Economics 160: Solutions to Midtem

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1 Problem 1: Basic Concepts

1.1 Part A

Dominated strategies come in two varieties, strictly dominated strategies and weakly dominated strategies. Suppose player i has pure strategy set S_i.

A strategy for player i, s_i, is strictly dominated if there's another strategy $\widehat{s}_i$ such that $\forall s_{-i} \in S_{-i}$, $u_i(\widehat{s}_i, s_{-i}) > u_i(s_i, s_{-i})$. Thus, a strictly dominated strategy is one that delivers a strictly lower payoff than some other specific alternative strategy no matter what other players do.

A strategy for player i, s_i, is weakly dominated if there's another strategy $\widehat{s}_i$ such that $\forall s_{-i} \in S_{-i}$, $u_i(\widehat{s}_i, s_{-i}) \geq u_i(s_i, s_{-i})$. Thus, a weakly dominated strategy is one that delivers a weakly lower payoff than some other specific alternative strategy no matter what other players do.

One can define dominance by mixed strategies analogously.

1.2 Part B

A Nash equilibrium is a profile of strategies (pure or mixed), s^*, in which each player plays a best response given the other players' strategies, i.e. in which $\forall i \in N$, $s_i^* \in BR_i(s_{-i}^*)$. One could equivalently say that a Nash equilibrium is a strategy profile such that no player can strictly increase her utility by unilaterally changing her strategy. So s^* is a NE in pure strategies if $\forall i \in N$ and $\forall s_i \in S_i$, $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$. The definition for NE in mixed strategies is similar.

1.3 Part C

The statement is false. One can easily construct a counter-example using a game with simultaneous moves. Suppose players i and j move simultaneously. One can reverse the order of i's and j's moves in the extensive form of the game without changing the normal form of the game.

1.4 Part D

The statement is true. The game is a game of perfect information, so we can solve for the SPE of the game using backward induction. Since no player is indifferent between any two outcomes, each player has a unique optimal action at each node at which she moves. Therefore, backward induction yields only a single SPE.

This question essentially asks you to prove a simple version of the theorem that says that all games of perfect information have unique SPE if no player is indifferent between any two outcomes.

1.5 Part E

The statement is true. We'll show that at least one of the Nash equilibria in pure strategies is not Pareto dominated by any other strategy profile, and therefore must be Pareto optimal.

Before we begin, recall the definition of Pareto optimality. A strategy profile s is Pareto optimal if there is no other strategy profile that Pareto dominates s. A strategy profile $\widetilde{s}$ Pareto dominates strategy profile s if and only if $\forall i \in N$, $u_i(\widetilde{s}) \geq u_i(s)$ and $\exists j \in N$ such that $u_j(\widetilde{s}) > u_j(s)$. In words, $\widetilde{s}$ Pareto dominates s if and only if moving from s to $\widetilde{s}$ makes no player worse off and makes at least one player better off. Unfortunately many students confused the concepts of Pareto dominance and Pareto optimality. You show a strategy profile s is Pareto optimal by showing no other profile Pareto dominates s, not by showing that s Pareto dominates all other profiles. Showing that neither of two strategy profiles dominates the other does not show that neither is Pareto optimal. In fact, both may be Pareto optimal if no other strategy profile dominates either one.

Suppose the payoff matrix for the game is as follows.

		Player 2	
		L	R
Player 1	T	a, b	c, d
	B	e, f	g, h

Step 1: Neither player can have a strictly dominant strategy. If either player had a strictly dominant strategy, then she would have to play that strategy in both Nash equilibria in pure strategies; you can't play a strictly dominated strategy in a Nash equilibrium. So if player i has a strictly dominant strategy, then she must play it in both Nash equilibria. But since the other player, j, isn't indifferent between any two outcomes, j cannot be indifferent between the two Nash equilibria in pure strategies. Since j strictly prefers one Nash equilibrium to the other, and since i plays the same strategy in both NE, j would deviate from the NE that she likes less to the one she likes more. Thus the NE that j likes less cannot be a NE, and we've reached a contradiction. We conclude that neither player can have a strictly dominant strategy.

Step 2: No profile of pure strategies that isn't a NE can Pareto dominate a NE. We've shown that neither player has a strictly dominant strategy, so without loss of generality, we can assume that the two NE in pure strategies are (T, L) and (B, R). If either player played the same strategy in both NE, she would have to have a strictly dominant strategy since we assumed no player is indifferent between any two outcomes. Therefore, no player plays the same strategy in both NE, and we can assume that the two NE are (T, L) and (B, R).

