

# Value at risk and efficiency under dependence and heavy-tailedness: models with common shocks

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Received: 20 November 2009 / Accepted: 11 August 2010 / Published online: 28 August 2010  
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**Abstract** This paper presents an analysis of diversification and portfolio value at risk for heavy-tailed dependent risks in models with multiple common shocks. We show that, in the framework of value at risk comparisons, diversification is optimal for moderately heavy-tailed dependent risks with common shocks and finite first moments, provided that the model is balanced, i.e., that all the risks are available for portfolio formation. However, diversification is inferior in balanced extremely heavy-tailed risk models with common factors. Finally, in several unbalanced dependent models, diversification is optimal, even though there is extreme heavy-tailedness in common shocks or in idiosyncratic parts of the risks. Analogues of the obtained results further hold for efficiency comparisons of linear estimators in random effects models with dependent and heavy-tailed observations.

**Keywords** Portfolio analysis · Value at risk · Power laws · Heavy-tailedness · Diversification · Dependence · Common shocks · Factor models · Riskiness · Majorization · Random effects · Linear estimators · Efficiency

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We thank the NUS Risk Management Institute for support, and Daniel Ahn, Shashibhushan Borade, John Campbell and Tian Tian Qiu for valuable discussions. Ibragimov gratefully acknowledges research support provided by the National Science Foundation grant SES-0820124. We also thank an anonymous referee for many helpful comments and suggestions.

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**JEL Classification** G11 · C13**1 Introduction**

Many important economic and financial variables are influenced by common macroeconomic, political, legal and environmental shocks. Examples of such shocks are given by (see the review and examples in [Andrews 2003, 2005](#)) financial crises affecting individual and firm consumption, investment and production decisions; stock market shocks influencing individual wealth and firm assets; oil price shocks and business cycles affecting firm factor costs and production; inflation affecting nominal wages; and employment shocks influencing economic decisions of market participants. Theoretical and empirical frameworks with common shocks discussed in the literature include models with output, investment and savings affected by common shocks due to, e.g., technology changes or financial crises; microeconomic models with wages influenced by prices for unmeasured skills and the worker and firm effects; factor models for financial asset returns (see the review and discussion in Sect. 2 in [Bai 2009](#)); health spending and health outcomes affected by technological advances, diseases, epidemics and other health shocks, and the implementation of new health policies (see [Moscone and Tosetti 2009](#)); co-movements of financial and insurance variables in different markets due to exogenous common shocks such as financial and economic crises (see, among others, [Embrechts et al. 2005](#), and references therein); and numerous others.

A number of studies in economics and finance have argued that many series encountered in these fields are heavy-tailed and can be modeled using risks  $X$  with distributions exhibiting power law decline<sup>1</sup>

$$P(|X| > x) \asymp x^{-\alpha} \quad (1.1)$$

(see, among others, the discussion in [Embrechts et al. 1997](#); [Rachev et al. 2005](#); [Gabaix 2009](#); [Ibragimov 2009a](#), and references therein). The parameter  $\alpha$  in (1.1) is referred to as the tail index, or the tail exponent, of the distribution of  $X$ . An important property of r.v.'s  $X$  satisfying (1.1) is that the absolute moments of  $X$  are finite if and only if their order is less than the tail index  $\alpha$ :  $E|X|^p < \infty$  if  $p < \alpha$  and  $E|X|^p = \infty$  if  $p \geq \alpha$ . We mention a sample of estimates of the tail index  $\alpha$  for returns on various stocks and stock indices:  $3 < \alpha < 5$  ([Jansen and de Vries 1991](#)),  $2 < \alpha < 4$  ([Loretan and Phillips 1994](#)),  $1.5 < \alpha < 2$  ([McCulloch 1997](#)),  $0.9 < \alpha < 2$  ([Rachev and Mittnik 2000](#)),  $\alpha \approx 3$  ([Gabaix et al. 2006](#)). Power laws (1.1) with  $\alpha \approx 1$  (Zipf laws) have been found to hold for city sizes and firm sizes (see [Gabaix 1999](#), and [Axtell 2001](#)). As discussed by [Nešlehová et al. \(2006\)](#), tail indices less than one are observed for empirical loss distributions of a number of operational risks. [Silverberg and Verspagen \(2007\)](#) report the tail indices  $\alpha$  to be significantly less than one for financial returns from technological innovations. The analysis in [Ibragimov et al. \(2009\)](#) indicates that the

<sup>1</sup> Here and throughout the paper,  $f(x) \asymp g(x)$  means that  $0 < c \leq f(x)/g(x) \leq C < \infty$  for large  $x$ , for constants  $c$  and  $C$ .

tail indices may be considerably less than one for economic losses from earthquakes and other natural disasters.

Several recent studies have analyzed value at risk and portfolio choice for heavy-tailed risks and related problems. [Bouchard and Potters \(2004\)](#), Ch. 12, present a detailed discussion of portfolio choice under various distributional and dependence assumptions and diversification measures, including the asymptotic results in the value at risk framework for heavy-tailed power law distributions. As was shown in [Ibragimov \(2005, 2009b\)](#) in a general context based on majorization theory and arbitrary portfolio weights comparisons (see also the review in [Ibragimov 2009a](#)), diversification may be inferior in the value at risk framework for heavy-tailed risks whose distributions satisfy power law (1.1) with  $\alpha < 1$  (see Proposition 3.2 in Sect. 3). As shown in [Ibragimov \(2005, 2009b\)](#), diversification is typically preferable in value at risk models with convolutions of stable heavy-tailed risks that follow (1.1) with  $\alpha > 1$  (see Proposition 3.1 in Sect. 3). Recently, [Ibragimov and Walden \(2007\)](#) showed that, with a value at risk approach with bounded risks concentrated on a sufficiently large interval, diversification may be suboptimal up to a certain number of risks and then become optimal. [Ibragimov et al. \(2009\)](#) demonstrate how this analysis can be used to explain low levels of reinsurance among insurance providers in markets for catastrophe reinsurance. [Ibragimov and Walden \(2008\)](#) study portfolio diversification for nonlinear transformations of heavy-tailed risks and for distributions that exhibit local or moderate deviations from power tails (1.1) in the form of additional slowly varying factors. Several examples that illustrate the phenomenon that diversification is not always preferable are presented in [Kaas et al. \(2004\)](#).

While the above works provide several extensions of the value at risk analysis for the case of dependence, including the case of multiplicative common shocks (see [Ibragimov 2005, 2009b](#); [Ibragimov and Walden 2007](#), and Proposition 7.1 in Sect. 7), *exact closed-form solutions are usually available only in the setting with uncorrelated risks*. Our objective with this paper is to extend the analysis beyond independence and uncorrelatedness. We obtain general results on portfolio value at risk comparisons for correlated heavy-tailed risks exhibiting additive common shocks structures of the type

$$Y_{ij} = R_i + C_j + U_{ij}, \quad i = 1, \dots, r, \quad j = 1, \dots, c. \quad (1.2)$$

In (1.2), the “row effects” common shocks  $R_i$ , the “column effects” common shocks  $C_j$  and the “error” variables  $U_{ij}$  are assumed to be independent of each other and to be independent and identically distributed among themselves.

We also present the value at risk analysis for a particular case of (1.2) given by

$$Y_{ij} = R_i + U_{ij}, \quad i = 1, \dots, r, \quad j = 1, \dots, c. \quad (1.3)$$

Together with their multiplicative analogues

$$Y_{ij} = R_i U_{ij}, \quad R_i > 0, \quad (1.4)$$

(see [Ibragimov 2005, 2009b](#) and [Ibragimov and Walden 2007](#)), models (1.2) and (1.3) provide a natural framework for modeling risks subject to (additive) common shocks

$R_i$  and  $C_j$  (such as political or macroeconomic ones, see the discussion in Andrews 2003, 2005). The common shocks  $R_i$  affect all risks  $Y_{ij}$ ,  $j = 1, \dots, c$ , in the  $i$ th row (say, in the  $i$ th country) and the common shocks  $C_j$  affect all risks  $Y_{ij}$ ,  $i = 1, \dots, r$ , in the  $j$ th column (say, in the  $j$ th industry).<sup>2</sup>

The results obtained in the paper cover the case of dependent and possibly non-identically distributed risks  $R_i$ ,  $C_j$  and  $U_{ij}$  (see Sect. 7). Furthermore, although we mainly work with models of the form (1.2) and (1.3), our results can be generalized to models with varying factor loadings, such as

$$Y_{ij} = \beta_{ij}^{(r)} R_i + \beta_{ij}^{(c)} C_j + U_{ij}, \quad i = 1, \dots, r, \quad j = 1, \dots, c, \quad (1.5)$$

(see Bai 2009). For simplicity, we first consider the unit beta case (1.2) and (1.3) and then discuss the extensions to general factor settings (1.5) in Sect. 7. We also note that the results in the paper can be further used to obtain value at risk and diversification comparisons for the usual fixed effects models in the form  $Y_{it} = X'_{it}\beta + R_i + C_t + U_{it}$ , where  $X_{it}$  is a  $p \times 1$  vector of (possibly heavy-tailed) regressors,  $\beta$  is a  $p \times 1$  vector of unknown regression coefficients, and  $R_i$  and  $C_t$  are, respectively, individual and time effect variables.

The paper is organized as follows. Section 2 introduces the notation and definitions of classes of moderately heavy-tailed and extremely heavy-tailed distributions dealt with throughout the paper.

Section 3 contains the main results of the paper on value at risk analysis and optimal portfolio choice in common shocks models (1.2). We show that portfolio value at risk comparisons with the most diversified and the least diversified portfolio weights for dependent risks  $Y_{ij}$  in (1.2) under heavy-tailedness are similar to those in the case of independence (Theorem 3.1). In particular, under moderate heavy-tailedness with  $\alpha > 1$  in the shocks  $R_i$ ,  $C_j$  and  $U_{ij}$  in (1.2), the most diversified portfolio of risks  $Y_{ij}$  with equal weights is optimal with respect to portfolio value at risk comparisons. In addition, in such settings, the least diversified portfolio of  $Y'_{ij}$ s consisting of only one risk maximizes the value at risk over all portfolio weights. These conclusions are reversed under extreme heavy-tailedness with  $\alpha < 1$  in the common shocks  $R_i$  and  $C_j$  and the idiosyncratic risks  $U_{ij}$  in (1.2). Under these assumptions, the most diversified portfolio with equal weights has the maximal value at risk among all portfolios of  $Y_{ij}$ . The optimal portfolio under extreme heavy-tailedness in the variables in (1.2) is given by the least diversified portfolio that consists of only one risk. Theorem 3.2 shows that these results continue to hold separately for the common shock and idiosyncratic components of the returns on the portfolios of risks  $Y_{ij}$ . Theorem 3.2 thus allows one to compare value at risk for the portfolios of common shocks  $R_i$  or  $C_j$ , or for the portfolios of  $U_{ij}$  under heavy-tailedness in these variables.

Section 4 provides extensions of the results in Theorems 3.1 and 3.2 to value at risk comparisons between portfolios that are different from the most diversified and the least diversified portfolios (Theorem 4.1). The results provide value at risk

<sup>2</sup> The dependence properties of additive common shock models (1.2) and (1.3) are more complicated than those in relation (1.4) and its two-shock analogues. For instance, while multiplicative common shock models (1.4) imply uncorrelatedness of the risks  $Y_{ij}$ , this, evidently, does not hold for (1.3).

comparisons for the portfolios considered in the literature on efficiency of linear location estimators in models of type (1.2) (see the review in Appendix A). The value at risk comparisons for these portfolios under extreme heavy-tailedness in risks  $R_i$ ,  $C_j$  and  $U_{ij}$  are opposite to those in the case of moderate heavy-tailedness. In addition, some of the comparisons have a natural interpretation in terms of the optimal portfolio choice for indices of the risks  $Y_{ij}$  in (1.2).

In Section 5 we show that the majorization approach to value at risk analysis developed in Sections 3 and 4 can be applied in the case of unbalanced models (1.2) or (1.3) that have unequal number of rows for each column or unequal number of columns for each row. Building on the interpretation in Section 4, the results in Section 5 are presented in the framework of value at risk analysis for equally weighted indices of heavy-tailed risks in (1.3). Theorems 5.1 and 5.2 imply optimality of diversification patterns in unbalanced dependent models, even though there is extreme heavy-tailedness in common shocks or in idiosyncratic parts of the risks. These conclusions are in contrast to variance comparisons for portfolio returns implied by the results in the literature on linear location estimation in random effects models (see the discussion in Sect. 5 and Appendix A).

Section 6 discusses econometric and statistical applications of the results obtained in the paper. These applications are the analogues of the value at risk results in the framework of efficiency comparisons of linear estimators of location in random effects models.

To illustrate the main ideas and results, the portfolio value at risk analysis in Sections 3–5 is presented in the framework of risks (1.2) and (1.3) that are subject to two or less additive common shocks with independence among the variables  $R_i$ ,  $C_j$  and  $U_{ij}$ . In Sect. 7, we discuss how most of the results can be generalized to models of type (1.5) with varying factor loadings and to the case with more than two common factors. In addition, we show how the results in paper can also be extended to the case of dependence within the common shocks and the idiosyncratic risks. These dependence structures include convolutions of  $\alpha$ -symmetric distributions and models with multiple multiplicative common shocks. Section 7 further discusses the analogues of the results in the paper for non-identically distributed dependent risks. Section 8 makes some concluding remarks. Appendix A reviews the results in the statistics literature related to the problems considered in the paper. Appendix B contains proofs of the results obtained in the paper.

## 2 Notation and classes of distributions

A r.v.  $X$  with density  $f : \mathbf{R} \rightarrow \mathbf{R}$  and the convex distribution support  $\Omega = \{x \in \mathbf{R} : f(x) > 0\}$  is log-concavely distributed if  $\log f(x)$  is concave in  $x \in \Omega$ , that is, if for all  $x_1, x_2 \in \Omega$ , and any  $\lambda \in [0, 1]$ ,  $f(\lambda x_1 + (1 - \lambda)x_2) \geq (f(x_1))^\lambda (f(x_2))^{1-\lambda}$  (see An 1998, and Bagnoli and Bergstrom 2005). A distribution is said to be log-concave if its density  $f$  satisfies the above inequalities. Examples of log-concave distributions include the normal distribution, the uniform density, the exponential density, the Gamma distribution  $\Gamma(\alpha, \beta)$  with the shape parameter  $\alpha \geq 1$ , the Beta distribution  $\mathcal{B}(a, b)$  with  $a \geq 1$  and  $b \geq 1$ ; and the Weibull distribution  $\mathcal{W}(\gamma, \alpha)$  with the

shape parameter  $\alpha \geq 1$ . Log-concave distributions have many appealing properties that have been utilized in a number of works in economics and finance (see the surveys in [Karlin 1968](#); [Marshall and Olkin 1979](#); [An 1998](#), and [Bagnoli and Bergstrom 2005](#)). However, such distributions cannot be used in the study of heavy-tailedness phenomena since any log-concave density is extremely thin-tailed: in particular, if a r.v.  $X$  is log-concavely distributed, then its density has at most an exponential tail, that is,  $f(x) = O(\exp(-\lambda x))$  for some  $\lambda > 0$ , as  $x \rightarrow \infty$  and, therefore, all the power moments  $E|X|^\gamma$ ,  $\gamma > 0$ , of the r.v. exist (see Corollary 1 in [An 1998](#)). Throughout the paper,  $\mathcal{LC}$  denotes the class of symmetric log-concave distributions ( $\mathcal{LC}$  stands for “log-concave”).

For  $0 < \alpha \leq 2$ ,  $\sigma > 0$ ,  $\beta \in [-1, 1]$  and  $\mu \in \mathbf{R}$ , we denote by  $S_\alpha(\sigma, \beta, \mu)$  the stable distribution with the characteristic exponent (index of stability)  $\alpha$ , the scale parameter  $\sigma$ , the symmetry index (skewness parameter)  $\beta$  and the location parameter  $\mu$ . That is,  $S_\alpha(\sigma, \beta, \mu)$  is the distribution of a r.v.  $X$  with the characteristic function (c.f.)

$$E(e^{itX}) = \begin{cases} \exp\{i\mu t - \sigma^\alpha |t|^\alpha (1 - i\beta \operatorname{sign}(t) \tan(\pi\alpha/2))\}, & \alpha \neq 1, \\ \exp\{i\mu t - \sigma |t|(1 + (2/\pi)i\beta \operatorname{sign}(t) \ln |t|)\}, & \alpha = 1, \end{cases} \quad (2.1)$$

$t \in \mathbf{R}$ , where  $i^2 = -1$  and  $\operatorname{sign}(t)$  is the sign of  $t$  defined by  $\operatorname{sign}(t) = 1$  if  $t > 0$ ,  $\operatorname{sign}(0) = 0$  and  $\operatorname{sign}(t) = -1$  otherwise (expression (2.1) is one of several possible parameterizations of c.f.’s of stable distributions). In what follows, we write  $X \sim S_\alpha(\sigma, \beta, \mu)$ , if the r.v.  $X$  has the stable distribution  $S_\alpha(\sigma, \beta, \mu)$ .

