Econ 160 | Prof. Steve Tadelis
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Midterm Solutions

Question 1. Some Abstractions (25 points)

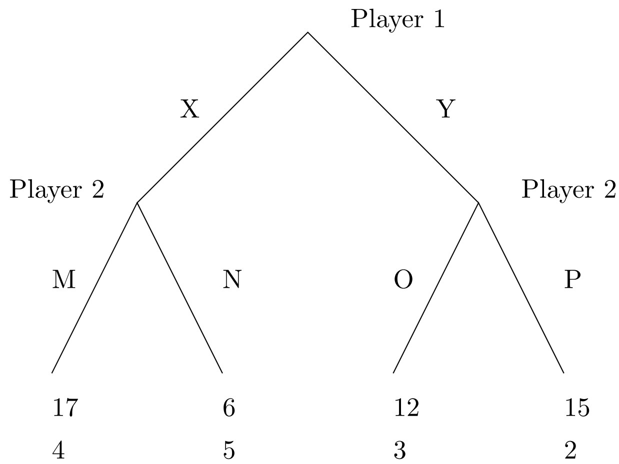
Part a. Normal Form

Player 1 acts at one information set, so his strategy set is $S_1 = \{X, Y\}$. Player 2 acts at two information sets, so his pure strategies are $S_2 = \{MO, MP, NO, NP\}$. The matrix representation of this game is

		Player 2			
		MO	MP	NO	NP
Player 1	X	a, b	a, b	c, d	c, d
	Y	e, f	g, h	e, f	g, h

Part b. Unique SPE

Here is an example of the game tree with payoffs leading to a unique SPE:



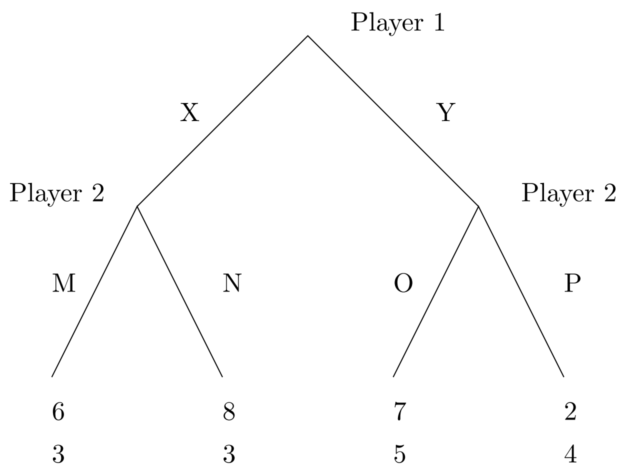
Player 2 has unique best-responses at both information sets, and must play NO in any SPE; this leaves player 1 with a unique best-response, Y, so the only SPE is (Y, NO). There are many other payoff choices that would lead to a unique SPE – as we show in the next section, any time $a \neq c \neq e \neq g$ and $b \neq d \neq f \neq h$, a unique SPE is guaranteed.

Part c. Uniqueness

If $b \neq d$, player 2 has a unique best-response if player 1 plays X. Let j be player 1's payoff when this happens. (That is, $j = a$ if $b > d$, $j = c$ if $d > b$.) Similarly, if $f \neq h$, then player 2 has a unique best-response following Y; let k be player 1's payoff ($k = e$ if $f > h$, and g if $h > f$.) Since payoffs are all distinct, $j \neq k$, so player 1 has a unique best-response between X and Y.

Part d. Multiple Equilibria

Here is an example of the game tree with payoffs leading to multiple SPE:



It is easy to verify that (X, NO) and (Y, MO) are both SPEs. Another one is for player 2 to play $\frac{1}{2}MO + \frac{1}{2}NO$, and for player 1 to mix with any ratio.

Part e. Infinite Equilibria

The point of the problem is to show that if a game of this form has more than one SPE, it must have an infinite number. So we will suppose we know all the SPE of a game and that there are more than two, and show that we can find an infinite number.