Remember that one strategy profile Pareto dominates another only if the first profile gives every player weakly greater utility than the second profile does, and the first profile gives at least one player strictly greater utility than does the second. Since neither player is indifferent between any two pure strategy combinations, one strategy profile must offer both players strictly greater utility in this game to Pareto dominate a second profile. Since (T, L) is a NE, we know that $a > e$ and $b > d$, so neither (B, L) nor (T, R) can Pareto dominate (T, L). Since (B, R) is a NE, we know that $g > c$ and $h > f$, so neither (B, L) nor (T, R) can Pareto dominate (B, R). We conclude that neither of the pure strategy combinations that isn't a NE can Pareto dominate either NE in pure strategies.

Step 3: Either one NE Pareto dominates the other or neither NE dominates the other. The preceding statement must be true because it's impossible for both NE to Pareto dominate each other.

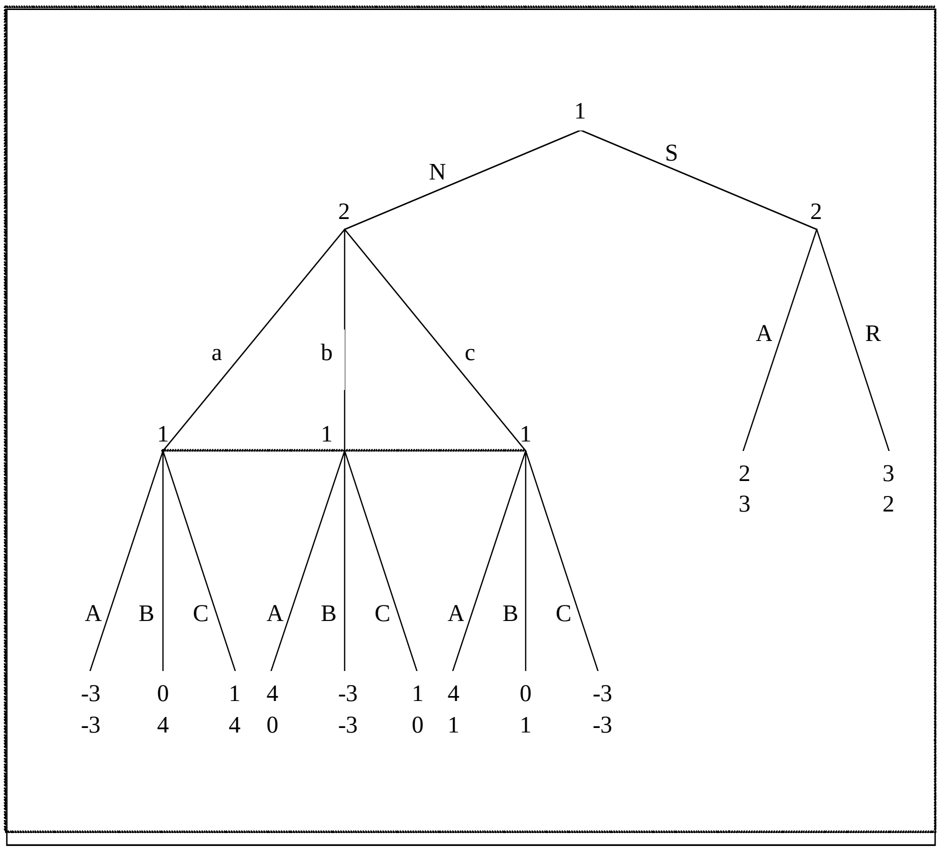
Step 4: No profile of mixed strategies can Pareto dominate both of the NE in pure strategies. So far, we've shown that at least one of the NE in pure strategies is not Pareto dominated by any other profile of pure strategies; it's not Pareto dominated by any strategy profile that's not a NE, and it's not Pareto dominated by the other NE in pure strategies. Our last step is to show that this NE is not Pareto dominated by any profile of mixed strategies, either.

