# Distortions and Efficiency in Production Economies with Heterogeneous Beliefs<sup>\*</sup>

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#### Abstract

We study consumption, saving, and asset prices in economies with disagreement and production, extending previous literature by focusing on settings with real effects of disagreement. Aggregate savings may be significantly distorted under disagreement, possibly related to the undersaving puzzle. Mispricing mainly manifests itself for idiosyncratic risk in the production economy, in contrast to the exchange economy where the risk-free rate and expected return on the market may be distorted. Potential policy implications include the introduction of investment taxes or subsidies. Overall, our results highlight the real effects of disagreement in financial markets, and the differences between economies with and without production.

**Keywords:** Heterogeneous beliefs, production economy, undersaving, speculation, idiosyncratic volatility puzzle, welfare, efficiency.

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# 1 Introduction

In the aftermath of the financial crisis, the effects of heterogeneous beliefs has received renewed attention, see Brunnermeier et al. (2014), Gilboa et al. (2014); Gayer et al. (2014), and Blume et al. (2018). These recent studies suggest that speculative trade between agents creates inefficiencies when some agents end up poor after speculating away much of their wealth. Speculation in the heterogeneous beliefs economy distorts consumption allocations and, instead of increasing welfare by allowing agents to hedge and share risks, generates volatility of consumption and wealth at the individual level (see Yan 2008, and Fedyk et al. 2013). It also distorts asset price dynamics, increasing volatility and potentially leading to mispricing (see Buraschi and Jiltsov 2006, David 2008, Dumas et al. 2009, Xiong and Yan 2010, Kubler and Schmedders 2012, Simsek 2013, Buss et al. 2016, and Ehling et al. 2018a, for recent contributions).<sup>1</sup>

The potential policy implications are, of course, huge, since the friction-free complete market equilibrium — even when it is implementable — actually may be inefficient with such welfare effects of speculation. For example, in a simple calibration, Blume et al. (2018) find that restrictions on the traded asset span, such as borrowing limits and transaction taxes, offer substantial welfare gains relative to the complete market benchmark.

Previous literature has mainly focused on the exchange economy setting, in which disagreement leads to speculation, redistribution, and inefficient risk sharing. These issues are important, but the role of financial markets in allocating resources to real productive assets is overlooked in the exchange economy. When there is speculation, the allocation of productive capital, i.e., real investments, may also be impacted. Hence, understanding the effects on the real economy becomes important once outside the exchange economy.

It is a priori unclear to what extent the distortions in the exchange economy carry over to the production economy setting. On the one hand, if distortions also arise in the real economy one may conjecture that these generate further distortions in financial markets. On the other hand, real investment opportunities are in general known to allow agents to smooth consumption across states and over time as, e.g., in Cox et al. (1985), which could potentially offset inefficient risk sharing in financial markets.

In this paper, we analyze the effects of disagreement in a competitive production economy, on consumption, savings, asset pricing, and welfare. We introduce the concept of a *distortion*, that all

<sup>&</sup>lt;sup>1</sup>See also Detemple and Murthy (1994), Zapatero (1998), Basak (2000), Basak (2005), Berrada (2006), Jouini and Napp (2007), Cvitanic and Malamud (2011), Cvitanic et al. (2012), Bhamra and Uppal (2014), Baker et al. (2016), Buraschi et al. (2014), Cujean and Hasler (2017), Ehling et al. (2018b), and Andrei et al. (2018).

agents with different beliefs can agree upon, and show that aggregate savings in general are distorted in equilibrium. Especially, when agents' elasticities of intertemporal substitution are lower than one, they save too little compared with what is socially optimal. We relate this result to the puzzling low savings rates that have been observed in the U.S. and many other countries. The distortions of savings are especially severe in markets with significant disagreement, with low inequality, and/or complete markets, leading to potentially testable implications of our theory. That savings distortions are high in markets with low inequality and/or complete markets, creates a tension for a policy maker between the objectives of increasing equality and completing markets, and the associated savings distortions generated by speculation. We discuss this tension and potential remedies in a separate section on policy implications.

We study the effect of disagreement on asset prices, and show how savings distortions generate asset price distortions, i.e., mispricing, in line with the previously conjectured link between distortions in the real economy and in financial markets. We relate this result to the idiosyncratic volatility puzzle, i.e., the observation that idiosyncratic risk is priced in real markets in contrast to what is predicted by standard asset pricing theory. We also show the conjectured reverse effect, namely that allowing for production decreases mispricing of some assets. Especially, when agents agree on aggregate risk, risk-free bonds are always mispriced in the exchange economy but never mispriced in the production economy. Altogether, our results show that the asset pricing implications of disagreement are different in economies with and without production.

The extension to production economies of the welfare analysis under disagreement creates novel challenges. For example, in the exchange economy agents' actual beliefs do not matter for whether an allocation is efficient. In contrast, in economies with production, these beliefs typically *do* matter. As a consequence, efficiency measures that work well in exchange economy settings may be inadequate in the production economy. Our welfare measure takes these challenges into account, providing a technical contribution. Interestingly, in our production economy the belief neutral inefficiency measure, which was introduced and analyzed in the context of an exchange economy, see Brunnermeier et al. (2014), is equivalent to our efficiency measure. In addition, our definitions of distortions and increased disagreement in general disagreement economies provide technical contributions that to our knowledge are novel.

Finally, we discuss potential policy implications. We cover the previously mentioned tensions between the welfare improvements associated with market completeness and limited inequality with the distortions created under disagreement, and also discuss additional challenges arising in a dynamic setting. An implication of our model is that taxes or subsidies on real investments may help mitigate real distortions generated by disagreement.

The rest of the paper is organized as follows: In the next section we develop a model of a competitive production economy with disagreement. In section 3 we formally introduce the concepts of efficiency and distortions. Section 4 contains our main results on consumption-savings and speculative distortions, whereas Section 5 analyzes asset pricing distortions, i.e., mispricing. Finally, in Section 6, we discuss policy implications. Throughout the paper, we introduce several examples that help explain and provide intuition for the general results. Some proofs are delegated to an Appendix, and the remaining proofs to an Internet Appendix.

## 2 A competitive production economy with disagreement

Consider a production economy with two dates,  $t \in T = \{0, 1\}$ . There are M > 1 possible states, and N > 1 agents with heterogeneous beliefs. Formally, we introduce the state space  $\Omega = \{\omega_m\}_{m=1,\ldots,M}$ , the filtration  $\mathcal{F}_{t\in T}$ , with  $\mathcal{F}_0 = \{\emptyset, \Omega\}$ ,  $\mathcal{F}_1 = 2^{\Omega}$ , and N probability spaces,  $(\mathcal{Q}^n, \Omega, \mathcal{F}_1)$ , where the probability measure  $\mathcal{Q}^n : \mathcal{F}_1 \to (0, 1)$ , represents agent n's beliefs about events in  $\mathcal{F}_1$ . We define  $q_m^n = \mathcal{Q}^n(\{\omega_m\}) > 0$  and the M-vectors  $q^n = (q_1^n, q_2^n, \ldots, q_M^n) \in S^M$ ,  $n = 1, \ldots, N$ . Here,  $S^M$  is the M-dimensional unit simplex,  $S^M \stackrel{\text{def}}{=} \{x \in \mathbb{R}_{++}^M : \sum_m x_m = 1\}$ . Agents' beliefs are summarized in the tuple  $\mathbf{q} = (q^1, q^2, \ldots, q^N) \in (S^M)^N$ .

In the special case when  $q^n = q \in S^M$  for all n, agents' have homogeneous beliefs, and we are in an agreement economy. Whenever  $q_m^n \neq q_m^{n'}$  for some n, n', and m, agents have heterogeneous beliefs, and we are in a disagreement economy. Agents are said to agree on events in  $\mathcal{F}_X \subset \mathcal{F}_1$ , if  $\mathbb{Q}^n(X) = \mathbb{Q}^{n'}(X)$  for all n, n', and events  $X \in \mathcal{F}_X$ . Moreover, agents are said to agree that  $\mathcal{F}_X$  and  $\mathcal{F}_Y$  are independent, if  $\mathcal{F}_X$  and  $\mathcal{F}_Y$  are independent under each agent's beliefs.

Agents are expected utility maximizers with constant relative risk aversion preferences. We focus on power utility, excluding the case of logarithmic utility. An agent's expected utility is then

$$U^{n} = U^{n}(c^{n}|q^{n}) = u(c_{0}^{n}) + \rho \sum_{m=1}^{M} u(c_{m}^{n})q_{m}^{n}, \qquad u(x) = \frac{x^{1-\gamma}}{1-\gamma}, \ \gamma \neq 1.$$
(1)

with  $\gamma > 0$ . Here, the  $\mathcal{F}_t$ -adapted process,  $\tilde{c}^n : T \times \Omega \to \mathbb{R}_{++}$ , represents the agent's consumption stream across different times and states,  $c_0^n = \tilde{c}_0^n(\omega_m)$ , and  $c_m^n = \tilde{c}_1^n(\omega_m)$ ,  $m = 1, \ldots, M$ . For simplicity, we assume that the personal discount rate is  $\rho = 1$ . We define the elasticity of intertemporal substitution (EIS),  $\psi = \frac{1}{\gamma}$ .