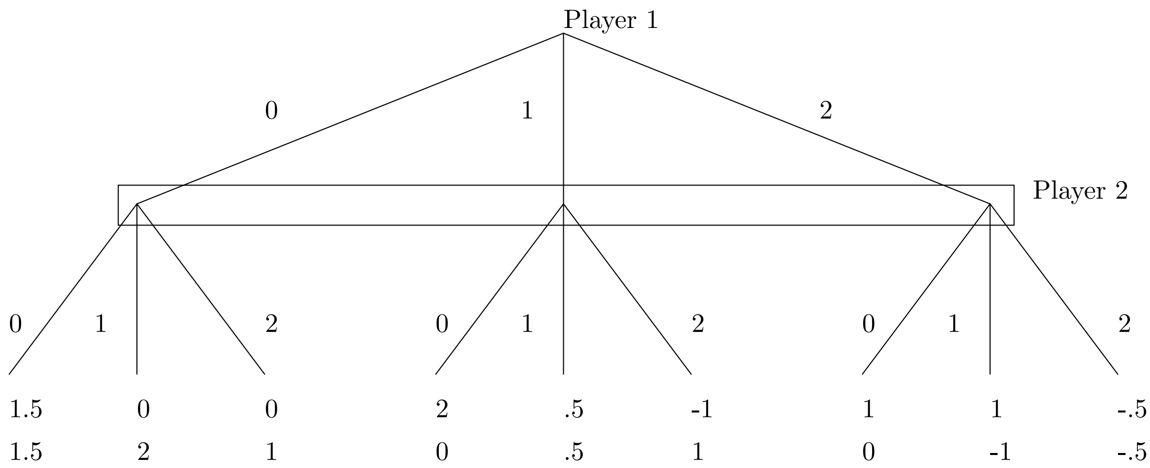
First, suppose that in one SPE, player 2 plays M with positive probability, and in another, player 2 plays N with positive probability. Then player 2 must be indifferent between them ($b = d$). Let Q be any best-response for player 2 following Y. (That is, Q is either O or P, whichever is better for player 2.) We can pick any probability $p \in [0, 1]$, assume that player 2 plays $p(MQ) + (1 - p)(NQ)$, find player 1's best-response, and know that this is an SPE. Since this works for any p, there are an infinite number of SPE.

Similarly, if player 2 plays O with positive probability in one SPE and P with positive probability in another, we make the same argument and show that there must be an infinite number of equilibria.

So now, suppose player 2 always plays the same strategy at each information set in every SPE. Then in order to have more than one SPE, player 1 must sometimes play X and sometimes play Y, even though player 2 is always playing the same strategy. This means that player 1 is indifferent between X and Y, so for any $p \in [0,1]$, player 1 playing $pX + (1-p)Y$, and player 2 playing whatever strategy he plays in his SPE, must be an SPE. Again, this works for any p, so there are an infinite number of equilibria.

Question 2. All-Pay Auction (35 points)

Part a. Game Tree



Part b. Matrix Representation

		Player 2		
		0	1	2
Player 1	0	$\frac{3}{2}, \frac{3}{2}$	$0, 2$	$0, 1$
	1	$2, 0$	$\frac{1}{2}, \frac{1}{2}$	$-1, 1$
	2	$1, 0$	$1, -1$	$-\frac{1}{2}, -\frac{1}{2}$

Part c. Dominated Strategies

There are no dominated strategies for either player.

Part d. NE with a Pure Strategy

We rule out equilibria where player 1 plays a pure strategy:

- If player 1 always bids 0, player 2 best-responds by always bidding 1, and player 1 would want to deviate and bid 2
- If player 1 always bids 1, player 2 best-responds by always bidding 2, and player 1 would want to deviate and bid 0
- If player 1 always bids 2, player 2 best-responds by always bidding 0, and player 1 would want to deviate and bid 1

Thus, there are no equilibria where player 1 plays a pure strategy. By the same arguments, there are no equilibria where player 2 plays a pure strategy.