The index of stability  $\alpha$  characterizes the heaviness (the rate of decay) of the tails of stable distributions  $S_\alpha(\sigma, \beta, \mu)$ . In particular, if  $X \sim S_\alpha(\sigma, \beta, \mu)$ ,  $\alpha \in (0, 2)$ , then its distribution satisfies power law (1.1). As discussed in the introduction, this implies that the  $p$ -th absolute moments  $E|X|^p$  of a r.v.  $X \sim S_\alpha(\sigma, \beta, \mu)$ ,  $\alpha \in (0, 2)$  are finite if  $p < \alpha$  and are infinite otherwise.

For two r.v.’s  $X$  and  $Y$ , we write  $X \stackrel{d}{=} Y$  if  $X$  and  $Y$  have the same distribution.

Distributions  $S_\alpha(\sigma, \beta, \mu)$  with  $\mu = 0$  for  $\alpha \neq 1$  and  $\beta = 0$  for  $\alpha = 1$  are called strictly stable. If  $X_i \sim S_\alpha(\sigma, \beta, \mu)$ ,  $\alpha \in (0, 2]$ ,  $i = 1, \dots, N$ , are i.i.d. strictly stable r.v.’s, then, for all  $w_i \geq 0$ ,  $i = 1, \dots, N$ , with  $\sum_{i=1}^N w_i \neq 0$ ,

$$\sum_{i=1}^N w_i X_i / \left( \sum_{i=1}^N w_i^\alpha \right)^{1/\alpha} \stackrel{d}{=} X_1 \quad (2.2)$$

(see [Zolotarev 1986](#); [Embrechts et al. 1997](#), and [Rachev and Mittnik 2000](#), for a detailed review of properties of stable distributions).

We denote by  $\overline{\mathcal{CS}}$  the class of distributions which are convolutions of symmetric stable distributions  $S_\alpha(\sigma, 0, 0)$  with characteristic exponents  $\alpha \in [1, 2]$  and  $\sigma > 0$  (here and below,  $\mathcal{CS}$  stands for “convolutions of stable”; the overline indicates that convolutions of stable distributions with indices of stability not less than 1 are taken). That is,  $\overline{\mathcal{CS}}$  consists of distributions of r.v.’s  $X$  for which, with some  $k \geq 1$ ,  $X = Y_1 + \dots + Y_k$ , where  $Y_i$ ,  $i = 1, \dots, k$ , are independent r.v.’s such that  $Y_i \sim S_{\alpha_i}(\sigma_i, 0, 0)$ ,  $\alpha_i \in [1, 2]$ ,  $\sigma_i > 0$ ,  $i = 1, \dots, k$ .

Further,  $\underline{\mathcal{CS}}$  stands for the class of distributions which are convolutions of symmetric stable distributions  $S_\alpha(\sigma, 0, 0)$  with indices of stability  $\alpha \in (0, 1]$  and  $\sigma > 0$  (the underline indicates considering stable distributions with indices of stability not greater than 1). That is,  $\underline{\mathcal{CS}}$  consists of distributions of r.v.'s  $X$  for which, with some  $k \geq 1$ ,  $X = Y_1 + \dots + Y_k$ , where  $Y_i, i = 1, \dots, k$ , are independent r.v.'s such that  $Y_i \sim S_{\alpha_i}(\sigma_i, 0, 0), \alpha_i \in (0, 1], \sigma_i > 0, i = 1, \dots, k$ .

Finally, we denote by  $\overline{\mathcal{CSLC}}$  the class of convolutions of distributions from the classes  $\mathcal{LC}$  and  $\overline{\mathcal{CS}}$ . That is,  $\overline{\mathcal{CSLC}}$  is the class of convolutions of symmetric distributions which are either log-concave or stable with characteristic exponents not less than one ( $\mathcal{CSLC}$  is the abbreviation of ‘‘convolutions of stable and log-concave’’). In other words,  $\overline{\mathcal{CSLC}}$  consists of distributions of r.v.'s  $X$  such that  $X = Y_1 + Y_2$ , where  $Y_1$  and  $Y_2$  are independent r.v.'s with distributions belonging to  $\mathcal{LC}$  or  $\overline{\mathcal{CS}}$ .

All the classes  $\mathcal{LC}, \overline{\mathcal{CSLC}}, \overline{\mathcal{CS}}$  and  $\underline{\mathcal{CS}}$  are closed under convolutions. In particular, the class  $\overline{\mathcal{CSLC}}$  coincides with the class of distributions of r.v.'s  $X$  such that, for some  $k \geq 1$ ,  $X = Y_1 + \dots + Y_k$ , where  $Y_i, i = 1, \dots, k$ , are independent r.v.'s with distributions belonging to  $\mathcal{LC}$  or  $\overline{\mathcal{CS}}$ .

In what follows, we write  $X \sim \mathcal{LC}$  (resp.,  $X \sim \overline{\mathcal{CSLC}}, X \sim \overline{\mathcal{CS}}$  or  $X \sim \underline{\mathcal{CS}}$ ) if the distribution of the r.v.  $X$  belongs to the class  $\mathcal{LC}$  (resp.,  $\overline{\mathcal{CSLC}}, \overline{\mathcal{CS}}$  or  $\underline{\mathcal{CS}}$ ).

The distributions of r.v.'s  $X$  in the class  $\overline{\mathcal{CSLC}}$  are moderately heavy-tailed in the sense that they have finite moments of all orders  $p \in (0, 1) : E|X|^p < \infty, p \in (0, 1)$ . In contrast, the distributions of r.v.'s  $X$  from the class  $\underline{\mathcal{CS}}$  are extremely heavy-tailed in the sense that, if  $X \sim \underline{\mathcal{CS}}$ , then, for some order  $p \in (0, 1)$ , the moment  $E|X|^p$  is infinite:  $E|X|^p = \infty$ . For a more extensive discussion on the classes of distributions discussed in this section and their generalizations, see [Ibragimov \(2005, 2009b\)](#) and [Ibragimov and Walden \(2007, 2008\)](#).

### 3 Portfolio value at risk for models with multiple additive common shocks

Given a loss probability  $0 < q < 1/2$  and a r.v. (risk)  $X$ , we denote by  $VaR_q[X]$  the value at risk (VaR) of  $X$  at level  $q$ , that is, the  $(1 - q)$ -quantile of  $X$ :  $VaR_q[X] = \inf\{x \in \mathbf{R} : P(X > x) \leq q\}$  (throughout the paper, we interpret the positive values of risks  $X$  as a risk holder's losses). For a risk  $X$  with finite second moment,  $var[X]$  will stand for its variance:  $var[X] = E(X - EX)^2$ .

In what follows,  $\mathbf{R}_+$  stands for  $\mathbf{R}_+ = [0, \infty)$ . For  $N \geq 1$ , denote  $\mathcal{I}_N = \{w = (w_1, \dots, w_N) \in \mathbf{R}_+^N : \sum_{i=1}^N w_i = 1\}$ . In addition, given the  $w = (w_1, \dots, w_N) \in \mathbf{R}_+^N$  and  $N$  risks  $X_1, \dots, X_N$ , we denote by  $X(w) = \sum_{i=1}^N w_i X_i$  the return on the portfolio of  $X_i$ 's with weights  $w$ .

Let  $\underline{w}_N = \underbrace{(1/N, 1/N, \dots, 1/N)}_N \in \mathcal{I}_N$  stand for the vector of equal portfolio weights and let  $\bar{w}_N = \underbrace{(1, 0, \dots, 0)}_N \in \mathcal{I}_N$  stand for the weights in the portfolio consisting of only one risk.

A vector  $a \in \mathbf{R}^N$  is said to be majorized by a vector  $b \in \mathbf{R}^N$ , written  $a < b$ , if  $\sum_{i=1}^k a_{[i]} \leq \sum_{i=1}^k b_{[i]}, k = 1, \dots, N - 1$ , and  $\sum_{i=1}^N a_{[i]} = \sum_{i=1}^N b_{[i]}$ , where  $a_{[1]} \geq \dots \geq a_{[N]}$  and  $b_{[1]} \geq \dots \geq b_{[N]}$  denote the components of  $a$  and  $b$  in decreasing order. The relation  $a < b$  implies that the components of the vector  $a$  are less

diverse than those of  $b$  (see Marshall and Olkin 1979). In this context, it is easy to see that the following relations hold for the vectors  $\underline{w}_N = (1/N, 1/N, \dots, 1/N)$  and  $\bar{w}_N = (1, 0, \dots, 0)$  with the least diverse and the most diverse components:

$$\underline{w}_N \prec w \prec \bar{w}_N \tag{3.1}$$

for all  $w \in \mathcal{I}_N$ .

A function  $\phi : A \rightarrow \mathbf{R}$  defined on  $A \subseteq \mathbf{R}^N$  is called *Schur-convex* (resp., *Schur-concave*) on  $A$  if  $(a \prec b) \implies (\phi(a) \leq \phi(b))$  (resp.  $(a \prec b) \implies (\phi(a) \geq \phi(b))$ ) for all  $a, b \in A$ . Evidently, if  $\phi$  is Schur-convex or Schur-concave on a symmetric set  $A$  (so that  $(a_1, \dots, a_N) \in A$  implies  $(a_{\pi(1)}, \dots, a_{\pi(N)}) \in A$  for all permutations  $\pi$  of the set  $\{1, \dots, N\}$ ), then  $\phi$  is symmetric on  $A$ , that is,  $\phi(a_{\pi(1)}, \dots, a_{\pi(N)}) = \phi(a_1, \dots, a_N)$  for all permutations  $\pi$ .

Suppose that  $v = (v_1, \dots, v_N) \in \mathbf{R}_+^N$  and  $w = (w_1, \dots, w_N) \in \mathbf{R}_+^N$ ,  $\sum_{i=1}^N v_i = \sum_{i=1}^N w_i$ , are the weights of two portfolios of  $N$  risks (indices or assets' returns). If  $v \prec w$ , it is natural to think about the portfolio with weights  $v$  as being more diversified than that with weights  $w$  (see the discussion in Ibragimov 2005, and Ibragimov 2009b). Thus, for example, the portfolio with equal weights  $\underline{w}_N$  in (3.1) is the most diversified among all the portfolios with weights  $w \in \mathcal{I}_N$ . In contrast, the portfolios with weights given by the components of  $\bar{w}_N$  or their permutations consist of one risk and are the least diversified among the portfolios with weights  $w \in \mathcal{I}_N$ . In this regard, the notion of one portfolio being more or less diversified than another one is, in some sense, the opposite to the majorization comparison for the vectors of weights of the two portfolios.

Recently, Ibragimov (2005, 2009b) obtained the following results on VaR and diversification comparisons for heavy-tailed portfolios of independent risks. According to the results in Proposition 3.1 provided by Theorem 4.1 in Ibragimov (2009b), in the case of independent moderately heavy-tailed risks  $X_i$ , diversification of a portfolio leads to a decrease in the riskiness of its return  $X(w)$ .

**Proposition 3.1** (Ibragimov 2009b, Theorem 4.1) *Let  $q \in (0, 1/2)$  and let  $X_i, i = 1, \dots, N$ , be i.i.d. risks such that  $X_i \sim \overline{CSLC}$ . Then*

- (i)  $VaR_q[X(v)] \leq VaR_q[X(w)]$  if  $v \prec w$  (in other words, the function  $\psi(w, q) = VaR_q[X(w)]$  is Schur-convex in  $w \in \mathbf{R}_+^n$ ).
- (ii) In particular,  $VaR_q[X(\underline{w}_N)] \leq VaR_q[X(w)] \leq VaR_q[X(\bar{w}_N)]$  for all weights  $w \in \mathcal{I}_N$ .

According to the results in Proposition 3.2 provided by Theorem 4.2 in Ibragimov (2009b), the results in Proposition 3.1 are reversed in the case of independent extremely heavy-tailed risks  $X_i$ . In such settings, diversification of a portfolio increases the riskiness of its return  $X(w)$ .

**Proposition 3.2** (Ibragimov 2009b, Theorem 4.2) *Let  $q \in (0, 1/2)$  and let  $X_i, i = 1, \dots, N$ , be i.i.d. risks such that  $X_i \sim \underline{CS}$ . Then*

- (i)  $VaR_q[X(v)] \geq VaR_q[X(w)]$  if  $v \prec w$  (in other words, the function  $\psi(w, q) = VaR_q[X(w)]$  is Schur-concave in  $w \in \mathbf{R}_+^n$ ).



(ii) *In particular,  $VaR_q[X(\bar{w}_N)] \leq VaR_q[X(w)] \leq VaR_q[X(\underline{w}_N)]$  for all weights  $w \in \mathcal{I}_N$ .*

*Remark 3.1* As discussed in Ibragimov (2009b), the VaR comparisons in Propositions 3.1 and 3.2 hold as equalities for i.i.d. stable risks  $X_i \in S_1(\sigma, 0, 0)$  with the tail index  $\alpha = 1$  that belong to both the classes  $\overline{CSLC}$  and  $\underline{CS}$ . Similar to the arguments for the results in the paper, this implies that all the VaR comparisons in them hold as equalities for risks that have distributions  $S_1(\sigma, 0, 0)$  with  $\alpha = 1$ . Similar to Propositions 3.1 and 3.2, throughout the paper, we present the results with non-strict inequalities for the values at risk of portfolios considered. All these results can be easily re-formulated in terms of strict inequalities.

*Remark 3.2* Let  $X_i \sim S_\alpha(\sigma_i, \beta, \mu)$  be not necessarily identically distributed stable risks with some scale parameters  $\sigma_i > 0$ . Similar to Marshall and Olkin (1979), denote  $\mathcal{D}_N = \{w = (w_1, \dots, w_N) \in \mathbf{R}_+^N : w_1 \leq w_2 \leq \dots \leq w_N\}$ . The extensions of the value at risk comparisons for the non-identically distributed case in Theorems A3.1 and A3.2 in Ibragimov (2009b), imply that Theorem 3.1 holds for  $X_i$  and  $v, w \in \mathcal{D}_N \cap \mathcal{I}_N = \{w \in \mathbf{R}_+^N : w_1 \leq w_2 \leq \dots \leq w_N, \sum_{i=1}^N w_i = 1\}$  if  $\alpha \in [1, 2]$  and  $\sigma_1 \leq \dots \leq \sigma_N$ . Theorem 3.2 holds for  $X_i$  and  $v, w \in \mathcal{D}_N \cap \mathcal{I}_N$  if  $\alpha \in (0, 1]$  and  $\sigma_1 \geq \dots \geq \sigma_N$ .

As discussed in the introduction, throughout this section and Sects. 4–6, we consider portfolios of risks  $Y_{ij}, i = 1, \dots, r, j = 1, \dots, c$ , in model (1.2) with identically distributed “row effects” common shocks  $R_i, i = 1, \dots, r$ , identically distributed “column effects” common shocks  $C_j, j = 1, \dots, c$ , and identically distributed idiosyncratic components  $U_{ij}, i = 1, \dots, r, j = 1, \dots, c$ . The r.v.’s  $R_i, C_j, U_{ij}, i = 1, \dots, r, j = 1, \dots, c$ , are assumed to be independent. Section 7 discusses extensions of the results to models of type (1.5) with varying factor loadings, the case of dependent and possibly non-identically distributed common shocks  $R_i, C_j$  and error variables  $U_{ij}$  and to models with multiple common shocks.

