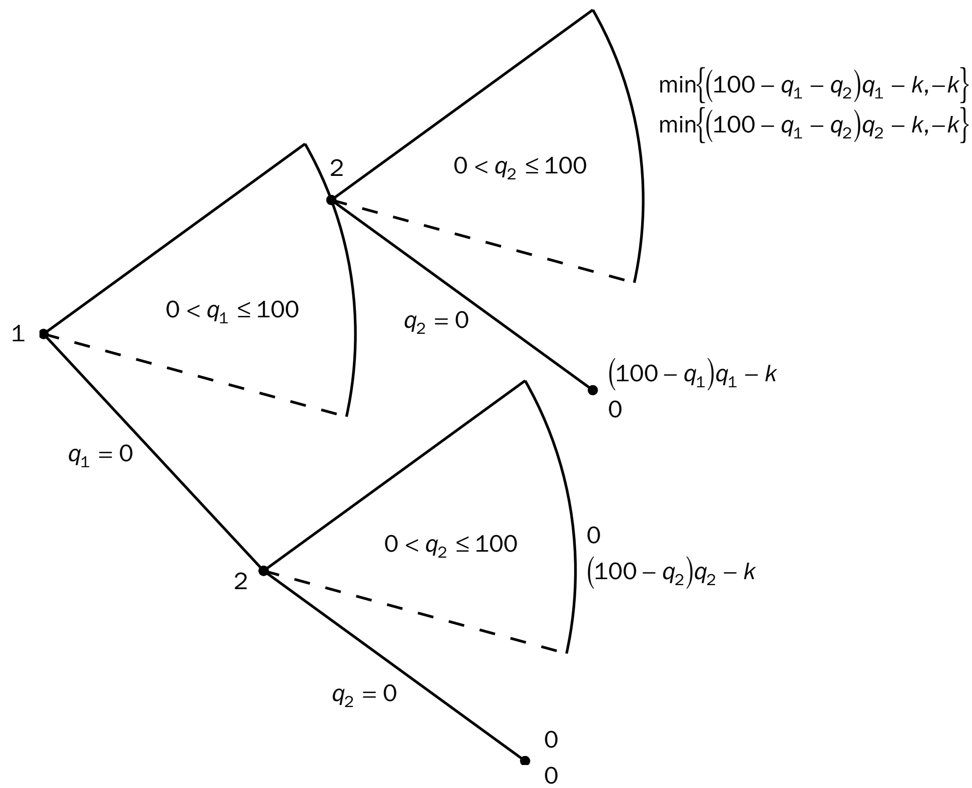
and the associated gross revenues are $10000/9 > 1000$, so this is in fact an equilibrium.

However, we must also note that the best response for player *i* to any $q_{-i} \ge \hat{q}_{-i}$ is $q_i = 0$, and that the best response for player *i* to $q_{-i} = 0$ is to produce the monopoly quantity $q_i = 50$. Furthermore, the net profits of competing against a firm that produces the monopoly quantity are

$$u_i(q_i = 25, q_{-i} = 50) = 625 - 1000 < 0 \tag{EQ 6}$$

Thus we have two more Nash equilibria, each of which involves one firm owning a monopoly in the market.

4.3 Extensive form for Stackelburg



Note that I have redefined the (inverse) demand function so that the firms can never realize negative gross revenues.

4.4 Subgame perfect equilibrium

We solve by backward induction.

In the subgame after firm 1 has chosen $q_1 = 0$, firm 2 has a strict best response of choosing the monopoly quantity unless monopoly gross revenues are less than k. The monopoly quantity is 50, as we know from lecture, and it provides gross revenues of 2500. Since in all cases we assume that $k < 2500$, this is the unique best response in the subgame.

In the subgame after firm 1 has chosen $q_1 > 0$, firm 2 has a strict best response of choosing $q_2 = \frac{1}{2}(100 - q_1)$, as we know from Cournot analysis, unless the resulting gross revenues (given by $\left(100 - q_1 - \frac{1}{2}(100 - q_1)\right)\frac{1}{2}(100 - q_1) = \frac{1}{4}(100 - q_1)^2$) are less than k.

Suppose that $k = 25$; then net revenues are positive if $q_1 < 90$, in which case $q_2 = \frac{1}{2}(100 - q_1)$ is the strict best response. If $q_1 > 90$ then $q_2 = 0$ is a the strict best response. If $q_1 = 90$ then $q_2 = 5$ and $q_2 = 0$ are both best responses, but we assume that firm 2 plays $q_2 = 0$.

Suppose that $k = 225$; then net revenues are non-negative if $q_1 < 70$ in which case $q_2 = \frac{1}{2}(100 - q_1)$ is the strict best response. If $q_1 > 70$ then $q_2 = 0$ is the strict best response, and if

$q_1 = 70$ then $q_2 = 15$ and $q_2 = 0$ are both best responses, but again we assume that firm 2 plays $q_2 = 0$.

Let $\hat{q}_1(k)$ be quantity for firm 1 above firm 2 will choose not produce. Then at the root node, firm 1 has three classes of options:

1. Play $q_1 = 0$ for a payoff of 0
2. Play $q_1 \in \left(0, \hat{q}_1(k)\right)$ for a payoff of $\left(100 - q_1 - \frac{1}{2}(100 - q_1)\right)q_1 - k$
3. Play $q_1 \in \left[\hat{q}(k), 100\right]$ for a payoff of $(100 - q_1)q_1 - k$

Suppose that $k = 25$; then $\hat{q}_1(25) = 90$ and the payoffs given the best responses in these four classes are:

1. 0
2. 1225 (we derived in lecture that $(50, 25)$ is the Stackelburg equilibrium outcome, and this gives firm 1 gross revenues of 1250)
3. $(100 - 90)90 - 25 = 875$

Thus firm 1's strict best response is $q_1 = 50$. The associated SPE is $\left(50, 1_{q_1 < 90}\, \frac{1}{2}(100 - q_1)\right)$, where 1_s is the indicator function of event s. Given the assumption that firm 2 plays $q_2 = 0$ if $q_1 = 90$, this SPE is unique by backward induction.

Suppose now that $k = 225$; then $\hat{q}_1(225) = 70$ and the payoffs given the best responses in these four classes are:

1. 0
2. 1025
3. $(100 - 70)70 - 225 = 1875$

Thus firm 1's strict best response is $q_1 = 70$. The associated SPE is $\left(70, 1_{q_1 < 70}\, \frac{1}{2}(100 - q_1)\right)$, where 1_s is the indicator function of event s. Given the assumption that firm 2 plays $q_2 = 0$ if $q_1 = 90$, this SPE is unique by backward induction.