Part e. NE Where A Player Mixes Between Two Strategies

We rule out equilibria where player 1 mixes between two of his three strategies:

- First, suppose player 1 mixes between 0 and 1. Then for player 2, bidding 0 is strictly dominated by bidding 1. But if player 2 never bids 0, then for player 1, bidding 1 is strictly dominated by 2, so player 1 can't play 1 in equilibrium.
- Next, suppose player 1 mixes between 0 and 2. Then for player 2, bidding 2 is strictly dominated by 0. If player 2 never bids 2, then for player 1, bidding 0 is strictly dominated by 1, so player 1 can't bid 0 in equilibrium.
- Finally, suppose player 1 mixes between 1 and 2. Then for player 2, bidding 1 is strictly dominated by bidding 2. But if player 2 never bids 1, then for player 1, bidding 2 is strictly dominated by 0, so player 1 can't bid 2.

Thus, there are no equilibria where player 1 mixes between exactly two strategies. By the same arguments, there are no equilibria where player 2 mixes between exactly two strategies.

Part f. Equilibria

In the last two parts, we showed that in any Nash equilibrium, both players must be mixing between all three of their strategies. Suppose that player 2 plays the generic mixed strategy $(a, b, 1-a-b)$. Then

$$\begin{array}{rcccl} u_1(0) & = & \frac{3}{2}a + 0b + 0(1-a-b) & = & \frac{3}{2}a \\ u_1(1) & = & 2a + \frac{1}{2}b - 1(1-a-b) & = & 3a + \frac{3}{2}b - 1 \\ u_1(2) & = & 1a + 1b - \frac{1}{2}(1-a-b) & = & \frac{3}{2}a + \frac{3}{2}b - \frac{1}{2} \end{array}$$

Now, since we know that in any equilibrium, player 1 mixes between all three strategies, we know that all three of these expected payoffs must be equal. Setting $u_1(0) = u_1(2)$ gives

$$\frac{3}{2}a = \frac{3}{2}a + \frac{3}{2}b - \frac{1}{2} \quad \longrightarrow \quad \frac{3}{2}b = \frac{1}{2} \quad \longrightarrow \quad b = \frac{1}{3}$$

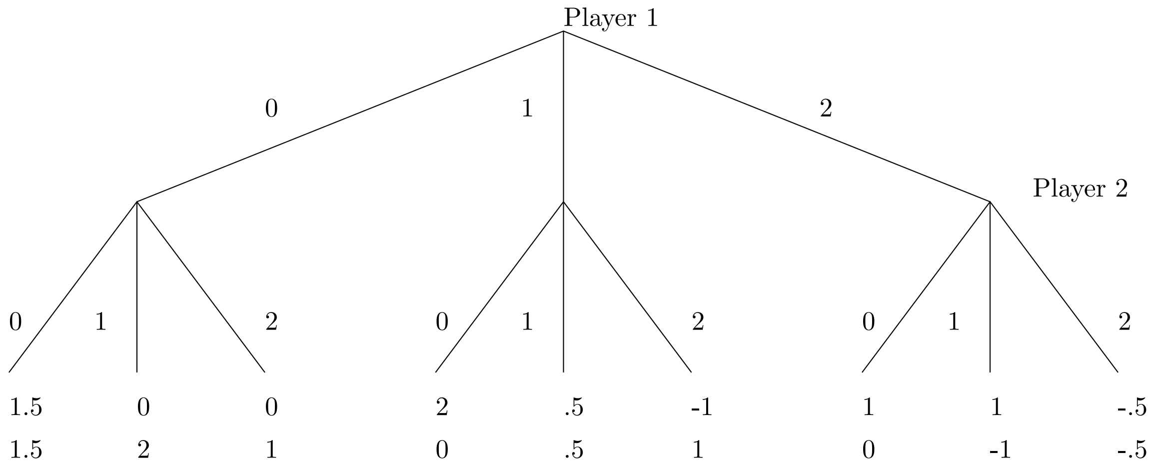
Setting $u_1(1) = u_1(2)$ similarly gives

$$3a + \frac{3}{2}b - 1 = \frac{3}{2}a + \frac{3}{2}b - \frac{1}{2} \quad \longrightarrow \quad \frac{3}{2}a = \frac{1}{2} \quad \longrightarrow \quad a = \frac{1}{3}$$

So in order for player 1 to mix between all three strategies, player 2 must be playing the mixed strategy $s_2^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$; since player 1 must play all three strategies in equilibrium, this is player 2's strategy in every equilibrium. By the same logic, in equilibrium, player 1 plays $s_1^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ as well.