For  $w = (w_{11}, \dots, w_{1c}, w_{21}, \dots, w_{2c}, \dots, w_{r1}, \dots, w_{rc}) \in \mathcal{I}_{rc}$ , denote  $w_{0j} = \sum_{i=1}^r w_{ij}, j = 1, \dots, c, w_{i0} = \sum_{j=1}^c w_{ij}, i = 1, \dots, r$ . Further, denote

$$w_0^{(row)} = (w_{10}, \dots, w_{r0}) \in \mathcal{I}_r, \tag{3.2}$$

$$w_0^{(col)} = (w_{01}, \dots, w_{0c}) \in \mathcal{I}_c. \tag{3.3}$$

For  $w = (w_{11}, \dots, w_{1c}, w_{21}, \dots, w_{2c}, \dots, w_{r1}, \dots, w_{rc}) \in \mathcal{I}_{rc}$ , the return on the portfolio of risks  $Y_{ij}$  in (1.2) with weights  $w$  is given by

$$\begin{aligned} Y(w) &= \sum_{i=1}^r \sum_{j=1}^c w_{ij} Y_{ij} = \sum_{i=1}^r w_{i0} R_i + \sum_{j=1}^c w_{0j} C_j + \sum_{i=1}^r \sum_{j=1}^c w_{ij} U_{ij} \\ &= R(w_0^{(row)}) + C(w_0^{(col)}) + U(w), \end{aligned} \tag{3.4}$$

where  $R(w_0^{(row)}) = \sum_{i=1}^r w_{i0} R_i, C(w_0^{(col)}) = \sum_{j=1}^c w_{0j} C_j$  and  $U(w) = \sum_{i=1}^r \sum_{j=1}^c w_{ij} U_{ij}$ .

Consider the vector of equal weights  $\underline{w}_{rc} = \underbrace{(1/(rc), 1/(rc), \dots, 1/(rc))}_{rc} \in \mathcal{I}_{rc}$  and the vector  $\bar{w}_{rc} = \underbrace{(1, 0, \dots, 0)}_{rc} \in \mathcal{I}_{rc}$  that corresponds to the portfolio of  $Y'_{ij}$ s that consists of only one risk.

Observe that the vectors  $\underline{w}_0^{(row)}$  and  $\underline{w}_0^{(col)}$  that correspond to  $\underline{w}_{rc}$  by (3.2) and (3.3) consist of equal weights. Namely,  $\underline{w}_0^{(row)} = \underbrace{(1/r, \dots, 1/r)}_r = \underline{w}_r \in \mathcal{I}_r$  and  $\underline{w}_0^{(col)} = \underbrace{(1/c, \dots, 1/c)}_c = \underline{w}_c \in \mathcal{I}_c$ .

Similarly, for the weights  $\bar{w}_0^{(row)}$  and  $\bar{w}_0^{(col)}$  corresponding to  $\bar{w}_{rc}$  by (3.2) and (3.3) we have  $\bar{w}_0^{(row)} = \underbrace{(1, 0, \dots, 0)}_r = \bar{w}_r \in \mathcal{I}_r$  and  $\bar{w}_0^{(col)} = \underbrace{(1, 0, \dots, 0)}_c = \bar{w}_c \in \mathcal{I}_c$ .

The following theorem provides value at risk comparisons for portfolio returns  $Y(w)$  in (3.4) under heavy-tailedness in the risk components  $R_i, C_j$  and  $U_{ij}$ .

**Theorem 3.1** *Let  $q \in (0, 1/2)$ .*

- (i) *If  $R_i, C_j, U_{ij} \sim \overline{CSLC}$ , then  $VaR_q[Y(\underline{w}_{rc})] \leq VaR_q[Y(w)] \leq VaR_q[Y(\bar{w}_{rc})]$  for all  $w \in \mathcal{I}_{rc}$ .*
- (ii) *If  $R_i, C_j, U_{ij} \sim \underline{CS}$ , then  $VaR_q[Y(\underline{w}_{rc})] \geq VaR_q[Y(w)] \geq VaR_q[Y(\bar{w}_{rc})]$  for all  $w \in \mathcal{I}_{rc}$ .*

Part (i) of Theorem 3.1 shows that, similar to the case of independence in Ibragimov (2005, 2009b) (see part (ii) of Proposition 3.1), the most diversified portfolio with equal weights  $\underline{w}_{rc}$  is preferred to any other portfolio of dependent risks  $Y_{ij}$  in (1.2) under moderate heavy-tailedness. In addition, the least diversified portfolio with weights  $\bar{w}_{rc}$  consisting of only one risk is dominated by any other portfolio of risks  $Y_{ij}$  with additive common shocks. Part (ii) of Theorem 3.1 implies that, similar to independence (part (ii) of Proposition 3.2), the conclusions are reversed under extreme heavy-tailedness. Extreme heavy-tailedness of common shocks  $R_i, C_j$  and idiosyncratic risks  $U_{ij}$  in (1.2) implies optimality of the least diversified portfolio with weights  $\bar{w}_{rc}$  with respect to the portfolio value at risk comparisons. In contrast, the portfolio value at risk is maximal for the most diversified portfolio with equal weights  $\underline{w}_{rc}$  under such assumptions.

As an immediate consequence of Propositions 3.1 and 3.2, one also obtains similar comparisons with the extremal portfolio weights  $\underline{w}_{rc}$  and  $\bar{w}_{rc}$  for the values at risk of the components  $R(w_0^{(row)})$ ,  $C(w_0^{(col)})$  and  $U(w)$  in decomposition (3.4) with weights  $w \in \mathcal{I}_{rc}$ . Namely, the following conclusions hold.

**Theorem 3.2** *Let  $q \in (0, 1/2)$ .*

- (i) *If  $U_{ij} \sim \overline{CSLC}$ , then  $VaR_q[U(\underline{w}_{rc})] \leq VaR_q[U(w)] \leq VaR_q[U(\bar{w}_{rc})]$  for all  $w \in \mathcal{I}_{rc}$ .*
- (ii) *If  $U_{ij} \sim \underline{CS}$ , then  $VaR_q[U(\underline{w}_{rc})] \geq VaR_q[U(w)] \geq VaR_q[U(\bar{w}_{rc})]$  for all  $w \in \mathcal{I}_{rc}$ .*

- (iii) If  $R_i \sim \overline{CSLC}$ , then  $VaR_q[R(\underline{w}_r)] \leq VaR_q[R(w_0^{(row)})] \leq VaR_q[R(\overline{w}_r)]$  for all  $w \in \mathcal{I}_{rc}$ .
- (iv) If  $R_i \sim \underline{CS}$ , then  $VaR_q[R(\underline{w}_r)] \geq VaR_q[R(w_0^{(row)})] \geq VaR_q[R(\overline{w}_r)]$  for all  $w \in \mathcal{I}_{rc}$ .
- (v) If  $C_j \sim \overline{CSLC}$ , then  $VaR_q[C(\underline{w}_c)] \leq VaR_q[C(w_0^{(col)})] \leq VaR_q[C(\overline{w}_c)]$  for all  $w \in \mathcal{I}_{rc}$ .
- (vi) If  $C_j \sim \underline{CS}$ , then  $VaR_q[C(\underline{w}_c)] \geq VaR_q[C(w_0^{(col)})] \geq VaR_q[C(\overline{w}_c)]$  for all  $w \in \mathcal{I}_{rc}$ .

As in the case of independence in [Ibragimov \(2009b\)](#) (see parts (i) of Propositions 3.1 and 3.2), it is of interest to also consider value at risk comparisons for general portfolio weights  $v, w \in \mathcal{I}_{rc}$  satisfying  $v < w$  (so that the portfolio with weights  $v$  is more diversified than that with weights  $w$ ). However, such general comparisons cannot be obtained using majorization on  $\mathcal{I}_{rc}$ . This is because the values at risk

$$VaR_q \left[ R \left( w_0^{(row)} \right) \right] = VaR_q \left[ \sum_{i=1}^r w_{i0} R_i \right] = VaR_q \left[ \sum_{i=1}^r \left( \sum_{j=1}^c w_{ij} \right) R_i \right],$$

$$VaR_q \left[ C \left( w_0^{(col)} \right) \right] = VaR_q \left[ \sum_{j=1}^c w_{0j} C_j \right] = VaR_q \left[ \sum_{j=1}^c \left( \sum_{i=1}^r w_{ij} \right) C_j \right]$$

for the components  $R(w_0^{(row)})$  and  $C(w_0^{(col)})$  in (3.4) are not symmetric functions of  $w_{ij}$ 's. Thus, these functions are neither Schur-concave nor Schur-convex in  $w \in \mathcal{I}_{rc}$ .<sup>3</sup>

Nevertheless, the above value at risk comparisons with  $v < w$  are possible for certain portfolio weights  $v$  and  $w$  that are different from the most diversified portfolio  $\underline{w}_{rc}$  and do not correspond to the least diversification with weights  $\overline{w}_{rc}$ . Some of these value at risk comparisons have a natural interpretation in terms of value at risk analysis for portfolios of equally weighted indices of risks  $Y_{ij}$  in (1.2). These value at risk orderings and the settings where they arise are considered in the next two sections.

#### 4 Further applications: Portfolio component value at risk analysis

Let  $n_{ij} \in \{0, 1\}, i = 1, \dots, r, j = 1, \dots, c$ , be a set of indicator variables. Denote  $n_{i0} = \sum_{j=1}^c n_{ij}, n_{0j} = \sum_{i=1}^r n_{ij}, n = \sum_{i=1}^r n_{i0} = \sum_{j=1}^c n_{0j} = \sum_{i=1}^r \sum_{j=1}^c n_{ij}$ .

Section 13.B in [Marshall and Olkin \(1979\)](#) discusses applications of majorization theory in comparisons of variance components for linear estimators based on observations in specifications (1.2) referred to as two-way classification random effects

<sup>3</sup> This situation is similar to the majorization-based analysis of variance decompositions for linear estimators of location in two-way classification random effects models in Sect. 13.B in [Marshall and Olkin \(1979\)](#) (see also Sect. 4 and Appendix A). As indicated by [Marshall and Olkin \(1979\)](#), neither Schur-convexity nor Schur-concavity (in  $w \in \mathcal{I}_{rc}$ ) holds for the variances  $var[R(w_0^{(row)})] = var[\sum_{i=1}^r w_{i0} R_i]$  and  $var[C(w_0^{(col)})] = var[\sum_{j=1}^c w_{0j} C_j]$  because these functions are not symmetric functions of  $w'_{ij}$ s.

models (see also the review in Appendix A and references therein). Marshall and Olkin (1979) consider the equal weights  $w_{ij} = 1/(rc)$  dealt with in the previous section and also the portfolio weights  $\tilde{v}_{ij} = n_{i0}/(nc)$ ,  $\tilde{\tilde{v}}_{ij} = n_{0j}/(nr)$  and  $\tilde{w}_{ij} = \frac{(n-n_{i0}-n_{0j}+n_{ij})n_{ij}}{n^2-\sum_{i=1}^r n_{i0}^2-\sum_{j=1}^c n_{0j}^2+n}$ ,  $i = 1, \dots, r, j = 1, \dots, c$ , discussed in Koch (1967a,b).

Denote the weight vectors corresponding to the last three choices by  $\tilde{v}$ ,  $\tilde{\tilde{v}}$  and  $\tilde{w}$ .

In the context of portfolio choice, some of the properties of the portfolios with weights  $\underline{w}$ ,  $\tilde{v}$ ,  $\tilde{\tilde{v}}$  and  $\tilde{w}$  may be summarized as follows. Consider  $r$  equally weighted indices comprised, for  $i = 1, \dots, r$ , of risks  $Y_{ij}, j = 1, \dots, c$ , in the  $i$ th row of (1.2). The return on index  $i$  is thus given by  $Z_i^{(col)} = R_i + \frac{1}{c} \sum_{j=1}^c C_j + \frac{1}{c} \sum_{j=1}^c U_{ij}, i = 1, \dots, r$ . For  $w = (w_1, \dots, w_r) \in \mathcal{I}_r$ , the return on the portfolio of the indices  $i = 1, \dots, r$  with returns  $Z_i^{(col)}$  and weights  $w$  is given by

$$Z^{(col)}(w) = \sum_{i=1}^r w_i Z_i^{(col)} = \sum_{i=1}^r w_i R_i + \frac{1}{c} \sum_{j=1}^c C_j + \sum_{i=1}^r \frac{(U_{i1} + \dots + U_{ic})}{c} w_i. \tag{4.1}$$

Similarly, consider, for  $j = 1, \dots, c$ , the equally weighted indices comprised of risks  $Y_{ij}, i = 1, \dots, r$ , in the  $j$ th column of (1.2). The return on index  $j$  is thus given by  $Z_j^{(row)} = C_j + \frac{1}{r} \sum_{i=1}^r R_i + \frac{1}{r} \sum_{i=1}^r U_{ij}, j = 1, \dots, c$ . For  $w = (w_1, \dots, w_c) \in \mathcal{I}_c$ , the return on the portfolio of the indices  $j = 1, \dots, c$  with returns  $Z_j^{(row)}$  and weights  $w$  is given by

$$Z^{(row)}(w) = \sum_{j=1}^c w_j Z_j^{(row)} = \frac{1}{r} \sum_{i=1}^r R_i + \sum_{j=1}^c w_j C_j + \sum_{j=1}^c \frac{(U_{1j} + \dots + U_{rj})}{r} w_j. \tag{4.2}$$

The risk  $Y(\tilde{v})$  obtained using (3.4) with weights  $\tilde{v}$  is the same as the return on the portfolio of the indices  $Z_i^{(col)}, i = 1, \dots, r$ , with weights  $w = \underbrace{(n_{10}/n, \dots, n_{r0}/n)}_r \in$

$\mathcal{I}_r$  in (4.1). The return  $Y(\tilde{\tilde{v}})$  in (3.4) with weights  $\tilde{\tilde{v}}$  is the same as the return on the portfolio of the risks  $Z_j^{(row)}, j = 1, \dots, c$ , with weights  $w = \underbrace{(n_{01}/n, \dots, n_{0c}/n)}_c \in \mathcal{I}_c$

in (4.2).

As discussed in the previous section, the equal weights  $\underline{w}_{rc} = \underbrace{(1/(rc), 1/(rc), \dots, 1/(rc))}_{rc} \in \mathcal{I}_{rc}$  correspond to the most diversified portfolio of

the risks  $Y_{ij}$ . The return  $Y(\underline{w}_{rc})$  with these weights in (3.4) is the same as the returns  $Z^{(col)}(\underline{w}_r)$  and  $Z^{(row)}(\underline{w}_c)$  on the portfolios of indices in (4.1) and (4.2) with equal weights  $\underline{w}_r = \underbrace{(1/r, \dots, 1/r)}_r \in \mathcal{I}_r$  and  $\underline{w}_c = \underbrace{(1/c, \dots, 1/c)}_c \in \mathcal{I}_c$ .

The weights  $\tilde{w}$  correspond to a portfolio of  $Y_{ij}$  where, in contrast to  $\underline{w}_{rc}$ ,  $\tilde{v}$  and  $\tilde{\tilde{v}}$ , some of the risks  $Y_{ij}$  are taken with zero weights that may be due to the risks' unavailability.

Lemma 13.B.2.a in Marshall and Olkin (1979) and relation (3.1) with  $N = rc$  show that the following majorization comparisons hold for the vectors  $\underline{w}_{rc}$ ,  $\tilde{v}$ ,  $\tilde{\tilde{v}}$ ,  $\tilde{w}$  and  $\overline{w}_{rc}$  :

$$\underline{w}_{rc} < \tilde{v} < \tilde{w} < \overline{w}_{rc}, \tag{4.3}$$

$$\underline{w}_{rc} < \tilde{\tilde{v}} < \tilde{w} < \overline{w}_{rc}. \tag{4.4}$$

Theorems 3.1 and 3.2 imply value at risk comparisons for the portfolio returns  $Y(w)$  and its components  $R(w_0^{(row)})$ ,  $C(w_0^{(col)})$  and  $U(w)$  in (3.4) between an arbitrary  $w \in \mathcal{I}_{rc}$  (for instance,  $w = \tilde{v}$ ,  $\tilde{\tilde{v}}$ ,  $\tilde{w}$ ) and the extremal portfolio weights  $\underline{w}_{rc}$  and  $\overline{w}_{rc}$ . In particular, from parts (iii)–(vi) of Theorem 3.2 it follows that the following comparisons hold for all  $q \in (0, 1/2)$  and all  $w \in \mathcal{I}_{rc}$  (e.g., for  $w = \tilde{v}$ ,  $\tilde{\tilde{v}}$ ,  $\tilde{w}$ ). These comparisons are direct consequences of the results for the independent case in Proposition 3.1 and 3.2.

$$VaR_q[R(\tilde{v}_0^{(row)})] = VaR_q[R(\underline{w}_r)] \leq VaR_q[R(w_0^{(row)})] \text{ if } R_i \sim \overline{\mathcal{CSLC}}, \tag{4.5}$$

$$VaR_q[C(\tilde{v}_0^{(col)})] = VaR_q[C(\underline{w}_c)] \leq VaR_q[C(w_0^{(col)})] \text{ if } C_j \sim \overline{\mathcal{CSLC}}, \tag{4.6}$$

$$VaR_q[R(\tilde{v}_0^{(row)})] = VaR_q[R(\underline{w}_r)] \geq VaR_q[R(w_0^{(row)})] \text{ if } R_i \sim \underline{\mathcal{CS}}, \tag{4.7}$$

$$VaR_q[C(\tilde{v}_0^{(col)})] = VaR_q[C(\underline{w}_c)] \geq VaR_q[C(w_0^{(col)})] \text{ if } C_j \sim \underline{\mathcal{CS}}. \tag{4.8}$$

Similar value at risk comparisons for  $VaR_q[U(w)]$  are also obtained between the portfolio with weights  $\tilde{w}$  and those with weights  $w = \tilde{v}$ ,  $\tilde{\tilde{v}}$ . Namely, the following result holds.