Suppose, without loss of generality, that (T, L) is not Pareto dominated by any other pure strategy profile. Then either $a > g$ or $b > h$, or both. The payoffs for a mixed strategy profile are a weighted average of the payoffs for pure strategy profiles, so a player's payoff from a mixed strategy profile is less than the maximum payoff she can receive from a pure strategy profile. Suppose $a > g$. Then if the players play mixed strategies, $u_1 < \max(a, c, e, g) = a$. Suppose $b > h$. Then if the players play mixed strategies, $u_2 < \max(b, d, f, h) = b$. It follows that whether $a > g$ or $b > h$, the mixed strategy profile cannot Pareto dominate (T, L), since at least one player is better off playing (T, L) than playing the mixed strategy profile. We conclude that (T, L) is not Pareto dominated by any mixed strategy.

We've shown that the game has at least one NE in pure strategies that is not Pareto dominated by any other strategy profile – pure or mixed. We conclude that this NE must be Pareto optimal.

2 Problem 2: Angry Friends

2.1 Part A



The only subtlety in the game tree is that because the two players move simultaneously after 1 chooses N, neither observes the other's choice of friend. Accordingly, one can draw the game tree so that either player moves first after 1 chooses N. Player 1 doesn't know which friend player 2 has chosen, so all of player 1's nodes after player 2's choice of friend belong to the same information set.

2.2 Part B

This game has three proper subgames. A proper subgame begins at each of the two nodes at which player 2 moves. The entire game is also a proper subgame.

One cannot divide the subgame with simultaneous moves into more subgames, because attempting to do so would separate nodes in the same information set for player 1. If a subgame contains one node in an information set, it must contain all other nodes in the information set as well.

2.3 Part C

Player 1 has 2 possible actions at his first information set and three possible actions at his second information set, so he has a total of $2 \times 3 = 6$ pure strategies. Similarly, player 2 has 2 possible actions at her first information set and three possible actions at her second information set, so she has a total of $2 \times 3 = 6$ pure strategies.

2.4 Part D

Recall that, by definition, a SPE induces a NE in every proper subgame. We'll solve for all NE in each proper subgame assuming that the players play NE in the other subgames.

Begin with the subgame with simultaneous moves. Notice that B is not a rationalizable strategy for 1 because it is never a best response to any strategy of 2. Similarly, b is not a rationalizable strategy for 2 because it is never a best response to any strategy of 1. Recall that all strategies played in a NE must be rationalizable. Therefore, we can eliminate strategies B and b from this subgame. (Note that we can make this simplification only because we've ruled out mixed strategies! We have to do more work to show that B and b aren't rationalizable if mixed strategies are possible.) The subgame becomes:

		Player 2	
		a	c
Player 1	A	$-3,-3$	$\underline{\overline{4,1}}$
	C	$\underline{\overline{1,4}}$	$-3,-3$

Once we've eliminated B and b, it's easy to see that (C, a) and (A, c) are NE of this subgame.

Next, consider the subgame in which 2 chooses between A and R. Clearly the only NE for this subgame occurs when 2 chooses A, since A yields 2 a payoff of 3 and R yields a payoff of 2.

Now, consider the subgame consisting of the entire game. 1 knows that if he chooses S, he'll receive a payoff of 2 because the only NE in the subgame after S is for 1 to choose A. 1 also knows that if he chooses N, he'll get 1 if the players play (C, a), and 4 if the players play (A, c). In a NE for the game as a whole, the players correctly forecast each other's strategies. Therefore, if player 1 anticipates (C, a), he'll choose S, since S gives him 2 and N gives him only 1. If player 1 anticipates (A, c), however, he'll choose N, since N gives him 4 and S gives him 2.

Designate each player's strategy with two letters, each indicating his action at one of his information sets; the second letter indicates the player's action in the subgame with simultaneous moves. We conclude that the only SPE in pure strategies are (NA, Ac) and (SC, Aa).

2.5 Part E

Let's write out the matrix representation of the normal form of this game.

		Player 2					
		Aa	Ab	Ac	Ra	Rb	Rc
	NA	$-3,-3$	$\underline{4},0$	$\underline{4},\underline{1}$	$-3,-3$	$\underline{4},0$	$\underline{4},\underline{1}$
	NB	$0,\underline{4}$	$-3,-3$	$0,1$	$0,\underline{4}$	$-3,-3$	$0,1$
Player 1	NC	$1,\underline{4}$	$1,0$	$-3,-3$	$1,\underline{4}$	$1,0$	$-3,-3$
	SA	$\underline{2},\underline{3}$	$2,\underline{3}$	$2,\underline{3}$	$\underline{3},2$	$3,2$	$3,2$
	SB	$\underline{2},\underline{3}$	$2,\underline{3}$	$2,\underline{3}$	$\underline{3},2$	$3,2$	$3,2$
	SC	$\underline{2},\underline{3}$	$2,\underline{3}$	$2,\underline{3}$	$\underline{3},2$	$3,2$	$3,2$

We can see that (NA, Rc) and (SA, Aa) are NE that are not SPE.