Finally, since there are no proper subgames other than the entire game, this is an SPE as well; since any SPE in a NE, there aren't any other SPE.

Part g. Game Tree



Part h. SPE

This is a game of perfect information; we can solve for the SPE by backward induction.

If player 1 bids 0, player 2's best response is 1. If player 1 bids 1, player 2's best response is 2. If player 1 bids 2, player 2's best response is 0. Thus, in any SPE, player 2 must play $1, 2, 0$. Given this, player 1's best move is to play 2. So the only SPE is $s_1^* = 2$, $s_2^* = 1, 2, 0$.

Part i. Other Equilibria

One example of another equilibrium is $s_1^* = 0$, $s_2^* = 1, 2, 2$. Player 2 is playing a best-response on the equilibrium path, and player 1 chooses between payoffs of 0, 0, and $-\frac{1}{2}$, so he plays a best-response by playing 0.

Part j. The Professor's Problem

In the static problem, the professor gives away three dollars, and (assuming the unique NE is played) takes in an average of two dollars in bids, losing money. In the dynamic problem, the professor again gives away three dollars, and this time (assuming the unique SPE is played) receives exactly two dollars in bids, losing money. The professor is better off not offering the game.

Question 3. Technology and Politics (40 points)

Part a. Perfect or Imperfect Information

Since the firms set quantities simultaneously, the game tree contains information sets which are not singletons, so it is a game of imperfect information.

Part b. Normal Form Game

The *Players* are $N = \{1, 2\}$.

The *Strategies* are $S_1 = S_2 = \mathbf{R}^+$.

The *Payoffs* are $u_i(q_i, q_j) = q_i(30 - q_i - q_j - c_i)$.

Part c. Best Responses

Firm i maximizes his profit function

$$\pi_i(q_i, q_j, c_i) = q_i(30 - q_i - q_j - c_i)$$

First, note that when $q_j + c_i \geq 30$, any positive quantity gives negative profits, so the best-response of firm i is $q_i = 0$. The rest of the time, there is an interior solution characterized by the first-order condition

$$\frac{\partial \pi_i}{\partial q_i} = 30 - 2q_i - q_j - c_i = 0$$

giving the best-response correspondence

$$BR_i(q_j, c_i) = \max\left\{0, \frac{30 - q_j - c_i}{2}\right\}$$

Part d. Nash Equilibrium

We consider the usual case where both firms produce positive amounts. (This will occur, for instance, whenever both c_1 and c_2 are less than 15.) Nash equilibrium is defined by both firms playing best-responses, so we know that

$$\begin{array}{rcl} q_1 & = & \frac{30 - q_2 - c_1}{2} \\ q_2 & = & \frac{30 - q_1 - c_2}{2} \end{array}$$

We can solve these simultaneously by noting that

$$\begin{array}{rcl} 2q_1 + q_2 & = & 30 - c_1 \\ q_1 + 2q_2 & = & 30 - c_2 \end{array}$$

Twice the first equation minus one times the second gives

$$3q_1 = 30 - 2c_1 + c_2$$

or $q_1 = 10 - \frac{2}{3}c_1 + \frac{1}{3}c_2$; similarly, we find $q_2 = 10 - \frac{2}{3}c_2 + \frac{1}{3}c_1$. Thus, the unique Nash equilibrium is

$$(q_1, q_2) = \left(10 - \frac{2}{3}c_1 + \frac{1}{3}c_2, 10 - \frac{2}{3}c_2 + \frac{1}{3}c_1\right)$$

(If either of these quantities is negative, then the equilibrium quantities are either $(0, BR_2(0)) = \left(0, \frac{30-c_2}{2}\right)$ or $(BR_1(0), 0) = \left(\frac{30-c_1}{2}, 0\right)$. For the rest of this problem, we consider the case where both firms produce positive quantities in equilibrium.)