**Theorem 4.1** *Let  $q \in (0, 1/2)$ . The following value at risk comparisons hold for the component  $U(w)$  of decomposition (3.4):*

- (i) *If  $U_{ij} \sim \overline{\mathcal{CSLC}}$ , then  $VaR_q[U(\tilde{w})] \geq VaR_q[U(\tilde{v})]$  and  $VaR_q[U(\tilde{w})] \geq VaR_q[U(\tilde{\tilde{v}})]$ .*
- (ii) *If  $U_{ij} \sim \underline{\mathcal{CS}}$ , then  $VaR_q[U(\tilde{w})] \leq VaR_q[U(\tilde{v})]$  and  $VaR_q[U(\tilde{w})] \leq VaR_q[U(\tilde{\tilde{v}})]$ .*

The main conclusions from the above value at risk comparisons for the portfolio components  $U(w)$ ,  $R(w_0^{(row)})$  and  $C(w_0^{(col)})$  in (3.4) with weights  $\underline{w}_{rc}$ ,  $\tilde{v}$ ,  $\tilde{\tilde{v}}$ ,  $\tilde{w}$  and  $\overline{w}_{rc}$  are summarized as follows. In the case of moderately heavy-tailed common shocks  $R_i$ ,  $C_j$  and idiosyncratic risks  $U_{ij}$ , we conclude that full diversification on the level of underlying  $Y_{ij}$  with weights  $\underline{w}_{rc}$  is preferred to any other portfolio choice, provided all the  $rc$  risks are available (left inequality in part (i) of Theorem 3.1). In particular, the equal weights  $\underline{w}_{rc}$  are preferred to less diversification with weights  $\tilde{w}$  where some of the risks may be taken with zero weights. Full diversification with weights  $\underline{w}_{rc}$  at  $Y_{ij}$  is also preferred to the portfolio of equally weighted indices  $Z_i^{(col)}$ ,

$i = 1, \dots, r$ , with weights  $w = \underbrace{(n_{10}/n, \dots, n_{r0}/n)}_r \in \mathcal{I}_r$  and the portfolio of indices

with returns  $Z_j^{(row)}$ ,  $j = 1, \dots, c$ , and the weight vector  $w = \underbrace{(n_{01}/n, \dots, n_{0c}/n)}_c \in$

$\mathcal{I}_c$ . In turn, comparisons (4.5) and (4.6) for the common shock parts  $R(w_0^{(row)})$  and  $C(w_0^{(row)})$  in (3.4) and part (i) of Theorem 4.1 for  $U(w)$  suggest that the weight vectors  $\tilde{v}$  and  $\tilde{\tilde{v}}$  may be preferred to the vector  $\tilde{w}$  where some of the risks are included with zero weights. These conclusions are similar to those in Sect. 13.B in Marshall and Olkin (1979) for variance comparisons of linear estimators based on observations (1.2) with  $ER_i = \mu$ ,  $EC_j = 0$ ,  $EU_{ij} = 0$ ,  $var(R_i) = \sigma_R^2$ ,  $var(C_j) = \sigma_C^2$ ,  $var(U_{ij}) = \sigma_{U_j}^2$ ,  $i = 1, \dots, r$ ,  $j = 1, \dots, c$ .

These results are reversed for extremely heavy-tailed risks  $R_i$ ,  $C_j$  and  $U_{ij}$ . In such settings, the equal weights  $\underline{w}_{rc}$  are dominated by any other portfolio weights (left inequality in part (ii) of Theorem 3.1). In contrast, the smallest VaR is achieved at the weights  $\bar{w}_{rc}$  and the portfolio consisting of only one risk. In particular, the portfolio of indices with returns  $Z_i^{(col)}$ ,  $i = 1, \dots, r$ , and weights  $w = \underbrace{(n_{10}/n, \dots, n_{r0}/n)}_r \in$

$\mathcal{I}_r$  and the portfolio of indices with returns  $Z_j^{(row)}$ ,  $j = 1, \dots, c$ , and  $w = \underbrace{(n_{01}/n, \dots, n_{0c}/n)}_c \in \mathcal{I}_c$  are preferred to the fully diversified portfolio of  $Y'_{ij}$ s with

equal weights  $\underline{w}_{rc}$ . Inequalities (4.7) and (4.8) for  $R(w_0^{(row)})$  and  $C(w_0^{(row)})$  and part (ii) of Theorem 4.1 for  $U(w)$  suggest that the weights  $\tilde{w}$  may dominate the weights  $\tilde{v}$  and  $\tilde{\tilde{v}}$ .

The results in (4.5)–(4.8) and Theorem 4.1 also indicate that, especially in the cases where heavy-tailedness of the common shocks  $R_i$  and  $C_j$  and that of the idiosyncratic risks  $U_{ij}$  is of different degree (say, in the case of the assumptions  $U_{ij} \sim \overline{CSLC}$  combined with  $R_i \sim \underline{CS}$  and  $C_j \sim \underline{CS}$ ), the VaR comparisons for the components  $R(w_0^{(row)})$ ,  $C(w_0^{(col)})$  and  $U(w)$  in (3.4) do not point out to an optimal portfolio choice among  $\tilde{v}$ ,  $\tilde{\tilde{v}}$  and  $\tilde{w}$ . Thus, the optimal portfolio selection depends crucially on the distributional properties of common shocks  $R_i$  and  $C_j$  and idiosyncratic risks  $U_{ij}$ .

### 5 When heavy-tailedness helps: value at risk for financial indices

In many real world situations, the sets of available risks in (1.2) or (1.3) are unbalanced and include unequal number of rows for each column or unequal number of columns for each row. In this section we show how the approach presented in the paper can be applied in the analysis of such settings. We obtain the results for unbalanced analogues of models (1.3) with one set of “row effects” common shocks  $R_i$  and focus on the framework of value at risk comparisons for indices based on risks  $Y_{ij}$  in such models. The results can be extended to more general models, including those corresponding to settings with two sets of common shocks in (1.2), risks in form (1.5) with varying factor loadings and multiple common shock models (see the discussion and the results in Sect. 7).

Let  $n_1 \geq n_2 \geq \dots \geq n_r \geq 1$ ,  $\sum_{i=1}^r n_i = n$ . Similar to the interpretation of weights  $\tilde{v}_{ij}$  and  $\tilde{v}_i$  in Sect. 4, consider  $r$  equally weighted indices comprised, for  $i = 1, \dots, r$ , of risks

$$Y_{ij} = R_i + U_{ij}, \quad j = 1, \dots, n_i, \tag{5.1}$$

with identically distributed common shocks  $R_i, i = 1, \dots, r$ , and identically distributed idiosyncratic risks  $U_{ij}, j = 1, \dots, n_i, i = 1, \dots, r$ .<sup>4</sup> As before, the r.v.'s  $R_i, U_{ij} j = 1, \dots, n_i, i = 1, \dots, r$ , are assumed to be independent.. The return on the index  $i$  is given by

$$Z_i = \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij} = R_i + \frac{1}{n_i} \sum_{j=1}^{n_i} U_{ij}. \tag{5.2}$$

As in Sects. 3 and 4, for  $w = (w_1, \dots, w_r) \in \mathcal{I}_r$ , denote by  $Z(w)$  the return on the portfolio of the indices  $i = 1, \dots, r$  with returns  $Z_i$  in (5.2) and weights  $w$ :

$$Z(w) = \sum_{i=1}^r w_i Z_i = \sum_{i=1}^r \frac{(Y_{i1} + \dots + Y_{in_i})}{n_i} w_i = \sum_{i=1}^r w_i R_i + \sum_{i=1}^r \frac{(U_{i1} + \dots + U_{in_i})}{n_i} w_i. \tag{5.3}$$

In what follows, for  $N \geq 1, e_N = \underbrace{(1, \dots, 1)}_N \in \mathbf{R}^N$  will denote the  $N$ -vector of ones. In addition, for  $m$  row vectors  $x^{(k)} \in \mathbf{R}^{1 \times N_k}, N_k \geq 1, k = 1, \dots, m$ , we will denote by  $x = (x^{(1)}, \dots, x^{(m)}) \in \mathbf{R}^{1 \times N}, N = N_1 + \dots + N_m$ , the vector with the first  $N_1$  components equal to those of  $x^{(1)}$ , the next  $N_2$  components equal to those of  $x^{(2)}$ , and so on:  $x_i = x_i^{(1)}, i = 1, \dots, N_1; x_{N_1+i} = x_i^{(2)}, i = 1, \dots, N_2; \dots, x_{N_1+N_2+\dots+N_{m-1}+i} = x_i^{(m)}, i = 1, \dots, N_m$ .

Decomposition (5.3) can be written as

$$Z(w) = R(w) + U(\tilde{w}), \tag{5.4}$$

where  $R(w)$  is the return on the portfolio of common shocks  $R_i$  with weights  $w = (w_1, \dots, w_r) \in \mathcal{I}_r$ , and  $U(\tilde{w}) = \sum_{i=1}^r \sum_{j=1}^{n_i} \tilde{w}_{ij} U_{ij}$  is the return on the portfolio of idiosyncratic risks  $U_{ij}$  with weights  $\tilde{w}_{ij} = w_i/n_i, j = 1, \dots, n_i, i = 1, \dots, r$ , and

<sup>4</sup> Equally weighted indices are quite common in financial markets, and their examples include the majority of hedge fund indices (see Lhabitant 2008), the Value Line Arithmetic Index, Global Dow, the Thomson Reuters Equal Weight Continuous Commodity Index, the Thomson Reuters/Jefferies CRB (Commodity Research Bureau) Index, and others. We also note that since the results in the paper can be generalized to models with varying factor loadings (see Sect. 7 and the discussion in the introduction), non-equal weighted indices can also be covered by the analysis if certain restrictions are met.

the corresponding weight vector

$$\tilde{w} = (\tilde{w}_{11}, \dots, \tilde{w}_{1n_1}, \dots, \tilde{w}_{r1}, \dots, \tilde{w}_{rn_r}) = \left( \frac{w_1}{n_1} e_{n_1}, \dots, \frac{w_r}{n_r} e_{n_r} \right) \in \mathcal{I}_n. \tag{5.5}$$

The return on the portfolio of the indices with equal weights

$$w^{(1)} = \underline{w}_r = \underbrace{(1/r, \dots, 1/r)}_r \in \mathcal{I}_r \tag{5.6}$$

is given by the sample mean of the risks  $Z_i, i = 1, \dots, r$ :

$$\begin{aligned} Z(w^{(1)}) &= \frac{1}{r} \sum_{i=1}^r Z_i = \frac{1}{r} \sum_{i=1}^r \frac{(Y_{i1} + \dots + Y_{in_i})}{n_i} \\ &= \frac{1}{r} \sum_{i=1}^r R_i + \frac{1}{r} \sum_{i=1}^r \frac{(U_{i1} + \dots + U_{in_i})}{n_i}. \end{aligned} \tag{5.7}$$

Similarly, the choice of portfolio weights

$$w^{(2)} = \left( \frac{n_1}{n}, \frac{n_2}{n}, \dots, \frac{n_r}{n} \right) \tag{5.8}$$

produces the return  $Z(w^{(2)})$  equal to the sample mean of the underlying risks  $Y_{ij}$  in (5.1):

$$Z(w^{(2)}) = \sum_{i=1}^r \frac{n_i}{n} Z_i = \frac{1}{n} \sum_{i=1}^r \sum_{j=1}^{n_i} Y_{ij}, \tag{5.9}$$

that is, in the notations of Sections 3 and 4,  $Z(w^{(2)}) = Y(\underline{w}_n)$ , where  $\underline{w}_n$  is the portfolio with equal weights  $\underline{w}_n = \underbrace{(1/n, \dots, 1/n)}_n \in \mathcal{I}_n$ . In other words, while the

weights  $w^{(1)}$  reflect diversification on the level of indices with returns  $Z_i$ , the weights  $w^{(2)}$  correspond to (full) diversification on the level of the underlying risks  $Y_{ij}$  that comprise these indices (see the discussion in Sections 3 and 4 for formalization of the notions of diversification in terms of majorization relations for portfolio weights at  $Y_{ij}$  and  $Z_i$ ).

Following the works in the statistics literature reviewed in Appendix A, we also consider the portfolio weights  $w^{(3)} = (w_1^{(3)}, w_2^{(3)}, \dots, w_r^{(3)}) \in \mathcal{I}_r$  and  $w(c) = (w_1(c), \dots, w_r(c)) \in \mathcal{I}_r, c \in [0, 1]$ , with

$$w_i^{(3)} = \frac{n_i(n - n_i)}{n^2 - \sum_{s=1}^m n_s^2}, \quad i = 1, \dots, r, \tag{5.10}$$



and

$$w_i(c) = \frac{n_i[(n_i - 1)c + 1]^{-1}}{\sum_{i=1}^r n_i[(n_i - 1)c + 1]^{-1}}. \tag{5.11}$$

For the weights  $w(c)$ , one has  $w(0) = w^{(2)}$  and  $w(1) = w^{(1)} = \underline{w}_r$ . We also note that the weight vectors  $w^{(1)}$ ,  $w^{(2)}$ ,  $w^{(3)}$  and  $w(c)$  in (5.6), (5.8), (5.10) and (5.11) become the same in the special case of balanced models with  $n_1 = n_2 = \dots = n_r : w^{(l)} = w(c) = \underline{w}_r$  for all  $l = 1, 2, 3$ , and  $c \in [0, 1]$ .

Theorems 5.1 and 5.2 provide value at risk comparisons for equally weighted indices comprised of risks  $Y_{ij}$  spanned by heavy-tailed common shocks  $R_i$  and idiosyncratic risks  $U_{ij}$ . In both of them the degree of heavy-tailedness of common shocks  $R_i$  is different from that for idiosyncratic risks  $U_{ij}$ .

Theorem 5.1 concerns the case of extremely heavy-tailed  $R_i$  and moderately heavy-tailed  $U_{ij}$ .

**Theorem 5.1** *Let  $q \in (0, 1/2)$ . Suppose that, in (5.1),  $R_i \sim \underline{\mathcal{CS}}$ ,  $i = 1, \dots, r$ , and  $U_{ij} \sim \overline{\mathcal{CSLC}}$ ,  $j = 1, \dots, n_i$ ,  $i = 1, \dots, r$ . Then the function  $\text{VaR}_q[Z(w(c))]$  is non-decreasing in  $c \in [0, 1]$ . In particular,*

$$\text{VaR}_q[Z(w^{(1)})] \geq \text{VaR}_q[Z(w(c))] \geq \text{VaR}_q[Z(w^{(2)})]$$

for all  $c \in [0, 1]$ . In addition,

$$\text{VaR}_q[Z(w^{(1)})] \geq \text{VaR}_q[Z(w^{(3)})] \geq \text{VaR}_q[Z(w^{(2)})].$$

Theorem 5.2 shows that the conclusions of Theorem 5.1 are reversed in the case where the common shocks  $R_i$  are moderately heavy-tailed and the idiosyncratic risks  $U_{ij}$  are extremely heavy-tailed.