At t = 0, a divisible investment good is available, that can either be immediately transformed

into consumption — such that a unit of the investment good yields a unit of consumption — or it can be invested in a linear risky production technology, which each agent has access to, that yields consumption at t = 1. The return function  $R : \Omega \to \mathbb{R}_{++}$  summarizes the output of consumption in the M states at t = 1, yielded by a unit of investment at t = 0. Agent n is initially endowed with the amount  $K^n > 0$  of the investment good. Total initial endowments are represented by the vector,  $K = (K^1, \ldots, K^N)$ , which is normalized such that one unit of the investment good is available in total,  $K \in S^N$ . In autarky, agent n's budget constraint is then

$$c_0^n = K^n - I^n, \qquad c_m^n = I^n R_m, \ m = 1, 2, \dots, M,$$
(2)

where  $I^n$  denotes the amount the agent invests in the production technology. The aggregate investment in the economy is  $I = \sum_n I^n$ , aggregate consumption at time 0 is

$$C_0 = \sum_m (K^n - I^n) = 1 - I,$$
(3)

and aggregate consumption at time 1 in state m is

$$C_m = \sum_n I^n R_m = I R_m, \quad m = 1, 2, \dots, M.$$
 (4)

The primitives of the competitive economy is summarized by the quadruple  $\mathcal{E}' = (K, R, \mathbf{q}, \gamma)$ .

We decompose the state space into aggregate and idiosyncratic components, a distinction that will be important in our subsequent analysis. Specifically, we assume that  $\Omega = \Omega^A \times \Omega^I$ , so that  $\omega_m = (\omega_a^A, \omega_i^I) \in \Omega$ , with associated  $\sigma$ -algebras  $\mathcal{F}^A = 2^{\Omega^A}$  and  $\mathcal{F}^I = 2^{\Omega^I}$ , so that  $\mathcal{F}_1 = \mathcal{F}^A \otimes \mathcal{F}^I$ , and that the return function, R, is measurable with respect to  $\mathcal{F}^A$ . Thus, the investment return does *not* depend on idiosyncratic risk,  $\mathcal{F}^I$ . Agents agree that aggregate and idiosyncratic risks are independent. When  $|\Omega^A| = 1$ , there is no uncertainty about aggregate returns,  $R_m \equiv R$ . We call this case the *savings economy*, since investments are equivalent to risk-free savings in this case.

#### 2.1 Market and Equilibrium

At t = 0, agents trade in a market for state-contingent Arrow-Debreu securities on each state, with the *m*th security paying off 1 unit of the consumption good at time t = 1 if state *m* occurs. The price of the *m*th Arrow-Debreu security, is  $p_m$ . Absence of arbitrage implies<sup>2</sup>

$$\sum_{m} p_m R_m = 1$$

The optimization problem for agent n is then

$$\max_{c_0^n, \dots, c_M^n} u(c_0^n) + \sum_{m=1}^M u(c_m^n) q_m^n, \qquad \text{s.t.},$$
(5)

$$c_0^n = K^n - \sum_{m=1}^M c_m^n p_m.$$
 (6)

Here,  $c_m^n$  is agent *n*'s demand for state *m* consumption, which we use to define the demand vector  $d^n = (c_1^n, \ldots c_M^n) \in \mathbb{R}^M$ . Note that absence of arbitrage ensures that the agent's budget constraint (6) covers real investments. Specifically, a real investment of  $I^n$  changes consumption at time 0 by  $\Delta c_0^n = -I^n$ , and at time 1 by  $\Delta c_m^n = R_m I^n$ , so  $-\sum_m \Delta c_m^n p_m = -\sum_m R_m I^n p_m = -I^n = \Delta c_0^n$  in line with (6), and the budget constraint therefore covers such investments.

We restrict our attention to the case with strictly positive Arrow-Debreu security prices, since arbitrage opportunities would arise if some prices were nonpositive, inconsistent with equilibrium. We define the state price vector  $p = (p_1, \ldots, p_M) \in \mathbb{R}^M_{++}$ . The strict concavity and smoothness of agents' preferences implies that the demand vector  $d^n$  is a unique and smooth function of  $K^n$ ,  $q^n$ , and p, so we can write

$$d^n = D(K^n, q^n | p),$$

where D is a smooth function.

**Definition 1** (Equilibrium). Given the economy,  $\mathcal{E}'$ , a competitive (Walrasian) equilibrium is defined by a state price vector, p, such that

$$\sum_{m} p_m R_m = 1, \qquad and$$

$$\sum_{n} (D(K^n, q^n | p))_m = IR_m, \qquad m = 1, \dots, M,$$

for some  $I \in (0, 1)$ .

<sup>&</sup>lt;sup>2</sup>Strictly speaking, the implication only holds for interior aggregate investment allocations, such that 0 < I < 1, a condition that will always be satisfied in equilibrium in our model, because our utility specification satisfies the Inada conditions.

We say that p is an equilibrium state price vector, and we then have the following existence and uniqueness result:

**Proposition 1.** In the economy,  $\mathcal{E}' = (K, R, \mathbf{q}, \gamma)$ , there exists an equilibrium state price vector, p. Moreover, there exists a  $\underline{\gamma} < 1$ , such that p is unique when  $\gamma \geq \underline{\gamma}$ .

In our subsequent analysis, we assume that risk aversion is sufficiently high so that equilibrium is guaranteed to be unique, i.e., that  $\gamma \geq \gamma$ .<sup>3</sup>

#### 2.2 Idiosyncratic endowment shocks

It is straightforward to allow for idiosyncratic endowment shocks in our model. Each agent faces a shock,  $e^n$ , at t = 1, where  $e_m^n$  is the size of agent n's shock in state m. These shocks do not affect aggregate output, i.e., they are such that

$$\sum_{n=1}^{N} e_m^n = 0, \qquad m = 1, \dots, M.$$

We ensure that each agents' total wealth, including the value of endowment shocks, is strictly positive by the following extension: Given the equilibrium state price vector in an economy with no endowment shocks and initial allocation K, this state price vector also determines an equilibrium in any economy with initial endowments K', and endowment shocks e, such that

$$K^{n} = (K')^{n} + \sum_{m} e^{n}_{m} p_{m}, \qquad n = 1, \dots, N,$$

in which agent n solves the optimization problem

$$\max_{d_{1}^{n},...,d_{M}^{n}} U(c^{n}, |q^{n}), \quad \text{s.t.},$$

$$c_{0}^{n} = (K')^{n} - \sum_{i=1}^{M} d_{m}^{n} p_{m},$$

$$c_{m}^{n} = d_{m}^{n} + e_{m}^{n}.$$
(7)

Our analysis therefore covers such economies with idiosyncratic shocks. A sufficient condition for

<sup>&</sup>lt;sup>3</sup>Our uniqueness proof, which uses an approach that to the best of our knowledge is novel, does not rely on the gross substitutes property of demand functions (see Arrow et al. 1959), which would restrict risk aversion to be less than unity in this setting. It is commonly believed that  $\gamma > 1$  in practice. Hence, uniqueness is guaranteed for reasonable levels of risk aversion in our model, which we view as a strength.

agent wealth to be strictly positive in the economy with endowment shocks is that  $K^n R_m + e_m^n > 0$ for all n and m.

#### 2.3 An equivalent characterization of equilibrium

As shown in the proof of Proposition 1, every equilibrium is associated with an aggregate maximization problem. Specifically, the equilibrium consumption allocation is the solution to the maximization problem

$$\max_{c^1,\dots,c^N} \sum_{n=1}^N \lambda^n U(c^n | q^n), \tag{8}$$

for some weights  $\lambda = (\lambda^1, \ldots, \lambda^N) \in S^N$ , where  $I, C_0$  and  $C_m, m = 1, 2, \ldots, M$ , satisfy the feasibility constraints in (3,4). Thus, there exists a mapping from initial capital allocation and state price vector to the optimization weights,  $(K, p) \to \lambda$ .<sup>4</sup>

In the proof of Proposition 1, we also show that for  $\gamma \geq \underline{\gamma}$ , there is actually a bijection between K and  $\lambda$  on  $S^N$ ,  $K \in S^N \leftrightarrow \lambda \in S^N$ . It follows that for such  $\gamma \geq \underline{\gamma}$ , we can equivalently characterize the economy by the quadruple  $\mathcal{E} = (\lambda, R, \mathbf{q}, \gamma)$ . We will henceforth mainly use the  $\mathcal{E}$  characterization in our analysis, since it is more tractable.