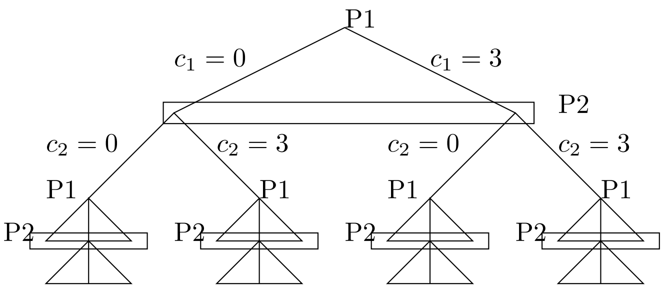
Part e. Payoffs

Plugging the equilibrium quantities into the profit function, equilibrium payoffs are

$$\begin{aligned} u_i(q_i^*, q_j^*, c_i) &= q_i^*(30 - q_i^* - q_j^* - c_i) \\ &= \left(10 - \tfrac{2}{3}c_i + \tfrac{1}{3}c_j\right)\left(30 - \left(10 - \tfrac{2}{3}c_i + \tfrac{1}{3}c_j\right) - \left(10 - \tfrac{2}{3}c_j + \tfrac{1}{3}c_i\right) - c_i\right) \\ &= \left(10 - \tfrac{2}{3}c_i + \tfrac{1}{3}c_j\right)^2 \end{aligned}$$

Part f. Extensive Form

The following is one way to draw the game tree of the two-stage game:



Part g. Payoffs

The payoffs are as follows: when both firms invest (set $c_i = 0$), payoffs are $(q_1(30 - q_1 - q_2) - k, q_2(30 - q_1 - q_2) - k)$. When firm 1 invests and 2 does not, payoffs are $(q_1(30 - q_1 - q_2) - k, q_2(30 - q_1 - q_2 - 3))$. When firm 2 invests and 1 does not, payoffs are $(q_1(30 - q_1 - q_2 - 3), q_2(30 - q_1 - q_2) - k)$. Finally, when neither firm invests, payoffs are $(q_1(30 - q_1 - q_2 - 3), q_2(30 - q_1 - q_2 - 3))$.

Part h. "Reduced" Normal Form

We know that if the firms play the unique SPE in the second stage, the payoffs of the second-stage game (from part (e)) are $\left(\left(10 - \frac{2}{3}c_1 + \frac{1}{3}c_2\right)^2, \left(10 - \frac{2}{3}c_2 + \frac{1}{3}c_1\right)^2\right)$.

In addition, firms which make the investment to lower their marginal costs incur a cost of k, which is subtracted from their payoffs. Plugging in the different combinations gives us the following "reduced-form" for the first-stage game:

		Player 2: 0	Player 2: 3
Player 1	0	$100-k, 100-k$	$121-k, 64$
	3	$64, 121-k$	$81, 81$

Part i. SPE with $k = 20$

Plugging in $k = 20$ gives the following simultaneous-move game in the first round:

		Player 2: 0	Player 2: 3
Player 1	0	$80, 80$	$101, 64$
	3	$64, 101$	$81, 81$

Making the investment (reducing marginal costs to 0) is strictly dominant for both players, so the unique equilibrium in stage 1 is for both players to make the investment. To be complete, the unique SPE is defined by the strategies

$$\begin{aligned} s_1^* &= \text{Invest, set } q_1 = 10 - \tfrac{2}{3}c_1 + \tfrac{1}{3}c_2 \\ s_2^* &= \text{Invest, set } q_2 = 10 - \tfrac{2}{3}c_2 + \tfrac{1}{3}c_1 \end{aligned}$$

Part j. Changing the Investment Costs

If k were changed to 50, we would be left with the first-stage game

		Player 2: 0	Player 2: 3
Player 1	0	$50, 50$	$71, 64$
	3	$64, 71$	$81, 81$

In this game, *not* making the investment is strictly dominant, so the equilibrium is for both firms to leave marginal costs at 3. This leads to payoffs of $(81, 81)$, as opposed to $(80, 80)$ without the bill. Thus, the firms do not campaign against the bill, since it leaves them slightly better off.

When the cost of investment was 20, the reduced-form first-stage game actually resembled a familiar game, the Prisoner's Dilemma. Making the investment was strictly dominant, but still led to a Pareto-inferior result. Increasing the cost of investment was analogous to introducing a punishment for playing Fink in the Prisoner's Dilemma; if the punishment is severe enough, the players would opt to play (Mum, Mum), leading to higher payoffs.