2.6 Part F

Before we begin, let me note that this part of the problem is tricky and complicated, so you weren't alone if you struggled with it.

We must now check whether allowing mixed strategies in the subgame with simultaneous moves creates new NE in this subgame. If so, we must then check whether any new NE in this subgame can be part of a SPE.

The difficulty with this problem is that we don't know what form the players' mixed strategies will take. We don't know whether one player will play all her actions in this subgame with positive probability, or only some actions. We have to break down the problem into cases.

I'll note one common mistake before continuing. Some students thought that B and b were not rationalizable and therefore could be eliminated from the players' strategy sets in the subgame with simultaneous moves. Unfortunately, while B and b are not rationalizable when only pure strategies are possible, they are rationalizable when mixed strategies are possible, so we cannot eliminate B and b.

2.6.1 Case 1: One player plays all actions in the subgame with positive probability.

Suppose player 1 plays all his strategies with positive probability. Then 2 must choose a mixed strategy that makes 1 indifferent toward all his pure strategies. Say that 2 plays $\sigma_2 = (p, q, 1-p-q)$, where $p, q \in [0,1]$ and $p + q \leq 1$. Note that we're allowing the possibility that 2 will assign zero probability to some of her pure strategies. To make 1 indifferent toward all his pure strategies, 2 must equate the payoffs to 1's pure strategies. The payoffs to 1's pure strategies are:

$$\begin{aligned}
u_1(A, \sigma_2) &= -3p + 4q + 4(1-p-q) = 4 - 7p \\
u_1(B, \sigma_2) &= -3q \\
u_1(C, \sigma_2) &= p + q - 3(1-p-q) = 4p + 4q - 3
\end{aligned}$$

Equating these three expected payoffs and solving for p and q gives us $p = 37/61$ and $q = 5/61$. Thus, 2 can only make 1 willing to play all his strategies with positive probability if 2 plays $\sigma_2 = (37/61, 5/61, 19/61)$. But 2 will only play this mixed strategy if she's indifferent toward all her pure strategies. By the same reasoning we just used, one can show that 2 will only be indifferent toward all her pure strategies if player 1 plays $\sigma_1 = (37/61, 5/61, 19/61)$.

We have shown that the only NE for the subgame with simultaneous moves in which one player plays all strategies with positive probability is $\sigma_1 = \sigma_2 = (37/61, 5/61, 19/61)$. Now we must check whether this NE for the subgame can be part of an SPE. Plug q into the expression for $u_1(B, \sigma_2)$ to show that 1 has an expected payoff of -15/61 in this NE. One can easily show that 2 has the same expected payoff. Anticipating these NE payoffs in the subgame with simultaneous moves, player 1 will choose S in his initial move, which gives him a payoff of $2 > -15/61$. We have found another SPE, $(S\sigma_1, A\sigma_2)$, where σ_1 and σ_2 are as defined above.

2.6.2 Case 2: A player plays only one action in subgame with positive probability.

Inspecting the subgame should convince you that there's no mixed strategy NE in which a player plays only one action with positive probability, because if i plays only one action with positive probability, j can never be indifferent toward any two or three of her strategies. So this case never occurs in a SPE.

2.6.3 Case 3: Both players play only two actions in the subgame with positive probability.

We have 9 subcases to consider since each player can choose any one of three pairs of pure strategies. In each case, we use exactly the same method of analysis as above. We find mixed strategies for one player that make the other indifferent (and therefore willing to mix) between the appropriate pure strategies, and then we repeat the process for the other player. Note one complication: Since each player assigns zero probability to one pure strategy, we must check to make sure that she doesn't prefer the omitted pure strategy to her mixed strategy, given that the other player is mixing. Since the analysis is extremely repetitious, I merely state the results below.

Subcase 3.1: 1 mixes between A and B, and 2 mixes between a and b. You can show that 1 would deviate to C if the players played a mixed strategy profile that made 1 indifferent between A and B. Therefore, this subcase cannot occur in any NE for the subgame.