#### 2.4 Equilibrium consumption and investments

The following proposition characterizes the equilibrium consumption of the agents in the economy, in terms of their beliefs,  $\mathbf{q}$ , and the weights,  $\lambda$ :

#### **Proposition 2.** The equilibrium consumption of agent n satisfies

$$c_0^n = f_0^n C_0, \qquad and \qquad c_m^n = f_m^n C_m, \qquad where \tag{9}$$

<sup>&</sup>lt;sup>4</sup>This maximization problem is technically that of a social planner with an *ex ante* welfare measure, leading to a so-called Arrow optimum, see the discussions in Starr (1973) and Harris (1978). Indeed, the utility specifications used for welfare analysis in Arrow (1951) and Debreu (1951), technically, also cover expected utility specifications with heterogeneous beliefs (where states replace goods in the specification). We do not use the social planner and welfare terms in this context, to avoid confusion since welfare in our context, similar to in Brunnermeier et al. (2014), Gilboa et al. (2014), and Blume et al. (2018), will not be based on such an *ex ante* measure, see Section 3.

$$f_0^n = \frac{(\lambda^n)^{\frac{1}{\gamma}}}{\sum_{i=1}^N (\lambda^i)^{\frac{1}{\gamma}}},$$
(10)

$$f_m^n = \frac{(q_m^n)^{\frac{1}{\gamma}} (\lambda^n)^{\frac{1}{\gamma}}}{\sum_{i=1}^N (q_m^i)^{\frac{1}{\gamma}} (\lambda^i)^{\frac{1}{\gamma}}},$$
(11)

and where  $C_0$  and  $C_m$ ,  $m = 1, 2, \ldots, M$ , are defined in (3,4).

Here,  $f_0^n$ , and  $f_m^n$  define the consumption shares of the agents in different times and states. It follows from (9-11) that agents tend to consume relatively more (have higher consumption shares) in states that they are optimistic about, in contrast to equilibrium consumption in the agreement economy, in which agents' consumption shares are equal across states. As a consequence agents deviate from full risk sharing in equilibrium, showing speculative behavior by "betting" against others who have different beliefs. Moreover, consumption shares do not depend on the aggregate levels of consumption,  $C_0$  and  $C_m$ . Hence, how the agents split the total "consumption pie" in equilibrium is independent of the pie's size.

We next characterize how disagreement affects aggregate investments, I. We define the aggregate investment-to-consumption ratio,  $Z = \frac{I}{1-I}$ . Equivalently,  $I = \frac{1}{1+\frac{1}{Z}}$ . Note that in the savings economy, I represents the savings rate. We have:

**Proposition 3.** The equilibrium investment-to-consumption ratio is:

$$Z = \left(\sum_{m=1}^{M} \hat{q}_m R_m^{1-\gamma}\right)^{\frac{1}{\gamma}},\tag{12}$$

where

$$\hat{q}_m = \left(\sum_{n=1}^N f_0^n (q_m^n)^{\frac{1}{\gamma}}\right)^{\gamma}, \qquad m = 1, \dots, M.$$
 (13)

In the agreement economy,  $\hat{q}_m = q_m$ , and the investment-to-consumption ratio reduces to that in a representative agent economy,

$$Z = (E[R^{1-\gamma}])^{1/\gamma},$$
(14)

where the expectation is taken with respect to the homogeneous beliefs. Note that this outcome is independent of the weights  $\lambda_n$ , and therefore also of agent wealth, K. In the savings economy with homogeneous beliefs, this further reduces to  $Z = R^{\frac{1-\gamma}{\gamma}}$ , independently of q.

In the disagreement economy, the investment-to-consumption ratio is no longer defined via a

simple expectation, since  $\sum_{m=1}^{M} \hat{q}_m$  in general does not sum to one and therefore does not define a probability measure.<sup>5</sup> It follows that the investment-to-consumption ratio in general deviates from that in a representative agent economy. In Section 4, we analyze how this leads to real distortions in the economy.

#### 2.5 Exchange economy with disagreement

It will be useful to compare our results with those in an exchange economy with disagreement, in which aggregate consumption is exogenously specified at each date and in each state. Specifically, in the exchange economy, there is no production and agent n instead receives endowments  $e_0^n > 0$  at time 0, and  $e_m^n > 0$  at time 1, in state  $m = 1, \ldots, M$ . In all other respects, the economy is identical to our production economy. The aggregate endowment is defined as  $e_0 = \sum_{n=1}^{N} e_0^n$ , which we normalize to 1, and  $e_m = \sum_{n=1}^{N} e_m^n$ ,  $m = 1, \ldots, M$ .

Given a state price vector, p, the optimization problem of agent n is

$$\max_{c_0^n, c_1^n, \dots, c_M^n} \ u(c_0^n) + \sum_{m=1}^M u(c_m^n) q_m^n, \qquad \text{s.t.}$$
(15)

$$c_0^n + \sum_{m=0}^M p_m c_m^n = e_0^n + \sum_{m=0}^M p_m e_m^n.$$
 (16)

Here, equation (16) describes the agent's budget constraint. Similar arguments as those leading to Proposition 1 implies the existence of a unique equilibrium state price vector in the exchange economy.

#### 2.6 Examples

To illustrate the effects of disagreement, we introduce a few examples that we will subsequently use. We first consider a baseline example with three agents and two aggregate states, a high state  $(\omega_H^A)$ , in which return on the investment is  $R_H$ , and a low state  $(\omega_L^A)$ , in which the return is  $R_L < R_H$ . We focus on the case with  $R_H = 1.1$  and  $R_L = 0.95$ . The idiosyncratic state space also contains two states,  $\omega_1^I$  and  $\omega_2^I$ . Hence, in this baseline example there are four states in total,  $\Omega = \{(\omega_H^A, \omega_1^I), (\omega_H^A, \omega_2^I), (\omega_L^A, \omega_2^I), (\omega_L^A, \omega_2^I)\}$ .

Table 1 summarizes aggregate returns and agent beliefs in the baseline example. The parameter  $\Delta_A$  measures the degree of disagreement among agents about the aggregate states and  $\Delta_I$  the degree of disagreement about the idiosyncratic states (we will formalize the concept of degree of disagreement)

<sup>&</sup>lt;sup>5</sup>See discussions in Jouini and Napp (2007) and Ehling et al. (2018a) about this in the case of exchange economies.

in Section 4.1). In the general case, we allow agents to disagree on both types of risk simultaneously, but we will mostly focus on the case where one of the two disagreement parameters is zero. When both parameters are zero, there is trivially no disagreement and all agents agree that each of the four states is equally likely. We consider four different levels of disagreement, with  $\Delta_i \in \{0, 0.1, 0.3, 0.4\}$ for  $i \in \{A, I\}$ , and two levels of risk aversion  $\gamma \in \{0.5, 2\}$ .

		Agent beliefs		
State, $m$	Aggregate Return, $R_m$	$q_m^1$	$q_m^2$	$q_m^3$
$(\omega_H^A, \omega_1^I)$	1.1	$(0.5 - \Delta_I) (0.5 - \Delta_A)$	0.25	$(0.5 + \Delta_I) \left(0.5 + \Delta_A\right)$
$(\omega_{H}^{A},\omega_{2}^{I})$	1.1	$(0.5 + \Delta_I) (0.5 - \Delta_A)$	0.25	$(0.5 - \Delta_I) \left( 0.5 + \Delta_A \right)$
$(\omega_L^A, \omega_1^I)$	0.95	$(0.5 - \Delta_I) (0.5 + \Delta_A)$	0.25	$(0.5 + \Delta_I) \left( 0.5 - \Delta_A \right)$
$(\omega_L^{\overline{A}},\omega_2^{I})$	0.95	$(0.5 + \Delta_I) (0.5 + \Delta_A)$	0.25	$(0.5 - \Delta_I) \left( 0.5 - \Delta_A \right)$

Table 1: The table shows the beliefs of the three agents about the probability for each of states m = 1, 2, 3, 4 to occur. The parameters  $\Delta_A$  and  $\Delta_I$  determine the amount of disagreement about aggregate and idiosyncratic risk.

Figure 1 illustrates the effect of disagreement about the idiosyncratic states on the consumption share of agent 1 in states  $\omega_1^I$  (y-axis) versus  $\omega_2^I$  (x-axis), when varying the agent's initial wealth.<sup>6</sup> Each line represents a different degree of disagreement, the left part of each line represents low wealth for agent 1, and the right part high wealth for the agent.