**Theorem 5.2** *Let  $q \in (0, 1/2)$ . Suppose that, in (5.1),  $R_i \sim \overline{\mathcal{CSLC}}$ ,  $i = 1, \dots, r$ , and  $U_{ij} \sim \underline{\mathcal{CS}}$ ,  $j = 1, \dots, n_i$ ,  $i = 1, \dots, r$ . Then the function  $\text{VaR}_q[Z(w(c))]$  is non-increasing in  $c \in [0, 1]$ . In particular,*

$$\text{VaR}_q[Z(w^{(2)})] \geq \text{VaR}_q[Z(w(c))] \geq \text{VaR}_q[Z(w^{(1)})]$$

for all  $c \in [0, 1]$ . In addition,

$$\text{VaR}_q[Z(w^{(2)})] \geq \text{VaR}_q[Z(w^{(3)})] \geq \text{VaR}_q[Z(w^{(1)})].$$

Similar to Theorems 3.1 and 3.2, the results provided by Theorems 5.1 and 5.2 hold for the value at risk comparisons for the components  $R(w)$  and  $U(\tilde{w})$  in decomposition (5.4) for the portfolio returns  $Z(w)$  (here and below, for weights  $w = (w_1, \dots, w_r) \in \mathcal{I}_r$  at  $Z_i$ 's,  $\tilde{w} \in \mathcal{I}_n$  is the vector of weights at  $U_{ij}$  in (5.4) that corresponds to  $w$  by (5.5)). These comparisons are provided in Theorem 5.3.

By the left majorization comparisons in (3.1), the portfolio weight vector  $w^{(1)} = \underline{w}_r$  provides the most diversified portfolio of common shocks  $R_i$  and the vector  $\tilde{w}^{(2)} = \underline{w}_n$

provides the most diversified portfolio of idiosyncratic risks  $U_{ij}$ . The portfolio weights  $w_i$  at  $R_i$  in decomposition (5.4) that correspond to the weights  $\tilde{w}_{ij}^{(2)} = 1/n, j = 1, \dots, n_i, i = 1, \dots, r$ , at the risks  $U_{ij}$  (the risks  $Y_{ij}$  are given by  $w_i = n_i/n, i = 1, \dots, r$ , and are thus different from the equal components  $w_i^{(1)} = 1/r$  of  $w^{(1)}$ ). This provides the intuition for the opposite results for the common shocks and idiosyncratic risks in Theorem 5.3 and for different degrees of heavy-tailedness in  $R_i$  and  $U_{ij}$  needed for the VaR comparisons in Theorems 5.1 and 5.2 to hold.

**Theorem 5.3** *Let  $q \in (0, 1/2)$ . The following comparisons hold for the components  $R(w)$  and  $U(\tilde{w})$  in decomposition (5.4).*

- (i) *Suppose  $R_i \sim \overline{CS\mathcal{L}\mathcal{C}}$ . Then the function  $VaR_q[R(w(c))]$  is non-increasing in  $c \in [0, 1]$ . In particular,*

$$VaR_q[R(w^{(2)})] \geq VaR_q[R(w(c))] \geq VaR_q[R(w^{(1)})]$$

*for all  $c \in [0, 1]$ . In addition,  $VaR_q[R(w^{(3)})] \leq VaR_q[R(w^{(2)})]$  and  $VaR_q[R(w^{(1)})] \leq VaR_q[R(w)]$  for all  $w \in \mathcal{I}_r$ .*

- (ii) *Suppose  $R_i \sim \underline{CS}$ . Then the function  $VaR_q[R(w(c))]$  is non-decreasing in  $c \in [0, 1]$ . In particular,*

$$VaR_q[R(w^{(2)})] \leq VaR_q[R(w(c))] \leq VaR_q[R(w^{(1)})]$$

*for all  $c \in [0, 1]$ . In addition,  $VaR_q[R(w^{(3)})] \geq VaR_q[R(w^{(2)})]$  and  $VaR_q[R(w^{(1)})] \geq VaR_q[R(w)]$  for all  $w \in \mathcal{I}_r$ .*

- (iii) *Suppose  $U_{ij} \sim \overline{CS\mathcal{L}\mathcal{C}}$ . Then the function  $VaR_q[U(\tilde{w}(c))]$  is non-decreasing in  $c \in [0, 1]$ . In particular,*

$$VaR_q[U(\tilde{w}^{(2)})] \leq VaR_q[U(\tilde{w}(c))] \leq VaR_q[U(\tilde{w}^{(1)})].$$

*In addition,  $VaR_q[U(\tilde{w}^{(2)})] \leq VaR_q[U(\tilde{w}^{(3)})]$  and  $VaR_q[U(\tilde{w}^{(2)})] = VaR_q[U(\underline{w}_n)] \leq VaR_q[U(\tilde{w})]$  for all  $w \in \mathcal{I}_r$ .*

- (iv) *Suppose  $U_{ij} \sim \underline{CS}$ . Then the function  $VaR_q[U(\tilde{w}(c))]$  is non-increasing in  $c \in [0, 1]$ . In particular,*

$$VaR_q[U(\tilde{w}^{(2)})] \geq VaR_q[U(\tilde{w}(c))] \geq VaR_q[U(\tilde{w}^{(1)})].$$

*In addition,  $VaR_q[U(\tilde{w}^{(2)})] \geq VaR_q[U(\tilde{w}^{(3)})]$  and  $VaR_q[U(\tilde{w}^{(2)})] = VaR_q[U(\underline{w}_n)] \geq VaR_q[U(\tilde{w})]$  for all  $w \in \mathcal{I}_r$ .*

It is interesting to compare the results provided by Theorems 5.1–5.3 with the inequalities for the variances  $var[Z(w)]$  in relations (A.1)–(A.3) in Appendix A. If both the common shock variables  $R_i$  and the idiosyncratic risks  $U_{ij}$  have finite second moments and are thus thin-tailed, then solving the optimal portfolio choice problem with minimization of the variance  $var[Z(w)]$  is problematic in the following sense. First, the optimal solution is given by the portfolio weights

$w_i(\gamma)$  that depend on the value of the intraclass correlation  $\gamma$  which is typically unknown. Second, in (A.2) and (A.3), the contributions to the variances of the risks  $Z(w)$  from the common shock and the idiosyncratic risk parts  $V_R$  and  $V_U$  are ordered in the opposite way for the diversified portfolio weights  $w^{(1)} = \underline{w}_r$  and  $w^{(2)}$ . This further holds regardless of the values of the variances  $\sigma_R^2$  and  $\sigma_U^2$  of the r.v.'s  $R_i$  and  $U_{ij}$ . Thus, minimization of the variance  $\text{var}[Z(w)]$  does not point out, even in the case of identically distributed  $R_i$ 's and  $U_{ij}$ 's, to diversification either on the level of indices  $i = 1, \dots, r$  with returns  $Z_i$  (the case of weights  $w^{(1)} = \underline{w}_r$ ) or on the level of underlying risks  $Y_{ij}$  (the case of weights  $w^{(2)}$ ).

These conclusions are in sharp contrast with the results for the value at risk portfolio choice under independence discussed in the introduction and reviewed at the beginning of Sect. 3 (see Ibragimov 2009a,b; Ibragimov and Walden 2007, 2008). They are also in contrast with the results for balanced models (1.2) presented in Sects. 3 and 4. Namely, portfolio value at risk under independence or in balanced models (1.2) is minimized at equal weights for all moderately heavy-tailed risks with finite first moments (part (ii) of Proposition 3.1 and part (i) of Theorem 3.1). Similarly, the solution to the value at risk minimization in such settings is given by the portfolio consisting of one risk within the whole class of extremely heavy-tailed risks with infinite first moments (part (ii) of Proposition 3.2 and part (ii) of Theorem 3.1).

In the settings of Theorem 5.1, the value at risk  $\text{VaR}_q[Z(w(c))]$  of the portfolios of indices  $i = 1, \dots, r$ , with weights  $w(c)$  defined in (5.11) is non-decreasing in  $c \in [0, 1]$ . Thus, the choice of portfolio weights  $w(0) = w^{(2)}$  in (5.8) and diversification on the level of underlying risks  $Y_{ij}$  is preferred, in terms of value at risk comparisons, to  $w(c)$  with any  $c \in (0, 1]$ . In particular,  $w(0) = w^{(2)}$  is preferred to the portfolio of equal weights  $w(1) = w^{(1)} = \underline{w}_r$  and the implied diversification on the level of indices. In addition, as shown by Theorem 5.1, the weight vector  $w^{(2)}$  is preferred to  $w^{(3)}$  that, in turn, dominates  $w^{(1)} = \underline{w}_r$  in terms of the value at risk comparisons for  $Z(w)$ .

Under the assumptions of Theorem 5.2, the value at risk  $\text{VaR}_q[Z(w(c))]$  is non-increasing in  $c \in [0, 1]$  for the weights  $w(c)$  in (5.11). Thus, in terms of VaR comparisons, the choice of equal weights  $w(1) = w^{(1)} = \underline{w}_r$  and diversification on the level of indices  $i = 1, \dots, r$ , is preferred to  $w(c)$  with any  $c \in [0, 1)$ . The equal weights  $w(1) = w^{(1)} = \underline{w}_r$  for the portfolio of indices are preferred, in particular, to the weights  $w(0) = w^{(2)}$  and the implied diversification on the level of individual risks  $Y_{ij}$ . Theorem 5.2 shows that the weight vector  $w^{(1)} = \underline{w}_r$  is also preferred to  $w^{(3)}$  and  $w^{(3)}$  is preferred to  $w^{(2)}$ .

Parts (i) and (iii) of Theorem 5.3 show that, if all the variables  $R_i, U_{ij}$  (and, thus, the risks  $Y_{ij}$  in (5.1)) are moderately heavy-tailed, then the orderings of the value at risks for the portfolio components  $R(w)$  and  $U(\tilde{w})$  in (5.4) with weights  $w = w^{(1)}, w^{(2)}, w^{(3)}$  are the same as in the case of variance comparisons in relations (A.2) and (A.3) in Appendix A. In addition, these comparisons do not point out to optimality of  $w^{(1)}$  or  $w^{(2)}$  within these three weight vectors or among the weights  $w(c), c \in [0, 1]$ , in (5.11). In other words, similar to variance minimization used as the portfolio choice criterion, the value at risk comparisons do not point out to

diversification either on the level of indices  $i = 1, \dots, r$  with returns  $Z_i$  (the case of weights  $w^{(1)}$ ) or on the level of underlying risks  $Y_{ij}$  (the case of weights  $w^{(2)}$ ).

Parts (ii) and (iv) of Theorem 5.3 show that the above conclusions are reversed in the case where all the variables  $R_i, U_{ij}$  (and, thus, the risks  $Y_{ij}$  in (5.1)) are extremely heavy-tailed. In such setting, the orderings of the value at risks for the portfolio components  $R(w)$  and  $U(\tilde{w})$  in (5.4) with weights  $w = w^{(1)}, w^{(2)}, w^{(3)}$  are the opposite to those in the case of variance comparisons in (A.2) and (A.3) and the case of VaR under moderate heavy-tailedness given by parts (i) and (iii) of Theorem 5.3. However, again, these orderings do not imply optimality of  $w^{(1)}$  or  $w^{(2)}$  within these three weight vectors or within the weight vectors  $w(c), c \in [0, 1]$ . Thus, the orderings do not imply optimality of diversification either on the level of indices  $i = 1, \dots, r$  with returns  $Z_i$  or the underlying risks  $Y_{ij}$ .

### 6 From risk management to statistics and econometrics: efficiency of linear estimators in random effects models

As indicated in the introduction, the value at risk results presented in the paper can be reformulated in the framework of the analysis of efficiency of linear estimators in random effects models (see also the discussion and review in Appendix A). As an example, in this section we discuss the implications of the results in Sect. 5 for efficiency of linear estimators in unbalanced two-stage nested design random effects models like (5.1). Using the results in Sects. 3 and 4 and in the next section, similar extensions can be obtained for linear estimators of location in two-way classification random effects models (1.2) and their analogues with more than two common shocks and varying factor loadings.

Similar to (5.1), we consider observations from the model

$$Y_{ij} = \mu + R_i + U_{ij}, \quad j = 1, \dots, n_i, \quad i = 1, \dots, r, \tag{6.1}$$

where  $\mu \in \mathbf{R}$  and  $R_i$  and  $U_{ij}$  are symmetric and unimodal r.v.'s. Similar to Sects. 3–5, in this section it is assumed that the variables  $R_i$  and  $U_{ij}$  are independent of each other and are independent and identically distributed among themselves.

A natural approach to comparisons of estimators of a population parameter under heavy-tailedness is that based on the likelihood of observing large deviations of these estimators from the true value of the parameter.

Let  $\hat{\theta}^{(1)}$  and  $\hat{\theta}^{(2)}$  be two estimators of the location parameter  $\mu$  in model (6.1). Following the above approach, we say, similar to Ibragimov (2007), that the estimator  $\hat{\theta}^{(1)}$  is (weakly) more efficient than  $\hat{\theta}^{(2)}$  in the sense of peakedness ( $P$ –more efficient than  $\hat{\theta}^{(2)}$  for short) if  $P(|\hat{\theta}^{(1)} - \mu| > \epsilon) \leq P(|\hat{\theta}^{(2)} - \mu| > \epsilon)$  for all  $\epsilon > 0$ .

For  $w = (w_1, \dots, w_r) \in \mathcal{I}_r$ , consider the linear estimators  $Z(w)$  of the location parameter  $\mu$  in form (5.3), with  $Z_i, i = 1, \dots, r$ , defined in (5.2):

$$Z(w) = \sum_{i=1}^r w_i Z_i, \quad Z_i = \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij}. \tag{6.2}$$

As in the Sect. 5, we deal with the weight vectors  $w = w^{(1)}, w^{(2)}, w^{(3)}, w^{(c)}$  defined in (5.6), (5.8), (5.10) and (5.11). Theorems 6.1 and 6.2 below provide  $P$ -efficiency comparisons for linear estimators  $Z(w)$  with the above weights considered, in the context of value at risk analysis, in Sect. 5.

**Theorem 6.1** *Let  $\epsilon > 0$ . Suppose that, in (1.3),  $R_i \sim \underline{CS}$ ,  $i = 1, \dots, r$ , and  $U_{ij} \sim \overline{CSLC}$ ,  $j = 1, \dots, n_i$ ,  $i = 1, \dots, r$ . Then the function  $\tau(c) = P [|Z(w(c)) - \mu| > \epsilon]$  is non-decreasing in  $c \in [0, 1]$ . In particular,*

$$P \left[ \left| Z(w^{(1)}) - \mu \right| > \epsilon \right] \geq P \left[ \left| Z(w(c)) - \mu \right| > \epsilon \right] \geq P \left[ \left| Z(w^{(2)}) - \mu \right| > \epsilon \right].$$

In addition,

$$P \left[ \left| Z(w^{(1)}) - \mu \right| > \epsilon \right] \geq P \left[ \left| Z(w^{(3)}) - \mu \right| > \epsilon \right] \geq P \left[ \left| Z(w^{(2)}) - \mu \right| > \epsilon \right].$$

**Theorem 6.2** *Let  $\epsilon > 0$ . Suppose that, in (1.3),  $R_i \sim \overline{CSLC}$ ,  $i = 1, \dots, r$ , and  $U_{ij} \sim \underline{CS}$ ,  $j = 1, \dots, n_i$ ,  $i = 1, \dots, r$ . Then the function  $\tau(c) = P [|Z(w(c)) - \mu| > \epsilon]$  is non-increasing in  $c \in [0, 1]$ . In particular,*

$$P \left[ \left| Z(w^{(2)}) - \mu \right| > \epsilon \right] \geq P \left[ \left| Z(w(c)) - \mu \right| > \epsilon \right] \geq P \left[ \left| Z(w^{(1)}) - \mu \right| > \epsilon \right].$$

In addition,

$$P \left[ \left| Z(w^{(2)}) - \mu \right| > \epsilon \right] \geq P \left[ \left| Z(w^{(3)}) - \mu \right| > \epsilon \right] \geq P \left[ \left| Z(w^{(1)}) - \mu \right| > \epsilon \right].$$

Theorems 6.1 and 6.2 show that  $P$ -efficiency comparisons in models (6.1) under heavy-tailedness are similar to VaR results in risk models (5.1) dealt with in the previous section. In particular, the results in Theorems 6.1 and 6.2 are in contrast to the case of variance comparisons for linear estimators  $Z(w)$  in the literature discussed in Sect. 5 and Appendix A. Namely, in contrast to the results for variances, under extreme heavy-tailedness in the common shocks  $R_i$  and moderate heavy-tailedness in the idiosyncratic risks  $U_{ij}$ ,  $P$ -efficiency comparisons point out to optimality of  $Z(w^{(2)}) = Z(w(0))$  among the estimators  $Z(w(c))$ ,  $c \in [0, 1]$  (Theorem 6.1). The estimator  $Z(w^{(2)})$  is also more  $P$ -efficient than  $Z(w^{(1)})$  and  $Z(w^{(3)})$ .