Subcase 3.2: 1 mixes between A and B, and 2 mixes between a and c. You can show that 1 will be indifferent between A and B and 2 will be indifferent between a and c if $\sigma_1 = (3/7, 4/7, 0)$ and $\sigma_2 = (4/7, 0, 3/7)$. Neither player wants to deviate to a pure strategy. 1 earns a higher payoff from playing S than from playing N and reaching the subgame with simultaneous moves, so the resulting SPE is $(S\sigma_1, A\sigma_2)$.

Subcase 3.3: 1 mixes between A and B, and 2 mixes between b and c. It's easy to see that this subcase cannot occur in any NE for the subgame because 1 cannot be indifferent between A and B when 2 mixes between b and c.

Subcase 3.4: 1 mixes between B and C, and 2 mixes between a and b. This subcase cannot occur in any NE for the subgame because 2 cannot be indifferent between a and b when 1 mixes between B and C.

Subcase 3.5: 1 mixes between B and C, and 2 mixes between a and c. This subcase cannot occur in any NE for the subgame because 2 cannot be indifferent between a and c when 1 mixes between B and C.

Subcase 3.6: 1 mixes between B and C, and 2 mixes between b and c. If 2 mixes between b and c, then 1 will deviate to A. Therefore, this subcase cannot occur in any NE for the subgame.

Subcase 3.7: 1 mixes between A and C, and 2 mixes between a and b. This subcase is just subcase 3.2 with the roles of 1 and 2 reversed. Thus, the mixed strategies $\sigma_1 = (4/7, 0, 3/7)$ and $\sigma_2 = (3/7, 4/7, 0)$ form a NE for this subgame. The resulting SPE is $(S\sigma_1, A\sigma_2)$.

Subcase 3.8: 1 mixes between A and C, and 2 mixes between a and c. Suppose that 1 plays A with probability $p \in [0,1]$ and C with probability $1-p$. Suppose that 2 plays a with probability $q \in [0,1]$ and c with probability $1-q$. Call player i's mixed strategy σ_i. The payoffs to 2's pure strategies (in this subgame) are:

$$\begin{aligned} u_2(\sigma_1, a) &= -3p + 4(1-p) = 4 - 7p \\ u_2(\sigma_1, c) &= p - 3(1-p) = 4p - 3 \end{aligned}$$

If both players mix, each must choose a strategy that makes the other indifferent between his pure strategies. Therefore, in equilibrium, p must set $u_2(\sigma_1, a) = u_2(\sigma_1, c)$. We conclude that $p = 7/11$. By similar reasoning, one can show that $q = 7/11$. However, the strategy profile $\sigma_1 = (7/11)A + (4/11)C$, $\sigma_2 = (7/11)a + (4/11)c$ is NOT a NE of the subgame with simultaneous moves. The reason is that this strategy profile generates payoffs $(-5/11, -5/11)$, so player 1 would be better off deviating to B (which gives a payoff of 0), and player 2 would be better off deviating to b (which gives a payoff of 0). Since players have incentives to deviate, the mixed strategy profile is not a NE. We conclude that this subcase cannot occur in any SPE.

Subcase 3.9: 1 mixes between A and C, and 2 mixes between b and c. This subcase cannot occur in any NE for the subgame because 1 cannot be indifferent between A and C when 2 mixes between b and c.

We've now examined all possible cases. The bottom line for this question is that the set of SPE does change: We get three new SPE, as described above.

3 Junk Mail Advertising (30 points)

3.1 part (a)

This is a game of perfect information. In the extensive form representation, every information set is a singleton.

3.2 part (b)