The blue line shows the agent's consumption share in the agreement economy. Since there is perfect risk sharing, the share is the same in the two states, leading to a straight 45-degree line. In the disagreement economies (the other lines, for which for  $\Delta_I > 0$ ), the consumption shares across states deviate from the straight 45 degree line, representing an equilibrium outcome with lower consumption in the state an agent is pessimistic about. As agent 1 is relatively pessimistic about  $\omega_I^I$ , corresponding to states 1 and 3, the consumption share is lower in those states than in states 2 and 4. The deviation from the straight line is greater under higher disagreement. Thus, the purple line — for which  $\Delta_I = 0.4$ — is further away from the blue line than the yellow line — for which  $\Delta_I = 0.3$  — is, which is in turn further away than the red line — for which  $\Delta_I = 0.1$ . Note that the expressions for the consumption shares in Proposition 2 do not depend on whether the state is aggregate or idiosyncratic. Hence, the figures would be the same under disagreement about the aggregate state,  $\Delta_A$ .

We next introduce a large economy example with rich disagreement. There are N agents and M = N states, where N is large. The belief of agent n about state n is  $q_n^n = \frac{1}{N} + \frac{N-1}{N}\Delta$ ,  $0 \le \Delta < 1$ , and about the other states,  $m \ne n$ , it is  $q_m^n = \frac{1-\Delta}{N}$ . All agents have equal wealths,  $K^n = \frac{1}{N}$  for all n.

<sup>&</sup>lt;sup>6</sup>We vary  $\lambda^1$  for agent 1 between zero and one, and set  $\lambda^2 = \lambda^3 = \frac{1}{2} (1 - \lambda^1)$ .



Figure 1: The figures shows consumption share in states 2 and 4 (x-axis) against the consumption share in state 1 and 3 (y-axis) for four different levels of idiosyncratic disagreement,  $\Delta_I = \{0, 0.1, 0.3, 0.4\}$ . The left plot shows the case of  $\gamma = 0.5$  and the right plot shows the case of  $\gamma = 2$ .

Note that when  $\Delta = 0$ , there is no disagreement. It is natural to view an economy with higher  $\Delta$  as one with more disagreement.

Finally, we consider a version of the large economy which we call the inequality economy, in which there is consumption inequality.<sup>7</sup> There are two types of agents: S > N/2 poor agents who consume the share  $f_0^P$  at time 0, and N - S rich agents who consume the share  $f_0^R > f_0^P$ . Note that  $f_0^R$  is determined by S and  $f_0^P$ , from the constraint  $Sf_0^P + (N - S)f_0^R = 1$ . The degree of inequality in the economy can be (partially) ranked by S and  $f_0^P$ . Specifically, if  $S' \ge S$  and  $f_0^{P'}S' \le f_0^P S$ , with at least one of the inequalities strict, inequality is higher in the second economy with parameters S' and  $f_0^{P'}$ , than in the first economy with parameters S and  $f_0^P$ . This is because there is a larger number of poor agents who in total consume less in the economy with higher inequality. It is easily verified that standard inequality measures, like the Gini coefficient, the Palma ratio, and the 80/20 ratio agree with this ranking, when defined over consumption.

<sup>&</sup>lt;sup>7</sup>Attanasio and Pistaferri (2016) argue that consumption inequality may be a better suited measure of inequality than the more commonly studied income inequality and wealth inequality. We consider consumption inequality because it is more tractable to analyze in our model. Note that the consumption share is closely linked to the weight  $\lambda^n$  through Equation (10), and that a higher weight corresponds to higher initial wealth,  $K^n$ .

# **3** Distortions and efficiency

We introduce measures of distortions and efficiency, based on two components that we take as given. The first component is the view, also taken in several recent papers, that welfare should not be measured based on individuals' subjective ex ante expected utilities. All agents cannot be correct in their different beliefs (a fact that agents agree about), and welfare gains from speculation, as measured by aggregating subjective expected utilities, may therefore be spurious.<sup>8</sup>

The second component is based on a sober view of the ability of a social planner, whose goal is to maximize welfare, to identify the economy's "true" probabilities. Such a view may be considered quite pessimistic, but we note that there are plenty of historical examples when it would have been appropriate. For example, it is known that the presence of overoptimistic investors in the market together with short-sale constraints may give rise to price and investment bubbles (see, e.g., Scheinkman and Xiong 2003, and Gilchrist et al. 2005). However, although bubbles and overoptimism may be easy to identify in hindsight, there is often considerable ex ante uncertainty about their presence. There was no consensus, and relatively few warnings about there being a bubble in the run-up years before the crash of the U.S. housing market in 2007, for example. Even former Chairman of the Federal Reserve Board, Alan Greenspan, admitted to not "getting" that there was trouble on the horizon until very late.<sup>9</sup> During the New Economy boom and the associated dot-com bubble between 1997-2000, warnings about a bubble were issued by some, whereas others argued that discontinuous technological transition made "Old Economy" valuation formulas out-of-date. Shiller (2000) warned about a bubble generated by irrational exuberance, borrowing the term from Alan Greenspan. Pastor and Veronesi (2006), in contrast, argue that the uncertainty about future growth rates in the late 1990's may well have justified the valuation of the NASDAQ. Moreover, there are many examples throughout history of technological innovations that turned out to be transformative, but that were originally dismissed by many as fads, including the automobile, personal computers, mobile phones and the Internet.

#### 3.1 Distortions

As shown in the previous section, disagreement affect aggregate investments. Are equilibrium investment levels reasonable, or should they be viewed as unreasonable, distorted? In line with our

<sup>&</sup>lt;sup>8</sup>It has also been argued that policies, which instead of being based on ex ante expected utilities are based on ex post measures of utility, could be viewed as paternalistic when they restrict the actions of agents (see, e.g., Harris and Olewiler (1979), and Fleurbaey (2010)). For further discussion, we refer to the extensive literature, see Starr (1973), Harris and Olewiler (1979), Hammond (1981), Harris (1978), Portney (1992), Hausman and McPherson (1994), Pollak (1998), Salanie and Treich (2009), and also recent studies by Gilboa et al. (2014), Brunnermeier et al. (2014), Gayer et al. (2014), and Blume et al. (2018). Our focus is not on this question.

<sup>&</sup>lt;sup>9</sup>CBS 60 Minutes, interview, September 13, 2007.

sober view of the planner's ability to identify the correct beliefs, we introduce a concept of distortion that respects individual agents' beliefs, such that an outcome is only viewed as distorted if all agents agree that it is. Specifically, the planner views a whole nonempty set,  $Q_R \subset S^N$  of beliefs as "reasonable." The special case when  $q^n = q$  for all agents, n, reduces to the homogeneous beliefs setting, in which case we require the planner's reasonable belief set to be  $Q_R = \{q\}$ . We are agnostic about the choice of  $Q_R$  in the general case. Brunnermeier et al. (2014) suggests using the convex hull of agents' individual beliefs.<sup>10</sup> Specifically, for a set  $X \subset \mathbb{R}^K$ , the convex hull is  $CH(X) = \left\{\sum_{k=1}^{K} \rho_k x_k : \rho \in \bar{S}^K, x_k \in X, 1 \le k \le K\right\}$ . The reasonable belief set suggested in Brunnermeier et al. (2014) is then

$$\mathcal{Q}_R^{CH} = CH\left(\left\{q^1, q^n, \dots, q^N\right\}\right).$$
(17)

Our theory does not rely on this specific choice of the reasonable belief set, but for the most part only on the restrictions that the set is nonempty, and that it contains a single element if and only if agents have homogeneous beliefs, in which case it is  $Q_R = \{q\}$ . For simplicity, we henceforth follow Brunnermeier et al. (2014) and assume that the reasonable belief set is equal to the convex hull of the set of agent beliefs, which in turn implies that each agent's beliefs are considered reasonable.

An equilibrium quantity in the economy is defined as being distorted if it takes on a value that is inconsistent with any equilibrium outcome under homogeneous beliefs in the reasonable belief set.<sup>11</sup> We formalize this concept as follows: Let  $\Pi(\mathcal{Q}_R) = \left\{ \mathbf{q} \in \prod_{n=1}^N S^M : \mathbf{q} = (q, q, \dots, q), q \in \mathcal{Q}_R \right\}$  denote the set of reasonable homogeneous beliefs. Now, consider the quantity, v, of the equilibrium outcome. Since the equilibrium allocation and price vector depends on initial endowments (K) and beliefs  $(\mathbf{q})$ , we may write  $v = v(K, \mathbf{q})$ .

The set of *reasonable values* of v is defined as

$$F_v^U = \left\{ v(K, \mathbf{q}) : K \in S^N, \mathbf{q} \in \Pi(\mathcal{Q}_R) \right\},\tag{18}$$

leading us to:

# **Definition 2** (Distortion). The equilibrium quantity v is said to be distorted if $v \notin F_v^U$ . It is too low

<sup>&</sup>lt;sup>10</sup>This choice may be appropriate in many applications, but other choices may be too. For example, the planner may wish to exclude some agents' beliefs that are obviously incorrect. Moreover, if agents self-report their beliefs, they may for strategic reasons report beliefs that are different than the ones they actually hold, and the social planner may therefore wish to filter out some reported beliefs. Finally, under more complex information structures, nonlinear aggregation of probabilities may be appropriate, rather than the linear aggregation that the convex hull corresponds to.