Similarly, in the case of moderate heavy-tailedness in  $R_i$  and extreme heavy-tailedness in  $U_{ij}$ ,  $P$ -efficiency of the estimators  $Z(w(c))$ ,  $c \in [0, 1]$ , is maximal under equal weights  $w_r = w(1)$  (Theorem 6.2). The estimator  $Z(w^{(1)})$  is also more  $P$ -efficient than  $Z(w^{(2)})$  and  $Z(w^{(3)})$ .

### 7 Extensions: multiple additive and multiplicative common shocks

The analysis presented in this paper can be extended to the case where the underlying risks  $Y$  in the portfolios exhibit dependence with more than two common shocks. For instance, let  $m \geq 1$ , and let  $N_1, N_2, \dots, N_m \in \mathbf{N}$ . Denote  $L = \prod_{s=1}^m N_s$ .

One can show that the analogues of the results in Sect. 3 also hold for the multiple shock extensions of (1.2) given by

$$Y_{i_1, i_2, \dots, i_m} = \sum_{s=1}^m \sum_{1 \leq j_1 < \dots < j_s \leq m} U_{i_{j_1}, \dots, i_{j_s}}^{(j_1, \dots, j_s)}, \tag{7.1}$$

$1 \leq i_k \leq N_k, k = 1, \dots, m$ , where the variables  $U_{i_{j_1}, \dots, i_{j_s}}^{(j_1, \dots, j_s)}$  are independent over all the indices  $1 \leq j_1 < \dots < j_s \leq m, s = 1, \dots, m, 1 \leq i_k \leq N_k, k = 1, \dots, m$ . The underlying risks  $Y$  in (7.1) have dependence structures exhibited by sums of  $U$ -statistics. Such dependence structures and a number of probabilistic and statistical results for them are discussed, among others, in de la Peña et al. (2002, 2003) and references therein.

A particular case of models (7.1) with  $m = 3$  is given by the risks

$$Y_{i, j, k} = U_i^{(1)} + U_j^{(2)} + U_k^{(3)} + U_{ij}^{(4)} + U_{ik}^{(5)} + U_{jk}^{(6)} + U_{ijk}, \tag{7.2}$$

$1 \leq i \leq N_1, 1 \leq j \leq N_2, 1 \leq k \leq N_3$ , where the summands are independent over all the indices. Specifications (7.1) also include, for instance, multi-stage nested design random effects models with  $U_{i_{j_1}, \dots, i_{j_s}}^{(j_1, \dots, j_s)} = 0$  for  $(j_1, \dots, j_s) \neq (1, \dots, s)$  :  $Y_{i_1, i_2, \dots, i_m} = U_{i_1}^{(1)} + U_{i_1, i_2}^{(2)} + \dots + U_{i_1, i_2, \dots, i_m}^{(m)}$ , and multi-way classification random effects models and their analogues (see Koch, 1967a,b).

Consider the portfolios of risks  $Y_{i_1, i_2, \dots, i_m}$  in (7.1) with weights  $w_{i_1, i_2, \dots, i_m} \in \mathbf{R}_+$  such that  $\sum_{i_1=1}^{N_1} \dots \sum_{i_m=1}^{N_m} w_{i_1, i_2, \dots, i_m} = 1$ . Let  $w \in \mathcal{I}_L$  denote the corresponding weight vector with components  $w_{i_1, i_2, \dots, i_m}$ . The return on the portfolio with weights  $w_{i_1, i_2, \dots, i_m}$  is given by  $Y(w) = \sum_{i_1=1}^{N_1} \dots \sum_{i_m=1}^{N_m} w_{i_1, i_2, \dots, i_m} Y_{i_1, i_2, \dots, i_m}$ . As in Sect. 3, one concludes that, in the case of moderately heavy-tailed  $U_{i_{j_1}, \dots, i_{j_s}}^{(j_1, \dots, j_s)} \sim \overline{\mathcal{CSLC}}$ , the value at risk  $VaR_q[Y(w)]$  of  $Y(w), w \in \mathcal{I}_L$  is minimized in the case of the most diversified portfolio with equal weights  $\underline{w}_{i_1, i_2, \dots, i_m} = \underbrace{(1/L, \dots, 1/L)}_L \in \mathcal{I}_L$ . In such

settings, the value at risk  $VaR_q[Y(w)], w \in \mathcal{I}_L$ , is maximized in the case of the least diversified portfolio with weights  $\bar{w}_{i_1, i_2, \dots, i_m} = \underbrace{(1, 0, \dots, 0)}_L \in \mathcal{I}_L$  that consists of

only one risk.

These comparisons are reversed for extremely heavy-tailed  $U_{i_{j_1}, \dots, i_{j_s}}^{(j_1, \dots, j_s)} \sim \underline{\mathcal{CS}}$ . Under extreme heavy-tailedness, the equal weights  $\underline{w}_{i_1, i_2, \dots, i_m} = (1/L, \dots, 1/L) \in \mathcal{I}_L$  maximize the portfolio value at risk  $VaR_q[Y(w)]$  over  $w \in \mathcal{I}_L$ . In contrast, the minimal portfolio value at risk over  $w \in \mathcal{I}_L$  is achieved for the least diversified portfolio with weights  $\bar{w}_{i_1, i_2, \dots, i_m} = (1, 0, \dots, 0) \in \mathcal{I}_L$ .

The analysis in the paper can also be generalized to the settings where the summands  $R_i, C_j$  and  $U_{ij}$  in model (1.2) and its analogues (including the case of multiple additive common shocks  $U_{i_{j_1}, \dots, i_{j_s}}^{(j_1, \dots, j_s)}$  in (7.1)), exhibit dependence. For instance, using in the proof the extensions of the results in Propositions 3.1 and 3.2 to the case

of dependence discussed in Ibragimov (2005, 2009b) and Ibragimov and Walden (2007) one obtains that all the results in the paper also hold in settings where the risks  $R_i$ ,  $C_j$  and  $U_{ij}$  in (1.2) and (1.3) are dependent among themselves or are bounded. These generalizations include models (1.2) and (1.3) in which the vectors of common shocks  $(R_1, \dots, R_r)$  and  $(C_1, \dots, C_c)$  and the vector of idiosyncratic errors  $(U_{11}, \dots, U_{1c}, \dots, U_{r1}, \dots, U_{rc})$  have distributions which are convolutions of  $\alpha$ -symmetric distributions (see Fang et al. 1990, and the review in Ibragimov 2005, 2009b, and Ibragimov and Walden 2007).

An  $N$ -dimensional distribution is called  $\alpha$ -symmetric if its c.f. can be written as  $\phi[(\sum_{i=1}^N |t_i|^\alpha)^{1/\alpha}]$ , where  $\phi: \mathbf{R}_+ \rightarrow \mathbf{R}$  is a continuous function (with  $\phi(0) = 1$ ) and  $\alpha > 0$ . An important property of  $\alpha$ -symmetric distributions is that, similar to stable laws, they satisfy property (2.2). The class of  $\alpha$ -symmetric distributions contains, as a subclass, spherical (also referred to as spherically symmetric) distributions corresponding to the case  $\alpha = 2$  (see Fang et al. 1990, p. 184). Spherical distributions, in turn, include such examples as Kotz type, multinormal, multivariate  $t$  and multivariate spherically symmetric  $\alpha$ -stable distributions (Fang et al. 1990, Ch. 3). Spherically symmetric stable distributions have c.f.'s

$$\exp \left[ -\lambda \left( \sum_{i=1}^N t_i^2 \right)^{\gamma/2} \right], \quad 0 < \gamma \leq 2, \tag{7.3}$$

and are, thus, examples of  $\alpha$ -symmetric distributions with  $\alpha = 2$  (that is, of spherical distributions) and  $\phi(x) = \exp(-x^\gamma)$ . For any  $0 < \alpha \leq 2$ , the class of  $\alpha$ -symmetric distributions includes distributions of vectors of risks  $(X_1, \dots, X_N)$  that have the common factor representation

$$(X_1, \dots, X_N) = (ZY_1, \dots, ZY_N), \tag{7.4}$$

where  $Y_i \sim S_\alpha(\sigma, 0, 0)$  are i.i.d. symmetric stable r.v.'s with  $\sigma > 0$  and the index of stability  $\alpha$  and  $Z \geq 0$  is a nonnegative r.v. independent of  $Y_i$ 's (see Fang et al. 1990, p. 197). Although the dependence structure in model (7.4) alone is restrictive, convolutions of such vectors provide a natural framework for modeling of random environments with different common shocks  $Z$ , such as macroeconomic or political ones, that affect all risks  $X_i$  (see Andrews 2003, 2005, and the discussion in the introduction). In the case  $Z = 1$  (a.s.), model (7.4) represents vectors with i.i.d. symmetric stable components that have c.f.'s  $\exp \left[ -\lambda \sum_{i=1}^N |t_i|^\alpha \right]$  which are particular cases of c.f.'s of  $\alpha$ -symmetric distributions with  $\phi(x) = \exp(-\lambda x^\alpha)$ .

Multiplicative common shock specifications (7.4) provide extensions of models (1.2) with  $R_i = \sum_{s=1}^{m_1} F_s \tilde{R}_{is}$ ,  $C_j = \sum_{s=1}^{m_2} G_s \tilde{C}_{js}$ ,  $U_{ij} = \sum_{s=1}^{m_3} H_s \tilde{U}_{ijs}$ , where the risks  $F_s, G_s, H_s > 0$  and  $R_{is}, C_{js}, U_{ijs}$  are independent of each other and among themselves. In these extensions, in addition to the two common shocks  $\tilde{R}$  and  $\tilde{C}$  as in (7.4), the risks  $Y_{ij}$  in are also affected by  $m_1 + m_2 + m_3$  common multiplicative shocks  $F, G$  and  $H$ .

As discussed in Ibragimov (2005, 2009b), and Ibragimov and Walden (2007), convolutions of  $\alpha$ -symmetric distributions exhibit both heavy-tailedness in marginals and dependence among them. For instance, convolutions of models (7.4) with  $\alpha < 1$  have extremely heavy-tailed marginal distributions with infinite means. On the other hand, convolutions of such models with  $1 < \alpha \leq 2$  can have marginals with power moments finite up to a certain positive order (or finite exponential moments) depending on  $\alpha$  and the choice of the r.v.'s  $Z$ . For instance, convolutions of models (7.4) with  $1 < \alpha < 2$  and  $E|Z| < \infty$  have finite means but infinite variances. However, marginals of such convolutions have infinite means if  $E|Z| = \infty$ . The moments  $E|ZY_i|^p$ ,  $p > 0$ , of marginals in models (7.4) with  $\alpha = 2$  (that correspond to normal r.v.'s  $Y_i$ ) are finite if and only if  $E|Z|^p < \infty$ . In particular, all marginal power moments in models (7.4) with  $\alpha = 2$  are finite if  $E|Z|^p < \infty$  for all  $p > 0$ . Similarly, marginals of spherically symmetric (that is, 2-symmetric) distributions range from extremely heavy-tailed to thin-tailed ones. For example, marginal moments of spherically symmetric stable distributions with c.f.'s (7.3) are finite if and only if their order is less than  $\gamma$ . Marginal moments of a multivariate  $t$ -distribution with  $k$  degrees of freedom which is an example of a spherical distribution are finite if and only if the order of the moments is less than  $k$ . These distributions were used in a number of works to model heavy-tailedness phenomena with moments up to some order (see the reviews in Ibragimov 2009b, and Ibragimov and Walden 2007, and references therein).

Let  $\Phi$  stand for the class of c.f. generators  $\phi$  such that  $\phi(0) = 1$ ,  $\lim_{t \rightarrow \infty} \phi(t) = 0$ , and the function  $\phi'(t)$  is concave. For  $0 < \alpha \leq 2$ , denote by  $\mathcal{G}_N(\alpha)$  the class of random vectors  $(X_1, \dots, X_N)$  with dependent components that satisfy one of the following conditions:

- (C1)  $(X_1, \dots, X_N)$  is a sum of  $k$  independent random vectors  $(Y_{1j}, \dots, Y_{Nj})$ ,  $j = 1, \dots, k$ , where  $(Y_{1j}, \dots, Y_{Nj})$  has an absolutely continuous  $\alpha$ -symmetric distribution with  $\phi_j \in \Phi$  and  $\alpha_j \in (0, 2]$ ;
- (C2)  $(X_1, \dots, X_N)$  is a sum of  $k$  random vectors  $(Y_{1j}, \dots, Y_{Nj}) = (Z_j V_{1j}, \dots, Z_j V_{Nj})$ ,  $j = 1, \dots, k$ , in (7.4) with independent r.v.'s  $Z_j, V_{ij}$ ,  $j = 1, \dots, k$ ,  $i = 1, \dots, N$ , such that  $Z_j$  are positive and absolutely continuous and  $V_{ij} \sim S_{\alpha_j}(\sigma_j, 0, 0)$ ,  $\sigma_j > 0$ ,  $\alpha_j \in (0, 2]$ . That is,  $X_i = \sum_{j=1}^k Z_j V_{ij}$  for  $i = 1, \dots, N$ .

Theorems 5.1 and 5.2 in Ibragimov (2009b) provide the following extensions of the VaR and diversification comparisons in Propositions 3.1 and 3.2 to the case of dependence.

**Proposition 7.1** (Ibragimov 2009b, Theorems 5.1 and 5.2) *Let  $(X_1, X_2, \dots, X_N) \in \mathcal{G}_N(\alpha)$ ,  $\alpha \in (0, 2]$ . Then Proposition 3.1 holds if  $\alpha \in [1, 2]$ , and Proposition 3.2 holds if  $\alpha \in (0, 1]$ .*

Using the VaR comparisons under dependence given by Proposition 7.1, one can show that the results in the paper continue to hold for the case of dependent common shocks  $R_i$  and  $C_j$  and idiosyncratic risks  $U_{ij}$ . In particular, the following results hold.

**Theorem 7.1** *Let  $(R_1, R_2, \dots, R_r) \in \mathcal{G}_r(\alpha_r)$ ,  $(C_1, C_2, \dots, C_c) \in \mathcal{G}_c(\alpha_c)$  and  $(U_{11}, U_{12}, \dots, U_{rc}) \in \mathcal{G}_{rc}(\alpha_u)$  with  $\alpha_r, \alpha_c, \alpha_u \in (0, 2]$ . Part (i) of Theorem 3.1;*



parts (i), (iii) and (v) of Theorem 3.2; relations (4.5) and (4.6); part (i) of Theorem 4.1 and parts (i) and (iii) of Theorem 5.3 hold if  $\alpha_r, \alpha_c, \alpha_u \in [1, 2]$ . Part (ii) of Theorem 3.1; parts (ii), (iv) and (vi) of Theorem 3.2; relations (4.7) and (4.8); part (ii) of Theorem 4.1 and parts (ii) and (iv) of Theorem 5.3 hold if  $\alpha_r, \alpha_c, \alpha_u \in (0, 1]$ . Theorems 5.1 and 6.1 hold if  $\alpha_r \in (0, 1]$  and  $\alpha_u \in [1, 2]$ . Theorems 5.2 and 6.2 hold if  $\alpha_r \in [1, 2]$  and  $\alpha_u \in (0, 1]$ .

The results in the paper can also be extended to portfolio choice problems for non-identically distributed risks. In addition, the results in the paper continue to hold for risk settings more general than common shock structures (1.2) and (1.3) considered so far for simplicity of presentation and the arguments. In particular, the results continue to hold for models (1.5) with varying factor loadings. As an example of the above generalizations, Theorem 7.2 provides the analogues of the results in Theorems 5.1 and 5.2 for the unbalanced one-factor case of (1.5) given by

$$Y_{ij} = \beta_{ij}R_i + U_{ij}, \quad j = 1, \dots, n_i, i = 1, \dots, r, \tag{7.5}$$

$n_1 \geq n_2 \geq \dots \geq n_r, \sum_{i=1}^r n_i = n$ , where  $R_i \sim S_\alpha(\sigma_i, 0, 0), U_{ij} \sim S_{\alpha'}(\sigma'_{ij}, 0, 0), \alpha, \alpha' \in (0, 2], \sigma_i, \sigma'_{ij} > 0, j = 1, \dots, n_i, i = 1, \dots, r$ , are independent not necessarily identically distributed heavy-tailed stable r.v.'s. Denote  $\tilde{\sigma}_i = \frac{1}{n_i} \left( \sum_{j=1}^{n_i} \beta_{ij} \right) \sigma_i$ . Observe that the within-group orderings  $\sigma'_{i1} \leq \dots \leq \sigma'_{i,n_i}$  and  $\sigma'_{i1} \geq \dots \geq \sigma'_{i,n_i}, i = 1, \dots, r$ , for the scale parameters  $\sigma'_{ij}$  in Theorem 7.2 do not restrict generality.