Players. $N = \{1, 2\}$ or $\{Buyer, Seller\}$
Strategies. Player 1 (the buyer) acts at the first node of the game tree, choosing Commute or Stay Home; and then after each possible price offer p, choosing Buy or Don't. Thus, a strategy for player 1 is a choice from $\{C, S\}$ followed by a choice rule for every possible p. Thus,

$$S_1 = \{C, S\} \times \{s : \mathbf{R}^+ \to \{B, D\}\}$$

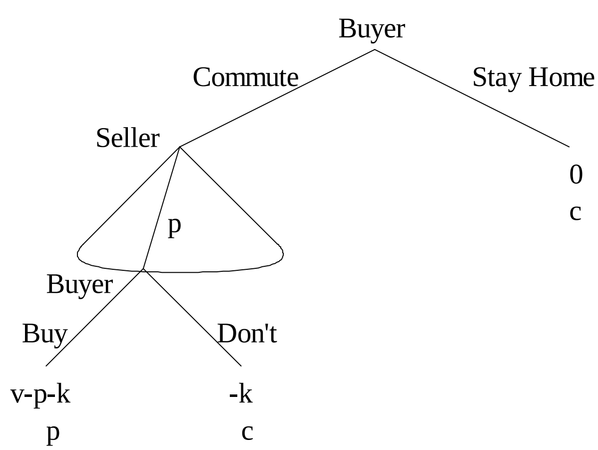
Player 2 (the buyer) acts at only one node, choosing p; thus, $S_2 = \mathbf{R}^+$.
Payoffs. The payoff functions can be written as

$$u_1(s_1, s_2) = \begin{cases} 0 & if \quad s_1^1 = S \\ -k & if \quad s_1^1 = C \quad and \quad s_1^2(s_2) = D \\ v - s_2 - k & if \quad s_1^1 = C \quad and \quad s_1^2(s_2) = B \end{cases}$$

$$u_2(s_1, s_2) = \begin{cases} c & if \quad s_1^1 = S \\ c & if \quad s_1^1 = C \quad and \quad s_1^2(s_2) = D \\ s_2 & if \quad s_1^1 = C \quad and \quad s_1^2(s_2) = B \end{cases}$$

where $s_1 = (s_1^1, s_1^2)$. (Note that you could alternatively set u_2 equal to 0 or $p - c$; however, using c and $p - c$ is not correct.)

3.3 part (c)



$$BR_1(s_2 = p) = \begin{cases} B & if \quad p < v \\ S & if \quad p > v \\ \text{either} & if \quad p = v \end{cases}$$

(Some people wrote that the best-response was to accept if $p \leq v - k$. However, once the buyer has already commuted, he is choosing between payoffs of $v - p - k$ and $-k$, not between $v - p - k$ and 0, so the best-response is what I wrote above.)

3.4 part (d)

Consider the subgame beginning at the node where the seller sets the price. If the buyer plans to not buy when $p = v$, or to buy with probability less than 1, the seller has no best-response. (He will want to set $p = v + \epsilon$, but since there is no smallest ϵ, he has no strategy that is a best-response.) Thus, the only SPE of this subgame is for the buyer to buy whenever $p \leq v$, and the seller to set $p = v$. Backing up a level, the buyer knows that commuting will lead to a payoff of $-k$, so he chooses to stay home. Thus, the equilibrium is the following:

s_1 is to Stay Home, then Buy if $p \leq v$; s_2 is to set $p = v$

This equilibrium is unique, because $p = v$ and Buy if $p \leq v$ is the only SPE of that subgame, and Stay Home is the unique best-response if that equilibrium will be played.

It is not Pareto-optimal. For any price $p \in (c, v - k)$, both buyer and seller would be strictly better off if the buyer commuted and then paid p for the good.

3.5 part (e)

Choose any value $\hat{p} \in (c, v - k)$. Suppose that in the subgame beginning with the seller's move, the following equilibrium is played:

s_1 is Buy if $p \leq \hat{p}$, Don't Buy otherwise; s_2 is to set $p = \hat{p}$

(This is a Nash equilibrium of the subgame beginning with the seller's move; it cannot be played in an SPE because the strategies played following $p \neq \hat{p}$ are not Nash equilibria of those subgames.) Then the best-response at the top of the game is to commute. Thus,

s_1 to Commute, then Buy if $p \leq \hat{p}$, s_2 to set $p = \hat{p}$

is a Nash equilibrium for any $\hat{p} \in (c, v - k)$, and gives strictly higher payoffs to both players. ($\hat{p} > c$ and $v - \hat{p} - k > 0$.)

3.6 part (f)

We assume that sending the card truly commits the seller to the advertised price, and that SPE are played. From our work above, we know that if the seller does not send a postcard, the buyer will not commute, so the payoffs will be $(0, c)$.