<sup>&</sup>lt;sup>11</sup>As the set of competitive equilibria in the homogeneous beliefs economy, via the welfare theorems, is equivalent to the set of efficient outcomes, we could equivalently have defined an outcome to be distorted with respect to the set of efficient outcomes under all homogeneous beliefs in the reasonable belief set.

if  $v < \inf F_v^U$ , and too high if  $v > \sup F_v^U$ . If the equilibrium quantity is not distorted, we say that it is reasonable.

If  $F_v^U = F_{v'}^U$ , and  $v' < v < \inf F_v^U$  or  $v' > v > \sup F_v^U$ , v' is said to be *more distorted* than v. Here, v and v' may represent the same equilibrium quantity in different economies, or different equilibrium quantities in the same economy. The definition of a distortion, which to the best of our knowledge is novel, applies to any real-valued equilibrium quantity, e.g., aggregate investments, consumption shares, and asset prices.

Returning to the baseline example with disagreement about idiosyncratic risk, as shown in Figure 1, it follows that any deviation from the blue 45-degree line corresponds to distorted consumption shares across states. Specifically, regardless of agent wealth, a reasonable consumption share of any agent will be the same in different states, since there is full risk sharing in the agreement economy. As shown in the figure, when there is disagreement, speculation makes the consumption shares of agent 1 in states 1 and 2 (which the agent is bullish about) higher than in states 3 and 4. We denote this deviation from full risk sharing a *speculative distortion*, and we use the following numerical measure of such distortions:  $SD = \frac{1}{N \times M} \sum_{m=1}^{M} \sum_{n=1}^{N} |f_m^n - f_0^n|$ , where  $f_0^n$ , and  $f_m^n$  define the consumption shares across agents and states, see Proposition 2. This measure is zero if and only if there is no speculation.

We examine aggregate investments in the baseline example, focusing on the parameters  $\gamma = 2$ ,  $\Delta_A = 0.3$ , and  $\Delta_I = 0$ . Using the expression for equilibrium investments in agreement economies (14), one verifies that the set of reasonable investments is  $F_I^U = [0.492, 0.503]$ . Under disagreement, when agents have equal wealth, equilibrium investments are I = 0.488 (which follows from (12,13)). So equilibrium investments are distorted — they are too low.

It is enlightening to compare our analysis with that in Malamud (2008), who derives bounds on prices in models with heterogeneous preferences and homogeneous beliefs, and shows that in such economies, prices might be lower or higher than what can be attained under homogeneous preferences. A similar observation for the risk-free rate is made in Wang (1996). These results complement ours, and the variation of preferences in those papers play a similar role as the variation of beliefs in our setting. The results are quite different from ours, though. Specifically, prices in agreement economies with heterogeneous preferences are never distorted according to our definition, even if they fall outside of the range obtainable with homogeneous preferences.

We carry out a systematic analysis of distortions in the disagreement economy in Sections 4 and 5.

#### 3.2 Efficiency

We define the set of feasible allocations,  $\mathcal{A} \subset \mathbb{R}^{M \times N \times T}_+$ . Specifically, for an element  $a \in \mathcal{A}$ ,  $a_{m,n,t}$  represents the allocation of the consumption good in state m to agent n at time t. In our economy, the set of feasible allocations are those with  $a_{m,n,0} = c_0^n$ , and  $a_{m,n,1} = c_m^n$ , where the consumption streams  $\tilde{c}^n$  satisfy the feasibility constraints (2-4).

Under homogeneous beliefs, the standard social planner's problem is to maximize

$$U = U(a|q,\lambda) = \sum_{n=1}^{N} \lambda^n U^n(a|q),$$
(19)

where  $\lambda \in S^N$  are Pareto weights in the planner's welfare function (see, e.g., Mas-Colell et al. 1995). Note that in this case with homogeneous beliefs, the social planner's problem has the same form as (8), since  $q^n = q$  for all n.

In the disagreement economy, agents' individual beliefs are used in the optimization problem (8), which could be used to define an efficiency measure. As mentioned, recent literature has challenged such measures and the speculative outcomes they lead to, see Gilboa et al. (2014), Brunnermeier et al. (2014), Gayer et al. (2014), and Blume et al. (2018), suggesting that efficiency should instead be defined based on beliefs that are common across agents. In line with this literature, we require a common belief, q, to be used for all agents in (19).

We use the following notation: If  $U(b|q,\lambda) > U(a|q,\lambda)$  for two allocations,  $a, b \in \mathcal{A}$ , we write  $b >_q^{\lambda} a$ , and if  $U(b|q,\lambda) \ge U(a|q,\lambda)$ , we write  $b \ge_q^{\lambda} a$ . Allocation b is not Pareto dominated by a, given q, if  $b \ge_q^{\lambda} a$  for some  $\lambda \in S^N$ , in which case we write  $b \succeq_q a$ . It is straightforward to verify that in the homogeneous beliefs economy, allocation b is Pareto efficient if and only if b is not Pareto dominated by any other feasible allocation, given q.

It is a priori unclear which beliefs to use in (19) under disagreement, since the social planner views a whole set of beliefs as reasonable. Our sober view of the planner's ability to identify the correct beliefs suggests a measure that requires a reallocation to be an improvement for *all* reasonable beliefs, for an initial allocation to be identified as inefficient. We formalize this idea as follows:

**Definition 3** (Dominance). Allocation b dominates a with respect to Pareto weights  $\lambda$ ,  $b \succ^{\lambda} a$ , if:

$$(\forall q \in \mathcal{Q}_R : b \ge_q^\lambda a) \text{ and } (\exists q \in \mathcal{Q}_R : b >_q^\lambda a).$$

From Definition 3 it follows that an allocation dominates another if, given Pareto weights, it is never

strictly dominated under any reasonable belief, and there exist a reasonable belief under which it strictly dominates the other allocation. It follows that under agreement (i.e., when  $Q_R$  is a singleton), allocation *a* dominates *b* in the classical sense of Pareto dominance, if and only if *a* dominates *b* with respect to Pareto weights  $\lambda$  (as per Definition 3), for all  $\lambda \in S^N$ .

We write  $a \succeq^{\lambda} b$  if it is not the case that  $b \succ^{\lambda} a$ . Efficiency is now defined as follows:

**Definition 4** (Efficiency). Allocation a is inefficient if  $\forall \lambda \in S^N, \exists b \in \mathcal{A}: b \succ^{\lambda} a$ . Equivalently, allocation a is inefficient if

$$\forall \lambda \in S^N, \exists b \in \mathcal{A}, \forall q \in \mathcal{Q}_R : b \ge_q^\lambda a, \tag{20}$$

where the inequality is strict for at least one q. An allocation that is not inefficient is called efficient.

An inefficient allocation is thus one for which whatever are the Pareto weights in the welfare function, there exists another allocation that is not dominated by the first allocation regardless of q in the reasonable belief set, and that dominates the first allocation for some such reasonable q.

The following result shows that the competitive equilibrium outcome is always inefficient under disagreement in the production economy:

#### **Proposition 4.** Competitive equilibrium is efficient if and only if beliefs are homogeneous.

The result is similar to what arises the exchange economy setting, as discussed in Brunnermeier et al. (2014). There are significant differences between efficiency in the exchange economy and the production economy, however, because of the possibility that multiple investment levels may be efficient in the production economy. Specifically, the efficiency concepts of *belief neutral efficiency* and *belief neutral inefficiency* were introduced in Brunnermeier et al. (2014). Belief neutral efficient allocations are those that are efficient regardless of which reasonable belief is correct. Belief neutral inefficiency for *one* that is *not* belief neutral *inefficiency* and *belief* neutral *inefficiency* are those that is not belief neutral *inefficient* may still not be belief neutral efficient, since efficiency for *one* reasonable belief is enough to rule out belief neutral inefficiency, but efficiency for *all* reasonable beliefs is needed for belief neutral efficiency. These differences are important in the production economy, as shown by the following result:

**Proposition 5.** The following relations between different efficiency measures hold:

(i) In the production economy, the set of belief neutral efficient allocations is empty, as long as at least two agents disagree about the optimal investment level.

- (ii) In the production economy, the set of efficient allocations is nonempty and identical to the set of not belief neutral inefficient allocations.
- (iii) In the exchange economy, the set of efficient allocations is nonempty, and identical to both the set of not belief neutral inefficient allocations and the set of belief neutral efficient allocations.

The proposition highlights the subtleties of conducting welfare analysis in production economies with disagreement. The first part shows that belief neutral efficiency in general is not well suited for the production economy, since it often identifies no efficient allocations. The second part shows that within our production economy, not belief neutral inefficient allocations are equivalent to those we define as being efficient. The result, which follows from duality, is a priori quite surprising, since belief neutral inefficiency requires the social planner to know correct beliefs, whereas our efficiency measure does not.<sup>12</sup> The third result shows that these subtleties do not arise in the exchange economy, in which all three concepts are equivalent.