**Theorem 7.2** *Theorem 5.1 holds for (7.5) if  $\alpha \leq 1, \alpha' \geq 1, \tilde{\sigma}_1 \leq \dots \leq \tilde{\sigma}_r$  and  $\sigma'_{11} \leq \dots \leq \sigma'_{1,n_1} \leq \sigma'_{21} \leq \dots \leq \sigma'_{2,n_2} \leq \dots \leq \sigma'_{r1} \leq \dots \leq \sigma'_{r,n_r}$ . Theorem 5.2 holds for (7.5) if  $\alpha \geq 1, \alpha' \leq 1, \tilde{\sigma}_1 \geq \dots \geq \tilde{\sigma}_r$  and  $\sigma'_{11} \geq \dots \geq \sigma'_{1,n_1} \geq \sigma'_{21} \geq \dots \geq \sigma'_{2,n_2} \geq \dots \geq \sigma'_{r1} \geq \dots \geq \sigma'_{r,n_r}$ .*

As discussed in Sect. 5, the degree of heavy-tailedness of the common shocks  $R_i$  in Theorems 5.1, 5.2 and 7.2 is different from that for the idiosyncratic risks  $U_{ij}$ . It is interesting to compare the theorems with the results in Theorem 7.3 below for models (7.5) with varying factor loadings and heavy-tailed risks  $R_i$  and  $U_{ij}$  with non-identical distributions. In contrast to Theorems 5.1, 5.2 and 7.2, the degrees of heavy-tailedness of the common factors  $R_i$  and the errors  $U_{ij}$  in Theorem 7.3 are the same. Denote  $\tilde{\sigma}'_i = \frac{1}{n_i} \left( \sum_{j=1}^{n_i} (\sigma'_{ij})^{\alpha'} \right)^{1/\alpha'}, i = 1, \dots, r$ .

**Theorem 7.3** *Theorem 5.1 holds for (7.5) if  $\alpha, \alpha' \leq 1, \tilde{\sigma}_1 \leq \dots \leq \tilde{\sigma}_r$  and  $\tilde{\sigma}'_1 \leq \dots \leq \tilde{\sigma}'_r$ . Theorem 5.2 holds for (7.5) if  $\alpha, \alpha' \geq 1, \tilde{\sigma}_1 \geq \dots \geq \tilde{\sigma}_r$  and  $\tilde{\sigma}'_1 \geq \dots \geq \tilde{\sigma}'_r$ .*

Theorems 7.2 and 7.3 illustrate that portfolio diversification decisions are affected by the risk structures dealt with and the interplay between dependence and heterogeneity properties of the risks in consideration. Note, in particular, that the assumptions in Theorems 7.2 include the homogeneous case  $\beta_{ij} = \beta, \sigma_i = \sigma, \sigma'_{ij} = \sigma', j = 1, \dots, n_i, i = 1, \dots, r$ . However, the assumptions in Theorem 7.3 require heterogeneity in the idiosyncratic risks  $U_{ij}$ . As is easy to see, since  $n_1 \geq n_2 \geq \dots \geq n_r$ ,

the orderings  $\tilde{\sigma}'_1 \leq \dots \leq \tilde{\sigma}'_r$  for  $\alpha' \leq 1$  and  $\tilde{\sigma}'_1 \geq \dots \geq \tilde{\sigma}'_r$  for  $\alpha' \geq 1$  in the latter theorem cannot hold under homogeneity  $\sigma'_{ij} = \sigma'$  in the r.v.'s  $U_{ij}$  unless  $n_1 = n_2 = \dots = n_r$  or  $\alpha' = 1$ . As indicated in Sect. 5, in the balanced case  $n_1 = n_2 = \dots = n_r$ , the vectors  $w^{(1)}, w^{(2)}, w^{(3)}$  and  $w^{(c)}$  in Theorems 5.1 and 5.2 become the same:  $w^{(l)} = w^{(c)} = \underline{w}_r$  for all  $l = 1, 2, 3$ , and  $c \in [0, 1]$ . In the special case  $\alpha' = 1$  with the stable risks  $U_{ij} \sim S_1(\sigma', 0, 0)$ , the conditions in Theorem 7.3 involve only the assumptions on the scale parameters  $\sigma_i$  and the tail index  $\alpha$  for the common shocks  $R_i$ . The latter assumptions on the degrees of heavy-tailedness of  $R_i$ 's ( $\alpha \leq 1$  or  $\alpha \geq 1$ ) are similar to those in Theorems 5.1 and 5.2. We further note that, as indicated in Remark 3.1, the VaR comparisons in Theorems 5.1 and 5.2 hold as equalities in the case  $\alpha = \alpha' = 1$ .

Similar to the arguments for the results in Sect. 6, from Theorems 7.2 and 7.3 it follows that Theorems 6.1 and 6.2 also hold (in the same assumptions on the common shocks  $R_i$  and the idiosyncratic risks  $U_{ij}$  as in Theorems 7.2 and 7.3) for the factor models  $Y_{ij} = \mu + \beta_{ij}R_i + U_{ij}$ ,  $j = 1, \dots, n_i$ ,  $i = 1, \dots, r$ . Furthermore, similar to the proof of Theorems 7.1–7.3, one can also obtain their analogues for risk models with more than two additive shocks, like those in (7.1). In addition, similar to Theorems 7.1–7.3, one can also obtain extensions of the results in the paper for dependent and possibly non-identically distributed risks, including convolutions of scaled  $\alpha$ -symmetric random vectors.

## 8 Conclusion

Our analysis illustrates the generality of the majorization-based approach to the study of portfolio diversification and value at risk. In particular, the results in this paper show that the approach can be used in a wide range of dependent models, including those with multiple additive common shocks.

Similar to the case of independence, the tail index threshold  $\alpha = 1$  and finiteness of first moments of some of the risk components is the boundary between the robustness and reversals of the standard results in the variance minimization framework. Usually, these reversals under extreme heavy-tailedness point away from diversification, like the results in Sects. 3 and 4.

Surprisingly, however, for some important problems—including the optimal portfolio choice for indices of dependent heavy-tailed risks—the implications are opposite and diversification is optimal when risks are extremely heavy-tailed, as discussed in Sect. 5. The value of diversification thus depends crucially on the interplay among heavy-tailedness, dependence and heterogeneity properties of the risks involved.

## Appendix A: Majorization and comparisons of location estimators

A number of works in statistics and its applications have focused on the estimation of location in models (6.1) that are referred to in the fields as two-stage nested design, random effects location models. Several authors have considered variance decompositions and efficiency comparisons for location estimators in such models (see the

discussion and reviews in Weiler and Culpin 1970; Section 13.B in Marshall and Olkin 1979; Birkes et al. 1981; El-Bassiouni and Abdelhafez 2000, and references therein).

Suppose that  $ER_i = 0$ ,  $var(R_i) = \sigma_R^2$ ,  $EU_{ij} = 0$ ,  $var(U_{ij}) = \sigma_U^2$ ,  $j = 1, \dots, n_i$ ,  $i = 1, \dots, r$ . Evidently, for the variables  $Z(w)$  in (6.2), one has

$$var[Z(w)] = \sigma_R^2 V_R(w) + \sigma_U^2 V_U(w), \tag{A.1}$$

where  $V_R(w) = \sum_{i=1}^r w_i^2$  and  $V_U(w) = \sum_{i=1}^r \frac{w_i^2}{n_i}$ .

In the framework of inference on the location  $\mu$  using linear unbiased estimators, Cochran (1954) recommends using the unweighted ( $Z(w^{(1)})$  in (6.2) with  $w^{(1)}$  in (5.6)) and weighted ( $Z(w^{(2)})$  with the weights  $w^{(2)}$  in (5.8)) averages of group means, for large and small values of the intraclass correlation  $\gamma = \sigma_R^2 / (\sigma_R^2 + \sigma_U^2)$ , respectively. Birkes et al. (1981) show that the minimal complete class of linear unbiased estimators of  $\mu$  is given by  $Z(w(c))$  in (6.2) with the weights  $w(c)$  in (5.11). It is straightforward to show that if the intraclass correlation  $\gamma = \sigma_R^2 / (\sigma_R^2 + \sigma_U^2)$  is known, then the variance  $var[Z(w)]$ ,  $w \in \mathcal{I}_r$ , in (A.1) is minimized under the choice of weights  $w(\gamma)$ . Birkes et al. (1981) further focus on the analysis of efficiency  $eff(c, \gamma)$  for estimators  $Z(w(c))$  defined as the ratio  $eff(c, \gamma) = var[Z(w(c))]/var[Z(w(\gamma))]$  of the variance of  $Z(w(c))$  to the least possible variance  $var[Z(w(\gamma))]$  of linear unbiased estimators of  $\mu$  in (6.1). The authors identify the maximin efficiency estimator  $Z(w(c^*))$  that maximizes (over  $c \in [0, 1]$ ) the minimum possible efficiency  $\min_{\gamma \in [0, 1]} eff(c, \gamma)$ . The value  $c^*$  is found from the equation  $nV_U(w(c^*)) = rV_R(w(c^*))$ , that is,  $n \sum_{i=1}^r w_i^2(c^*)/n_i = r \sum_{i=1}^r w_i^2(c^*)$ .<sup>5</sup>

Koch (1967a) discusses variance decompositions (A.1) for the averages  $Z(w^{(1)})$  and  $Z(w^{(2)})$  in (6.2) with  $w^{(1)}$  and  $w^{(2)}$  in (5.6) and (5.8). He shows that the statistics  $Z(w^{(1)})$  and  $Z(w^{(2)})$  have the opposite orderings of the contributions to their variances in (A.1) from the row effects and the idiosyncratic error parts  $V_R$  and  $V_U$ . More precisely, as shown in Koch (1967a),  $V_R(w^{(1)}) \leq V_R(w^{(2)})$  and  $V_U(w^{(1)}) \geq V_U(w^{(2)})$ .<sup>6</sup> Koch (1967a) further conjectures that for the weights  $w^{(3)}$  in (5.10) one has

$$V_R[Z(w^{(1)})] \leq V_R[Z(w^{(3)})] \leq V_R[Z(w^{(2)})] \tag{A.2}$$

and

$$V_U[Z(w^{(1)})] \geq V_U[Z(w^{(3)})] \geq V_U[Z(w^{(2)})]. \tag{A.3}$$

This conjecture was proven by Low (1970) using some inequalities implied by majorization theory. An alternative more direct proof of the conjecture is provided in

<sup>5</sup> If one compares the estimators  $Z(w)$  by variances instead of efficiencies, then it is easy to show that, as discussed in Birkes et al. (1981), the unweighted average  $Z(\underline{w}_r)$  of group means in (6.2) with  $\underline{w}_r$  in (5.6) has the optimal property of being the “minimax variance” linear unbiased estimator of  $\mu$  in models (6.1) with fixed  $\sigma_R^2 + \sigma_U^2$ . More precisely,  $var[Z(\underline{w}_r)] = \min_{w \in \mathcal{I}_r} \max_{\gamma \in [0, 1]} var[Z(w)]$ , where, from (A.1),  $var[Z(w)] = (\sigma_R^2 + \sigma_U^2)[\gamma V_R(w) + (1 - \gamma)V_U(w)]$ , and  $\max_{\gamma \in [0, 1]} var[Z(w)] = (\sigma_R^2 + \sigma_U^2)V_R(w)$  since  $V_R(w) \geq V_U(w)$  for all  $w \in \mathcal{I}_r$ .

<sup>6</sup> Due to a typo, the inequality sign in the second of these two relations is reversed in the review on p. 393 in Marshall and Olkin (1979).

Sect. 13.B in Marshall and Olkin (1979). As discussed by Birkes et al. (1981), the maximin efficiency of  $Z(w(c^*))$  compares favorably with efficiency of  $Z(w^{(k)})$ ,  $k = 1, 2, 3$ .

**Appendix B: Proofs**

The proof of Theorems 3.1, 5.1 and 5.2 is based on the results in Theorems 3.2 and 5.3; this explains the order of the arguments presented below.

*Proof of Theorem 3.2.* Parts (i) and (ii) of Theorem 3.2 follow from Propositions 3.1 and 3.2 and majorization comparisons  $\underline{w}_{rc} < w < \bar{w}_{rc}$  for all  $w \in \mathcal{I}_{rc}$  implied by (3.1) with  $N = rc$ .

Using (3.1) with  $N = r$  and  $N = c$  we conclude that, for the vectors  $w_0^{(row)}$  and  $w_0^{(col)}$  in (3.2) and (3.3), one has

$$\underline{w}_0^{(row)} < w_0^{(row)} < \bar{w}_0^{(row)}, \tag{B.1}$$

$$\underline{w}_0^{(col)} < w_0^{(col)} < \bar{w}_0^{(col)}, \tag{B.2}$$

where, as in Sect. 3,  $\underline{w}_0^{(row)} = \underline{w}_r = \underbrace{(1/r, 1/r, \dots, 1/r)}_r \in \mathcal{I}_r$ ,  $\underline{w}_0^{(col)} = \underline{w}_c = \underbrace{(1/c, 1/c, \dots, 1/c)}_c \in \mathcal{I}_c$ ,  $\bar{w}_0^{(row)} = \underbrace{(1, 0, \dots, 0)}_r = \bar{w}_r \in \mathcal{I}_r$  and  $\bar{w}_0^{(col)} = \underbrace{(1, 0, \dots, 0)}_c = \bar{w}_c \in \mathcal{I}_c$  are the vectors that correspond to  $\underline{w}_{rc}$  and  $\bar{w}_{rc}$  by (3.2) and (3.3). Majorization comparisons (B.1) and (B.2), together with Propositions 3.1 and 3.2, imply parts (iii)–(vi) of Theorem 3.2.  $\square$

*Proof of Theorem 3.1* Let  $R_i, C_j, U_{ij} \sim \mathcal{CS}, i = 1, \dots, r, j = 1, \dots, c$ , and let  $w \in \mathcal{I}_{rc}$ . From part (ii) of Theorem 3.2 it follows that the risks  $U(w)$  in decomposition (3.4) satisfy

$$VaR_q[U(\underline{w}_{rc})] \geq VaR_q[U(w)] \geq VaR_q[U(\bar{w}_{rc})], \quad q \in (0, 1/2). \tag{B.3}$$

In addition, from parts (iv) and (vi) of Theorem 3.2 we conclude that the following value at risk comparisons hold for the components  $R(w_0^{(row)})$  and  $C(w_0^{(col)})$  in decomposition (3.4):

$$VaR_q[R(\underline{w}_0^{(row)})] \geq VaR_q[R(w_0^{(row)})] \geq VaR_q[R(\bar{w}_0^{(row)})], \quad q \in (0, 1/2), \tag{B.4}$$

$$VaR_q[C(\underline{w}_0^{(col)})] \geq VaR_q[C(w_0^{(col)})] \geq VaR_q[C(\bar{w}_0^{(col)})], \quad q \in (0, 1/2), \tag{B.5}$$

where  $\underline{w}_0^{(row)} = \underline{w}_r \in \mathcal{I}_r$ ,  $\bar{w}_0^{(row)} = \bar{w}_r \in \mathcal{I}_r$ ,  $\underline{w}_0^{(col)} = \underline{w}_c \in \mathcal{I}_c$  and  $\bar{w}_0^{(col)} = \bar{w}_c \in \mathcal{I}_c$  are the vectors that correspond to  $\underline{w}_{rc}$  and  $\bar{w}_{rc}$  by (3.2) and (3.3).

From Theorem 2.7.6 in Zolotarev (1986), p. 134, and Theorems 1.6 and 1.10 in Dharmadhikari and Joag-Dev (1988), pp. 13 and 20, by induction it follows that

the densities of the r.v.'s  $R(w_0^{(row)})$ ,  $C(w_0^{(col)})$  and  $U(w)$  are symmetric and unimodal if the assumptions of Theorem 3.1 hold. From Lemma in Birnbaum (1948) (see also Theorem 3.D.4 on p. 173 in Shaked and Shanthikumar 2007) it follows that if  $X_1, \dots, X_n$  and  $Y_1, \dots, Y_n$  are independent absolutely continuous symmetric unimodal r.v.'s such that, for  $i = 1, 2, \dots, n$ , and all  $q \in (0, 1/2)$ ,  $VarR_q(X_i) \leq VarR_q(Y_i)$ , then  $VarR_q(\sum_{i=1}^n X_i) \leq VarR_q(\sum_{i=1}^n Y_i)$  for all  $q \in (0, 1/2)$ .