If the seller does send a postcard advertising a price p, the buyer's best-response will be

$$BR_1(s_2 = p) = \begin{cases} \text{Commute and Buy} & if \quad p < v - k \\ \text{Stay Home} & if \quad p > v - k \\ \text{Either} & if \quad p = v - k \end{cases}$$

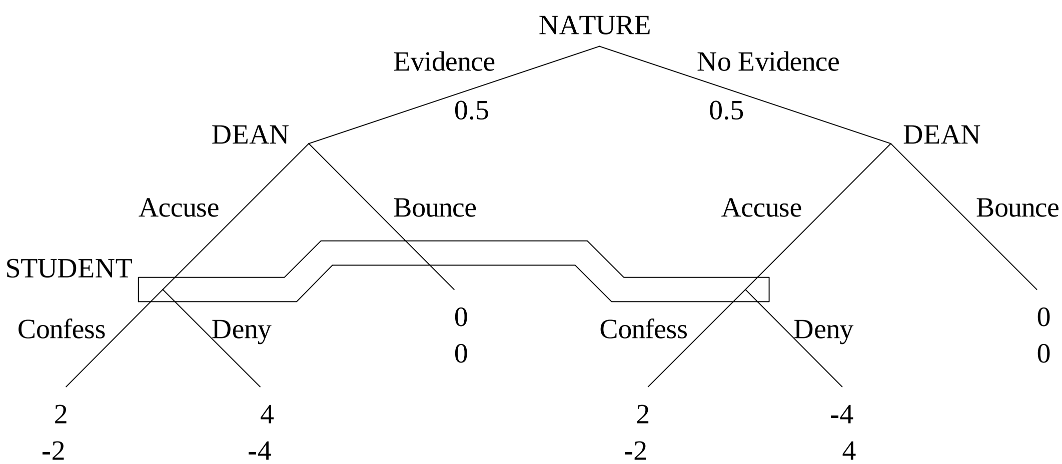
Once again, if the buyer stays home when $p = v - k$ is advertised, the seller has no best-response; the only SPE is when the buyer chooses to commute and buy when $p = v - k$, and the seller advertises $p = v - k$. (This is an SPE whenever $\epsilon \leq v - c - k$, which is is here; if the cost of advertising were too high, the seller would prefer to just consume the good.)

Thus, by sending the card, the seller gets payoff $v - k - \epsilon$, which is greater than c; thus, he does choose to send the card.

4 The Dean's Dilemma (25 points)

4.1 part (a)

Lots of people were not sure how to set up a problem like this. The convention is to introduce a third "player", Nature, who does not receive any payoffs, but moves first and randomly chooses whether evidence exists or not. Once Nature has moved, the Dean moves, followed by the student, who can see the Dean's move but not Nature's. Thus, the extensive-form representation is the following:



Since the student's information set cuts across the "branches" of the game tree, there are no proper subgames other than the entire game.

4.2 part (b)

The Dean acts at two information sets – following Nature's choice of Evidence or No Evidence – and in each case he has two actions – Accuse or Bounce – so his strategy set is $S_1 = \{AA, AB, BA, BB\}$. The Student acts at only one information set, so his strategy set is just $S_2 = \{C, D\}$. For each strategy profile $s \in S_1 \times S_2$, we calculate payoffs as the expected payoffs each player gets, that is, assuming Nature is mixing half-half. This leads to the following matrix:

		Student	
		C	D
Dean	AA	2,-2	0,0
	AB	1,-1	2,-2
	BA	1,-1	-2,2
	BB	0,0	0,0

4.3 part (c)

We can first notice that the Dean has two strictly dominated strategies – BA is dominated by AA, and BB is dominated by AB. Thus, we can eliminate these two strategies, and solve the smaller game

		Student	
		C	D
Dean	AA	2,-2	0,0
	AB	1,-1	2,-2

We can easily see that there are no pure-strategy Nash equilibria. Using our usual method (solve for one player's mixture by setting the other player's expected payoffs equal), we can calculate the mixed-strategy equilibrium, which is

$$s_1 = \frac{1}{3}AA + \frac{2}{3}AB, \quad s_2 = \frac{2}{3}C + \frac{1}{3}D$$

That is, the only Nash equilibrium of this game is for the Dean to accuse whenever he has evidence, and one-third of the time when he does not; and for the Student to confess two-thirds of the time when he is accused.

4.4 part (d)

Since the only proper subgame is the entire game, any Nash equilibrium is a Subgame Perfect equilibrium by definition. So no, you cannot find a NE that is not a SPE.