The following result provides additional insight, by showing how efficiency relates to distortions in the production economy and exchange economy:

**Proposition 6.** In the exchange economy, an outcome is efficient if and only if there is no speculative distortion. In the production economy, an outcome is efficient if and only if there is no speculative distortion and no investment distortion.

The challenge of identifying efficient allocations in the production economy can be seen in the baseline example. Consider the case we studied before, with  $\gamma = 2$ ,  $\Delta_A = 0.3$  and  $\Delta_I = 0$ , i.e., with disagreement about the aggregate state. Any probability for the high state between 0.2 and 0.8 is reasonable in this case which, as noted in the previous section section, leads to reasonable investment levels of  $0.492 \leq I \leq 0.503$ . The lower bound corresponds to the highest reasonable probability for the high state (0.8), and the upper bound to the lowest reasonable probability (0.2). Any chosen investment level is therefore dominated for some reasonable belief, so there is no level that is optimal for all reasonable beliefs, and therefore no belief neutral efficient allocations. In contrast, under our efficiency criteria, any allocation without speculative distortions and with aggregate investments in the range  $I \in [0.492, 0.503]$  is efficient. Any such allocation is also not belief neutral inefficient, in line with Proposition 5:(ii).

<sup>&</sup>lt;sup>12</sup>In a companion paper, we show that the two efficiency concepts may differ in more general production economies, see also an earlier version of this paper titled "Welfare in Economies with Production and Heterogeneous Beliefs."

## 4 Consumption-savings distortions and speculative distortions

As noted in the previous section, investments are distorted in the baseline example with parameters  $\gamma = 2$ ,  $\Delta_A = 0.3$ , and  $\Delta_I = 0$ . How special is this case? Figure 2 shows the investment-to-consumption ratio (x-axis), and speculative distortions (y-axis) for all possible agent wealth allocations and for three levels of disagreement about the aggregate state ( $\Delta_A \in \{0.1, 0.3, 0.4\}$ ). The left plot shows the case of  $\gamma = 0.5$  and the right plot  $\gamma = 2$ . In the figure, the boundaries are also shown of the set of reasonable investment-to-consumption ratios for the three levels of disagreement. The dotted, dashed and solid vertical lines correspond to  $\Delta_A$  equal to 0.1, 0.3 and 0.4, respectively. For instance, when  $\Delta_A = 0.4$  the set of reasonable investment-to-consumption ratios is  $F_Z^U = [0.97, 1.09]$  as shown in the figure. Note that as disagreement increases, the set grows. This is a consequence of the disagreement being about aggregate risk, leading to disagreement among agents about the optimal investment level.



Figure 2: The figures shows the investment-to-consumption ratio (x-axis) and speculative distortion (y-axis) for three different levels of aggregate disagreement:  $\Delta_A = 0.1$ (turquoise),  $\Delta_A = 0.3$  (green) and  $\Delta_A = 0.4$  (blue). Each point on the figure corresponds to a particular wealth vector, K. The left plot shows the case of  $\gamma = 0.5$  and the right plot shows the case of  $\gamma = 2$ . The dotted, dashed and solid vertical lines correspond to the range of reasonable investment-to-consumption ratios for the case of  $\Delta_A$  being 0.1, 0.3 and 0.4, respectively.

Figure 3 considers the case of disagreement about the idiosyncratic states,  $\Delta_I$ . In this case, the set of reasonable investment-to-consumption ratios is a singleton, represented by the vertical line in each plot, since beliefs about idiosyncratic states are irrelevant for the prospects of the production technology. In other words, in any agreement economy the investment-to-consumption ratio is the same. When there is disagreement, the equilibrium investment-to-consumption ratio is different as seen in the figure, leading to a distortion. since the equilibrium investment-to-consumption ratio is different from the single element in  $F_Z^U$ .



Figure 3: The figures shows the investment-to-consumption ratio (x-axis) and speculative distortion (y-axis) for three different levels of idiosyncratic disagreement:  $\Delta_I = 0.1$ (turquoise),  $\Delta_I = 0.3$  (green) and  $\Delta_I = 0.4$  (blue). Each point on the figure corresponds to a particular wealth vector, K. The left plot shows the case of  $\gamma = 0.5$  and the right plot shows the case of  $\gamma = 2$ . The gray vertical line represents the unique level of reasonable investment-to-consumption ratio.

The investment distortion arises because of speculation. All agents know aggregate consumption in the next period, but speculate on the idiosyncratic states, which they disagree about. As a consequence, all agents perceive their wealth from future expected consumption to be higher than in the agreement economy. As mentioned earlier, the speculative distortions are the same when disagreement is about aggregate risk. Agents' desires to make speculative bets are independent of whether risks are aggregate or idiosyncratic, but whether the equilibrium effect on aggregate investments should be considered a distortion will in general depend on the type of disagreement.

These results can be generalized. From our discussion in Section 2.4, it follows that in the agreement economy, there is a unique equilibrium level of aggregate investments,  $I^* = \frac{1}{1+\frac{1}{Z}}$ , where Z is given by (14), regardless of agents' beliefs about idiosyncratic risk. In contrast, in the disagreement economy we have:

**Proposition 7.** Consider a disagreement economy in which agents agree on aggregate risk:

- (i) If the elasticity of intertemporal substitution is less than one,  $\psi < 1$ , there is underinvestment (overconsumption) in equilibrium,  $I < I^*$ .
- (ii) If the elasticity of intertemporal substitution is greater than one,  $\psi > 1$ , there is overinvestment (underconsumption) in equilibrium,  $I > I^*$ .

In the case of a savings economy, we can interpret Proposition 7 in terms of consumption-savings distortions. The intuition for why there is an consensus among agents about the savings distortion is because of the wealth effect of disagreement: When agents disagree, they all feel wealthier because of their anticipated future gains from speculation. They therefore readjust their consumption and savings, decreasing savings if the elasticity of intertemporal substitution,  $\psi$ , is less than one and increasing savings if it is greater than one. The effect of increased wealth on savings is known to be closely connected to  $\psi$ . Specifically,  $\psi$  determines if the income or substitution effect of the perceived increased wealth dominates, leading to oversaving if  $\psi > 1$  (substitution effect dominates) and undersaving if  $\psi < 1$  (income effect dominates).

The savings distortion is a consequence of "spurious unanimity" (see Mongin 1995): Agents believe the distorted savings outcome to be optimal for themselves, although they base their conclusions on different — mutually exclusive — beliefs. We note that although the wealth effects on savings are well known in the literature, what is special in our framework is that all agents agree that there is no increase in aggregate wealth, but still jointly behave as if there is one because of disagreement. To the best of our knowledge this impact of disagreement on aggregate savings has not been documented in the previous literature. It creates a "wedge" between the behavior of a representative agent and the aggregate behavior of agents with heterogeneous beliefs.<sup>13</sup> The mechanism shows that there are potential real effects of speculation, that are not present in the exchange economy.

Proposition 7 suggests that disagreement may help explain the low personal savings rate that has been observed in the U.S. and many other countries. The savings rate is substantially lower than what is predicted predicted by standard models and, moreover, has decreased significantly over the last four decades, see Parker (1999) and Guidolin and Jeunesse (2007). A suggested explanation is given in Han et al. (2018), namely that individuals overestimate the consumption of their peers because of so-called *visibility bias*. Specifically, the paper argues that consumption is more salient than non-consumption, and agents who fail to correct for this infer that their peers are consuming more than they actually do. These agents use their observations to infer that the risk for adverse shocks in the future is low, and as a consequence decrease their precautionary savings. The visibility bias mechanism is quite different from what occurs in our model, in which undersaving arises because of subjective wealth effects due to disagreement, and the associated speculation it leads to. In our model, agents believe that other agents are not saving enough, but that their own saving is optimal. This difference from Han et al.

 $<sup>^{13}</sup>$ The income and substitution effect have recently been studied in relation to disagreement for exchange economies; in Ehling et al. (2018a), who look at the effect on the risk-free rate; in Guzman and Stiglitz (2016), who focus on the sum of individual wealth expectations being higher than total wealth; and in Iachan et al. (2017), who study the effect of introducing new speculative assets.

(2018) could be used to separate the two explanations empirically, using survey data. Other behavioral explanations for the observed savings rates include hyperbolic discounting, see Laibson (1996), and catching up with the Joneses preferences, see Harbaugh (1996).

We conclude this section by examining how investment distortions are related to i) amount of disagreement, ii) inequality, and iii) degree of market completeness.