This, together with inequalities (B.3)–(B.5) implies that, for all  $q \in (0, 1/2)$ ,

$$VarR_q[R(\underline{w}_0^{(row)}) + C(\underline{w}_0^{(col)}) + U(\underline{w}_{rc})] \geq VarR_q[R(w_0^{(row)}) + C(w_0^{(col)}) + U(w)] \geq VarR_q[R(\bar{w}_0^{(row)}) + C(\bar{w}_0^{(col)}) + U(\bar{w}_{rc})].$$

Consequently,

$$VarR_q[Y(\underline{w}_{rc})] \geq VarR_q[Y(w)] \geq VarR_q[Y(\bar{w}_{rc})] \tag{B.6}$$

for all  $q \in (0, 1/2)$ . Thus, part (ii) of Theorem 3.1 holds. Part (i) of Theorem 3.1 may be proven in a similar way, with the use of parts (i), (iii) and (v) of Theorem 3.2 instead of parts (ii), (iv) and (vi) of the theorem.  $\square$

*Proof of Theorem 4.1* The theorem follows from parts (i) of Propositions 3.1 and 3.2 and the majorization comparisons between  $\tilde{v}$  and  $\tilde{w}$  and between  $\tilde{v}$  and  $\tilde{\tilde{w}}$  given by (4.3) and (4.4).  $\square$

For the proof of Theorems 5.1–5.3 we need a lemma that follows from Proposition 5.B.1 in Section 5.B in Marshall and Olkin (1979) applied with condition (a') in that section.

**Lemma B.1** (Marshall and Olkin 1979, Proposition 5.B.1) *If  $a_1 \geq \dots \geq a_r > 0$ ,  $b_1 \geq \dots \geq b_r > 0$  and  $b_i/a_i$  is non-increasing in  $i = 1, \dots, r$ , then*

$$\left( \frac{a_1}{\sum_{i=1}^r a_i}, \dots, \frac{a_r}{\sum_{i=1}^r a_i} \right) < \left( \frac{b_1}{\sum_{i=1}^r b_i}, \dots, \frac{b_r}{\sum_{i=1}^r b_i} \right). \tag{B.7}$$

*If  $0 < a_1 \leq \dots \leq a_r$ ,  $0 < b_1 \leq \dots \leq b_r$  and  $b_i/a_i$  is non-decreasing in  $i = 1, \dots, r$ , then*

$$\left( \frac{a_1}{\sum_{i=1}^r n_i a_i} e_{n_1}, \dots, \frac{a_r}{\sum_{i=1}^r n_i a_i} e_{n_r} \right) < \left( \frac{b_1}{\sum_{i=1}^r n_i b_i} e_{n_1}, \dots, \frac{b_r}{\sum_{i=1}^r n_i b_i} e_{n_r} \right), \tag{B.8}$$

where, as in Sect. 5, for  $N \geq 1$ ,  $e_N = \underbrace{(1, \dots, 1)}_N \in \mathbf{R}^N$  denotes the  $N$ -vector of ones.

*Proof of Theorem 5.3* Consider the vectors  $w^{(1)} = \underline{w}_r = \underbrace{(1/r, \dots, 1/r)}_r \in \mathcal{I}_r$ ,  $w^{(2)}$ ,  $w^{(3)}$  and  $w^{(c)}$ ,  $0 \leq c \leq 1$ , defined in (5.6), (5.8), (5.10) and (5.11). From

the left majorization comparison in (3.1) it follows that

$$w^{(1)} \prec w \tag{B.9}$$

for all  $w \in \mathcal{I}_r$  and, since  $\left(\frac{w_1^{(2)}}{n_1}e_{n_1}, \dots, \frac{w_r^{(2)}}{n_r}e_{n_r}\right) = \underline{w}_n = \underbrace{(1/n, \dots, 1/n)}_n \in \mathcal{I}_n$ ,

$$\left(\frac{w_1^{(2)}}{n_1}e_{n_1}, \dots, \frac{w_r^{(2)}}{n_r}e_{n_r}\right) \prec w \tag{B.10}$$

for all  $w \in \mathcal{I}_n$ . Let us show, using Lemma B.1, that the following majorization relations hold:

$$w^{(3)} \prec w^{(2)} \tag{B.11}$$

(relation (B.11) is a part of Lemma 13.B.1.a in Marshall and Olkin 1979);

$$\left(\frac{w_1^{(3)}}{n_1}e_{n_1}, \dots, \frac{w_r^{(3)}}{n_r}e_{n_r}\right) \prec \left(\frac{w_1^{(1)}}{n_1}e_{n_1}, \dots, \frac{w_r^{(1)}}{n_r}e_{n_r}\right), \tag{B.12}$$

and

$$w(c') \prec w(c), \tag{B.13}$$

$$\left(\frac{w_1(c)}{n_1}e_{n_1}, \dots, \frac{w_r(c)}{n_r}e_{n_r}\right) \prec \left(\frac{w_1(c')}{n_1}e_{n_1}, \dots, \frac{w_r(c')}{n_r}e_{n_r}\right), \tag{B.14}$$

if  $0 \leq c < c' \leq 1$ .

To obtain (B.11), take  $a_i = n_i(n - n_i)$  and  $b_i = n_i, i = 1, \dots, r$ , in Lemma B.1. Under the assumptions of the theorem,  $b_1 \geq \dots \geq b_r$ . As indicated in the proof of Lemma 13.B.1.a in Marshall and Olkin (1979), because  $z_1 \geq z_2$  and  $z_1 + z_2 \leq 1$  together imply  $z_1(1 - z_1) \geq z_2(1 - z_2)$ , one also has  $a_1 \geq \dots \geq a_r$ . In addition, evidently,  $b_i/a_i = 1/(n - n_i)$  is non-increasing in  $i = 1, \dots, r$ . Consequently, by (B.7), (B.11) indeed holds.

To establish (B.12), take  $a_i = n - n_i$  and  $b_i = 1/(rn_i)$ . Then  $a_1 \leq \dots \leq a_r, b_1 \leq \dots \leq b_r$  and  $b_i/a_i = 1/(rn_i(n - n_i))$  is non-decreasing in  $i = 1, \dots, r$ . Consequently, (B.12) holds by (B.8).

Relation (B.13) is a consequence of (B.7) applied to  $a_i = n_i/((n_i - 1)c' + 1)$  and  $b_i = n_i/((n_i - 1)c + 1)$ .

Majorization (B.14) follows from (B.8) applied to  $a_i = 1/((n_i - 1)c + 1)$  and  $b_i = 1/((n_i - 1)c' + 1)$ .

Theorem 5.3 now follows from parts (i) of Propositions 3.1 and 3.2 and majorization comparisons (B.9)–(B.14). □

*Proof of Theorems 5.1 and 5.2* Suppose that, in (5.1),  $R_i \sim \overline{CSLC}$ ,  $i = 1, \dots, r$ , and  $U_{ij} \sim \underline{CS}$ ,  $j = 1, \dots, n_i$ ,  $i = 1, \dots, r$ . Let  $0 \leq c < c' \leq 1$ . Using parts (i) and (ii) of Theorem 5.3, we obtain

$$VaR_q[R(w(c'))] \leq VaR_q[R(w(c))], \tag{B.15}$$

$$VaR_q[U(\tilde{w}(c'))] \leq VaR_q[U(\tilde{w}(c))], \tag{B.16}$$

$$VaR_q[R(w^{(1)})] \leq VaR_q[R(w^{(3)})] \leq VaR_q[R(w^{(2)})], \tag{B.17}$$

$$VaR_q[U(\tilde{w}^{(1)})] \leq VaR_q[U(\tilde{w}^{(3)})] \leq VaR_q[U(\tilde{w}^{(2)})] \tag{B.18}$$

for all  $q \in (0, 1/2)$ .

Similar to the proof of Theorem 3.1, from Theorem 2.7.6 in Zolotarev (1986), p. 134, and Theorems 1.6 and 1.10 in Dharmadhikari and Joag-Dev (1988), pp. 13 and 20, we conclude that the densities of the r.v.'s  $R(w)$  and  $U(\tilde{w})$  are symmetric and unimodal under the assumptions of Theorems 5.1-5.3. As in the proof of Theorem 3.1, inequalities (B.15)–(B.18), together with Lemma in Birnbaum (1948) and Theorem 3.D.4 on p. 173 in Shaked and Shanthikumar (2007), imply

$$VaR_q[R(w(c')) + U(w(c'))] \leq VaR_q[R(w(c)) + U(w(c))], \tag{B.19}$$

$$\begin{aligned} VaR_q[R(w^{(1)}) + U(\tilde{w}^{(1)})] &\leq VaR_q[R(w^{(3)}) + U(\tilde{w}^{(3)})] \\ &\leq VaR_q[R(w^{(2)}) + U(\tilde{w}^{(2)})] \end{aligned} \tag{B.20}$$

for all  $q \in (0, 1/2)$ . That is,  $VaR_q[Z(w(c'))] \leq VaR_q[Z(w(c))]$  and  $VaR_q[Z(w^{(1)})] \leq VaR_q[Z(w^{(3)})] \leq VaR_q[Z(w^{(2)})]$  for all  $q \in (0, 1/2)$ . This proves Theorem 5.2.

Theorem 5.1 for  $R_i \sim \underline{CS}$  and  $U_{ij} \sim \overline{CSLC}$  may be proven in a similar way, with the reversals of the inequality signs in (B.15)–(B.20) implied by parts (ii) and (iv) of Theorem 5.3. □

*Proof of Theorems 6.1 and 6.2* As is easy to see, for symmetric r.v.'s  $X_1$  and  $X_2$ ,  $P(|X_1| > \epsilon) \leq P(|X_2| > \epsilon)$  for all  $\epsilon > 0$  if and only if  $VaR_q(X_1) \leq VaR_q(X_2)$  for all  $q \in (0, 1/2)$ . Therefore, Theorems 6.1 and 6.2 follow from the value at risk comparisons in Theorems 5.1 and 5.2. □

*Proof of Theorem 7.1* Denote  $\mathcal{R} = (R_1, R_2, \dots, R_r)$ ,  $\mathcal{C} = (C_1, C_2, \dots, C_c)$  and  $\mathcal{U} = (U_{11}, U_{12}, \dots, U_{rc})$ . The arguments for extensions of relations (4.5)–(4.8) and the results in Theorems 3.2, 4.1 and 5.3 for  $\mathcal{R} \in \mathcal{G}_r(\alpha_r)$ ,  $\mathcal{C} \in \mathcal{G}_c(\alpha_c)$  and  $\mathcal{U} \in \mathcal{G}_{rc}(\alpha_u)$  are completely similar to their proof in the case of classes  $\overline{CSLC}$ ,  $\overline{CS}$  and  $\underline{CS}$ , with the use of Proposition 7.1 instead of Propositions 3.1 and 3.2. As follows from the proof of Theorems 5.1 and 5.2 in Ibragimov (2009b), the portfolio return  $X(w) = \sum_{i=1}^N w_i X_i$ ,  $w_i \geq 0$ , is symmetric and unimodal for risks  $(X_1, \dots, X_N) \in \mathcal{G}_N(\alpha)$ ,  $\alpha \in (0, 2]$ . This property and the VaR comparisons in Theorems 3.2 and 5.3 for  $\mathcal{R} \in \mathcal{G}_r(\alpha_r)$ ,  $\mathcal{C} \in \mathcal{G}_c(\alpha_c)$  and  $\mathcal{U} \in \mathcal{G}_{rc}(\alpha_u)$  imply, by Lemma in Birnbaum (1948), the results in Theorems 3.1, 5.1 and 5.2 for  $\mathcal{R} \in \mathcal{G}_r(\alpha_r)$ ,  $\mathcal{C} \in \mathcal{G}_c(\alpha_c)$  and  $\mathcal{U} \in \mathcal{G}_{rc}(\alpha_u)$  (the arguments are completely similar to the proof of the latter theorems for the classes  $\overline{CSLC}$ ,  $\overline{CS}$  and  $\underline{CS}$ ).

As in the case of classes  $\overline{\mathcal{CSLC}}$ ,  $\overline{\mathcal{CS}}$  and  $\underline{\mathcal{CS}}$ , the extensions of Theorems 6.1 and 6.2 follow from the corresponding extensions of Theorems 5.1 and 5.2.  $\square$

*Proof of Theorem 7.2.* We have that the r.v.  $R(w)$  in decomposition (5.4) satisfies  $R(w) = \sum_{i=1}^r \frac{w_i}{n_i} \left( \sum_{j=1}^{n_i} \beta_{ij} \right) R_i =^d \sum_{i=1}^r w_i V_i$ , where  $V_i \sim S_\alpha(\tilde{\sigma}_i, 0, 0)$ ,  $i = 1, \dots, r$ , are independent stable r.v.'s independent of  $U_{ij}$ ,  $j = 1, \dots, n_i$ ,  $i = 1, \dots, r$ . Let  $\alpha \geq 1$ ,  $\alpha' \leq 1$ ,  $\tilde{\sigma}_1 \geq \dots \geq \tilde{\sigma}_r$  and  $\sigma'_{11} \geq \dots \geq \sigma'_{1,n_1} \geq \sigma'_{21} \geq \dots \geq \sigma'_{2,n_2} \geq \dots \geq \sigma'_{r1} \geq \dots \geq \sigma'_{r,n_r}$ . Since  $n_1 \geq \dots \geq n_r$ , the components  $w_1, \dots, w_r$  of the vectors  $w^{(1)}$ ,  $w^{(2)}$ ,  $w^{(3)}$  and  $w(c)$ ,  $c \in [0, 1]$ , in majorization comparisons (B.9), (B.11) and (B.13) satisfy  $w_1 \geq \dots \geq w_r$ . Similarly, the components  $w_1/n_1, \dots, w_r/n_r$  of the vectors  $\tilde{w}$  that corresponds to  $w^{(1)}$ ,  $w^{(2)}$ ,  $w^{(3)}$  and  $w(c)$  by (5.5) (see left majorization comparison (B.10) and relations (B.12) and (B.14)) satisfy  $w_1/n_1 \leq \dots \leq w_r/n_r$ . Using Remark 3.2 for the portfolio returns  $\sum_{i=1}^r w_i V_i$  and  $U(\tilde{w})$  similar to the proof of Theorem 5.3 we thus obtain that  $R(w)$  and  $U(\tilde{w})$  in (5.4) satisfy inequalities (B.15)–(B.18). Similar to the proof of Theorem 5.2, these inequalities imply (B.19) and (B.20) and the conclusion of the theorem for the portfolio return  $Z(w) = R(w) + U(\tilde{w})$ . The extension of Theorem 5.1 may be proven in a similar way.  $\square$

*Proof of Theorem 7.3* Similar to the proof of Theorem 7.2, using property (2.2), we have that the r.v.  $R(w)$  and  $U(\tilde{w})$  in decomposition (5.4) satisfy  $R(w) =^d \sum_{i=1}^r w_i V_i$ ,  $U(\tilde{w}) = \sum_{i=1}^r \frac{w_i}{n_i} \left( \sum_{j=1}^{n_i} U_{ij} \right) =^d \sum_{i=1}^r w_i W_i$ , where  $V_i \sim S_\alpha(\tilde{\sigma}_i, 0, 0)$ ,  $W_i \sim S_\alpha(\tilde{\sigma}'_i, 0, 0)$ ,  $i = 1, \dots, r$ , are independent stable r.v.'s. As noted in the proof of Theorem 7.2, the components  $w_1, \dots, w_r$  of the vectors  $w^{(1)}$ ,  $w^{(2)}$ ,  $w^{(3)}$  and  $w(c)$ ,  $c \in [0, 1]$ , satisfy  $w_1 \geq \dots \geq w_r$ . Let  $\alpha, \alpha' \geq 1$ ,  $\tilde{\sigma}_1 \geq \dots \geq \tilde{\sigma}_r$  and  $\tilde{\sigma}'_1 \geq \dots \geq \tilde{\sigma}'_r$ . Using Remark 3.2 applied to the portfolio returns  $\sum_{i=1}^r w_i V_i$  and  $\sum_{i=1}^r w_i W_i$ , we obtain that  $R(w)$  and  $U(\tilde{w})$  in (5.4) satisfy inequalities (B.15)–(B.18). Similar to the proof of Theorems 5.2 and 7.2, this implies that Theorem 5.2 holds for  $Z(w) = R(w) + U(\tilde{w})$ . The extension of Theorem 5.1 may be proven in a similar way.  $\square$

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