#### 4.1 Degree of disagreement

Figures 2 and 3 in the baseline example show that the investment-to-consumption distortion tends to increase as  $\Delta_i$  increases,  $i \in \{A, I\}$ . It is natural to interpret a higher  $\Delta_i$  as representing a higher degree of disagreement among the agents in this example, since the higher  $\Delta_i$  is, the further from each other are the agents beliefs about the different states. Alternatively, the greater is the differences between agents' perceived likelihood ratios between any pair of two states. Specifically, consider agent n's belief about the likelihood ratio between states m and m':  $\eta_{m,m'}^n = \frac{q_m^n}{q_{m'}^n}$ , and note that the higher  $\Delta_i$  is, the higher is the difference between high and low  $\eta_{m,m'}^n$ . For example, agent 1's likelihood ratio between states 4 and 1,  $\eta_{4,1}^1$  increases in  $\Delta_i$ , whereas agent 3's likelihood ratio decreases, as can be seen in Table 1. We use this alternative formulation as the basis for a general definition of higher degree of disagreement.

In the general economy, with N agents and M states, and beliefs  $\mathbf{q} = (q^1, q^2, \dots, q^N)$ , the set of agents with the highest beliefs about likelihood ratio for states  $m_1$  and  $m_2 \neq m_1$ , is defined by

$$\Gamma_{m_1,m_2} = \operatorname{argmax}_n \{\eta_{m_1,m_2}^n : n = 1, 2, \dots, N\}.$$

Beliefs  $\hat{\mathbf{q}} = (\hat{q}^1, \hat{q}^2, \dots, \hat{q}^N)$  — with associated likelihood ratios for states  $\hat{\eta}_{m_1,m_2}^n$  — are said to be obtained by a *disagreement increasing spread* of  $\mathbf{q}$ , if for some  $m_1$  and  $m_2 \neq m_1$ : (i)  $\hat{\eta}_{m_1,m_2}^n \geq \eta_{m_1,m_2}^n$ for all  $n \in \Gamma_{m_1,m_2}$  with at least one inequality being strict, and (ii)  $\hat{q}_m^n = q_m^n$  when  $n \notin \Gamma_{m_1,m_2}$ , and for all  $n \in \Gamma_{m_1,m_2}$  when  $m \notin \{m_1, m_2\}$ . Intuitively, a disagreement increasing spread is obtained by choosing some agents who have the highest beliefs about likelihood ratios between two specific states,  $m_1$  and  $m_2$ , and increasing these likelihood ratios even further, without affecting either of these agents' beliefs about other states, or the beliefs of other agents.

**Definition 5.** The degree of disagreement under beliefs  $\mathbf{q}^B$  is higher than under  $\mathbf{q}^A$ , if  $\mathbf{q}^B$  can be obtained by a sequence of disagreement increasing spreads of  $\mathbf{q}^A$ .

The key point here is that a disagreement increasing spread increases the likelihood ratio for the agent(s) with the *most extreme* likelihood ratio beliefs (which then become even more extreme). It is straightforward to show that any agent with such beliefs must lie on the boundary of the set of reasonable beliefs,  $\mathcal{Q}_R^{CH}$ , and that the disagreement increasing spread moves the boundary to include the agent's new beliefs (which were not previously included), in line with the intuition that disagreement increases. It is also easy to show that a disagreement increasing spread (weakly) increases the distance between the agent's beliefs and all other agents' beliefs, as measured by the  $\ell_1$ -norm  $||q^1 - q^2|| = \sum_m |q_m^1 - q_m^2|$ .

One can verify that the degree of disagreement is higher, the higher is  $\Delta_i$  in the baseline example, and the higher is  $\Delta$  in the large economy. We now have:

**Proposition 8.** In the savings economy, savings distortions are increasing in the degree of disagreement.

Our notion of increasing disagreement in general economies is to the best of our knowledge novel, and contrasts to most of the previous literature on disagreement, which has focused on economies with two agents, and/or used tight parametric specifications of belief heterogeneity. For example, Ehling et al. (2018a) links degree of disagreement to relative entropy in a two-agent economy.

It is useful to have a measure of disagreement that allows for many agents, since most empirical studies use panels with many forecasters. Iachan et al. (2017) link financial innovation and speculation to the savings rate and Ehling et al. (2018a) use survey date on both professional and households to study the link between the interest rate and disagreement. Propositions 7 and 8 provide potentially testable predictions of the impact of disagreement.

#### 4.2 Degree of inequality

Our model suggests lower savings distortions when there is higher inequality in the economy. In the extreme case, an agent who owns almost all capital will not have a counter-party to enter into significant speculative bets with and, as a consequence, distortions will be low. In contrast, in an economy with rich disagreement among many agents with similar wealth, there is room for significant speculation between agents, and thereby distortions. One can show the following result for the inequality economy introduced in Section 2.6:

**Proposition 9.** In the inequality economy, savings distortions are decreasing in the degree of inequality.

Similar to the results on degree of disagreement, this provides a potentially testable implication of our model.

#### 4.3 Degree of market completeness

Whether financial markets are complete or not should also be important in determining the degree of distortions, since it is through speculation in these markets that distortions arise. The analysis of incomplete markets is known to be quite challenging, see, e.g., the discussion in Geneakopolis (1990). Our model is no exception. For tractability, we consider the large economy example introduced in Section 2.6, focusing on the case with a moderate amount of disagreement,  $\Delta$ , and for the empirically relevant case of a risk aversion coefficient above unity,  $\gamma > 1$ . We vary the number of markets for Arrow-Debreu securities that are open, between the extreme cases of no open markets (autarky) and market completeness. One can show the following result:

**Proposition 10.** In the large economy with moderate levels of disagreement and risk aversion above unity,  $\gamma > 1$ , savings distortions increase when the number of open markets increases.

We have numerically verified the result in several other cases than those covered by the proposition, e.g., for  $\gamma < 1$ , for high levels of disagreement, and for other state spaces than assumed in the large economy example. The relation between market completeness and distortions provides another potentially testable implication of our model.

In light of Proposition 10, one may wonder whether a policy maker should simply shut down markets for contingent claims, since they generate distortions. Such a case for incomplete markets is discussed in Blume et al. (2018). Within our setting, shutting down markets for contingent claims help mitigate speculative and consumption-savings distortions. On the flip side, when agents are exposed to idiosyncratic endowment shocks, as in Section 2.2, closing markets hinders risk sharing and hedging, so a policy maker also needs to take these effects into account. In Section 6, we further discuss the actions a policy maker may take to improve welfare under disagreement.

# 5 Asset price distortions

It is a commonly held view that significant asset mispricing has been present during historical fads and bubble periods, see, for example, Shiller (2000). Mispricing, in our context corresponds to price distortions, and the bar for them to be present is high, since all agents need to agree on them. This is in contrast to many models in which behavioral assumptions make some agents' beliefs, actions, and valuations to be viewed as irrational when determining whether there is mispricing. Our objective in this section is to understand whether price distortions may arise in our model and, if so, under which conditions. One verifies that the price of the *m*-th Arrow-Debreu security,  $\delta^m$ , which pays one unit at time 1 if and only if state *m* is realized, is  $p_m = \hat{q}_m R_m^{-\gamma} Z^{-\gamma}$ , where  $\hat{q}_m$  is defined in (13). Let the vector  $a \in \mathbb{R}^M$ represent a general financial asset that pays  $a_m$  in state *m* at t = 1. By no-arbitrage arguments, its equilibrium price is  $P^a = \sum_m a_m p_m$ .

**Definition 6** (Price distortion). An (a, b) price distortion is defined by two financial assets, a and b, with equilibrium prices  $P^a$  and  $P^b \neq 0$ , such that  $\frac{P^a}{P^b}$  is distorted.

The asset b in this definition has the natural interpretation of being a numeraire, i.e., the price distortion is defined in terms of the price of asset a measured in number of units of asset b. When b = R, this numeraire is in terms of the consumption good, since  $P^R = 1$ . In this case we say that asset a is mispriced. Note that the price of a risk-free bond is  $P^1 = \sum_m p_m = \frac{1}{R^f}$ , where  $R^f$  is the (gross) risk-free rate. It follows that the risk-free rate is distorted if and only if the risk-free bond is mispriced. The expected return on asset a, according to agent n's beliefs, is  $\sum_m q_m^n a_m/P^a$ , and the expected return on the market is  $\sum_m q_m^n R_m$ , since  $P^R = 1$ .

The following proposition characterizes mispricing of Arrow-Debreu securities for states about which there is agreement, i.e., states for which all agents agree about  $q_m$ . The equilibrium price of such an asset is  $p_m = q_m R_m^{-\gamma} Z^{-\gamma}$ , leading to:

**Proposition 11.** An Arrow-Debreu security on a state, m, about which there is agreement, is mispriced if and only if investments are distorted. Specifically,

- (i)  $p_m$  is too high if there is underinvestment,
- (ii)  $p_m$  is too low if there is overinvestment.

It follows immediately from Propositions 7 and 8 that in the savings economy with disagreement, an Arrow-Debreu security on a state about which there is agreement is always mispriced, and that the degree of mispricing is increasing in the degree of disagreement in the economy.

To illustrate, consider the baseline example with  $\Delta_A = \Delta_I = 0.1$ . In this case, there is very little disagreement about the two middle states,  $(\omega_H^A, \omega_2^I)$  and  $(\omega_L^A, \omega_1^I)$ , as both agent 1 and 3 believe the probability for each of these states to be 0.24 and the middle agent believes it to be 0.25. One verifies that the two Arrow-Debreu prices related to these states are both distorted. For instance,  $p_{H,1} = 0.2237$  which falls outside the set of reasonable prices  $F_{p_{H,1}}^U = [0.2245, 0.2356]$ . For the states with high disagreement,  $(\omega_H^A, \omega_1^I)$  and  $(\omega_L^A, \omega_2^I)$ , there is no mispricing.<sup>14</sup>

 $<sup>^{14}</sup>$ The above example is technically not covered by Proposition 11, since it has disagreement about all states. However, when the beliefs of agent 2 are modified to be 0.24 for each of the two middle states and 0.26 for the other two states, there is no disagreement about these middle states, and the proposition therefore implies that they are mispriced.

Proposition 11 shows how investment distortions might generate mispricing in financial markets, as conjectured in the introduction. As a direct consequence of the result, a general asset (not necessarily an Arrow-Debreu security) that makes all its payments in states that agents agree on, is mispriced whenever investments are distorted. An example would be an asset which contains idiosyncratic risk about which there is no disagreement.

That mispricing arises for assets about which there is *agreement* may a priori be surprising. The reason is that such assets have the advantage — from a mispricing identification perspective — of having unique reasonable prices. This is in contrast to assets about which there is disagreement, for which a whole range of reasonable prices exists, making mispricing harder to identify.

We next focus on two important quantities: the expected return on the market and the risk-free interest rate. We have:

Proposition 12. In the production economy with disagreement

- (i) The expected return on the market is never distorted,
- (ii) The risk-free interest rate is never distorted when agents agree on aggregate risk.

It is useful to compare these results with those in the exchange economy. The following proposition shows that mispricing in the exchange economy in several ways arise for the opposites types of assets than in the production economy:

**Proposition 13.** In the exchange economy with disagreement

- (i) An Arrow-Debreu security on a state about which there is agreement is never mispriced,
- (ii) The expected return on the market may be distorted,
- (iii) The risk-free interest rate is always distorted when agents agree on aggregate risk.

The first part of the proposition illustrates that since investment distortions are not present in the exchange economy, the associated mispricing does not arise in the exchange economy either. The second and third parts suggest that distortions of the market's expected returns and the risk-free rate are more prevalent in the exchange economy than in the production economy.

Our results on distortions of aggregate risk are in line with the discussion in the introduction that the possibility to smooth consumption in the production economy mitigates some distortions. For example, in the savings economy, agents adjust their investments to smooth out perceived gains from speculation. In the exchange economy, such smoothing is not possible. Instead, prices have to adjust to keep aggregate consumption equal to total endowments (see Ehling et al. (2018a)), thereby distorting the risk-free rate. Altogether, our results highlight the differences between the exchange economy and production economy under disagreement.

#### 5.1 Idiosyncratic disagreement and mispricing

The previous results in this section suggest that disagreement in the production economy may be better at explaining mispricing of idiosyncratic risk than of aggregate risk. Goyal and Santa-Clara (2003) and Ang et al. (2006) analyze how idiosyncratic volatility affects expected returns. Ang et al. (2006) find that firms with high idiosyncratic volatility have low future expected returns, i.e., that a negative risk premium is associated with idiosyncratic volatility.

Standard asset pricing models predict that idiosyncratic volatility should not be priced, so this *idiosyncratic volatility puzzle* is thus consistent with mispricing of assets with idiosyncratic risk. Specifically, the puzzle is consistent with assets with high idiosyncratic volatility having too high prices. It follows from Proposition 11 that if agents agree on such assets with high idiosyncratic volatility, overpricing occurs when there is underinvestment, and that the degree of overpricing increases in the degree of underinvestment. Equivalently, abnormal expected returns of such assets are negatively related to the degree of underinvestment.

Whether there is underinvestment in equilibrium depends on the market's elasticity of intertemporal substitution (EIS),  $\psi$ . A short summary of empirical evidence suggests that the EIS various across markets and is often, but not always, less than one; for a recent survey, see Havranek et al. (2013). Also, the EIS is higher in markets with more sophisticated and wealthier investors. For example, Vissing-Jorgensen (2002), shows that the EIS implied from agents' Euler equations is consistently higher for bond market participants than for stock market participants and, in turn, higher for stock market participants than for nonparticipants. Blundell et al. (1994) find that the EIS is higher for rich than for poor households, and Havranek et al. (2013) furthermore find that the EIS is higher in wealthy countries than in poor countries.

Focusing on the case when the EIS is less than one, the idiosyncratic volatility puzzle suggests that idiosyncratic volatility is positively related to degree of disagreement, such that the payoffs of assets with high idiosyncratic volatility occur in states and/or times of high disagreement. This positive relation provides a potentially testable prediction.

# 6 Discussion

Our efficiency measure has the appealing feature of respecting individual beliefs. A policy maker may therefore be able to take steps that all agents agree improve efficiency. Such steps may be viewed as less paternalistic than those that violate the beliefs of agents identified as irrational and not acting in their own best interest.

As we have seen, the source of inefficiencies is speculative trades between agents, that both move equilibrium away from the risk sharing outcome and lead to investment distortions. In the literature, steps to discourage speculation are discussed, in terms of transaction taxes (see, e.g., Dávila 2020), and limiting the set of traded assets, i.e., shutting down markets (see Blume et al. 2018). Introducing a transaction tax in a purely speculative market, or even shutting down that market, improves efficiency. When the market in addition to speculation also serves a purpose to facilitate risk-sharing and hedging, there is a trade-off between welfare improvements and costs of introducing such measures. Quantifying this trade-off is outside of the scope of our model.

A novel implication of our model is that the policy maker may also consider steps that alleviate investment distortions generated by disagreement. Consider, for example, the baseline example, in which agents disagreed about aggregate risk, and thereby about the prospects of the investment technology, but agreed that equilibrium investments were too low. An investment subsidy, which may be designed to be revenue neutral via taxation, would make investments in the production technology more profitable, leading to higher equilibrium investments. At the right level, such a subsidy would lead to a reasonable equilibrium investment level, thereby resolving the investment distortion part of the inefficiency. In numerical examples, we have verified that such a policy can be designed to improve efficiency when there is an equilibrium investment distortion. Note that in the case of equilibrium overinvestment, i.e., when the EIS is greater than one, the policy would be implemented as an investment tax.

A related ongoing discussion is taking place in the academic literature and within international organizations, about how to create an investment friendly climate, see Decker et al. (2014). For dynamic young businesses, an environment that encourages risk-taking among the economy's entrepreneurs and innovators is especially important. As emphasized in Mukherjee et al. (2017) and Watson and Kaeding (2019), taxes and subsidies play an important role in this discussion. Our results suggest that the degree of disagreement about productive resources should matter in determining the optimal incentives.

In our analysis, we have not focused on the choice of reasonable belief set, but mainly followed previous literature and used the convex hull of individual agents' beliefs. In practice, the choice will be influenced by the amount of information available to policy makers, and their willingness to override individual beliefs — at the risk of being viewed as paternalistic. The smaller is the reasonable belief set, the stronger are the implications for ruling out inefficient allocations. If the set is sufficiently small, it will exclude some agents' beliefs. These beliefs are then viewed as unreasonable (irrational), which opens up for additional policy making via financial education, information sharing, and announcements. Such steps have been emphasized in the household finance literature. For example, in a recent paper D'Acunto et al. (2018) discuss policies that are efficient in affecting household beliefs, concluding that the simplicity of an announcement is a crucial factor that has been neglected in previous literature.

Our model is developed in a one-period economy. A more realistic dynamic environment with many dates would bring new dimensions to the analysis. Consider a *T*-period extension of the model, in which an agent receives the constant stream of non-perishing investment capital,  $K^n$  at each date,  $0, \ldots, T-1$ , and investment returns are i.i.d. over time. In autarky, the agent's budget constraint is then  $w^n(t+1) = K^n + (w^n(t) - c^n(t))R(t)$ , where  $w^n(t)$  defines the agent's accumulated stock of investment capital at time t,  $c^n(t)$  time-t consumption,  $I^n(t) = w^n(t) - c^n(t)$  time-t investments, and R(t) investment returns between t and t + 1. In the competitive market, one-period Arrow-Debreu securities on all state realizations over the next period are traded at all times, making the market dynamically complete. For simplicity, we assume that agents do not update their beliefs over time. It is straightforward to characterize equilibrium in this *T*-period extension, using backward induction.

The policy maker in the multi-period extension may wish to draw inferences about the reasonable belief set from observed returns. For example, the empirical distribution function at time t,  $\bar{q}^E(t)$  could be used to update the reasonable belief set, leading the policy maker to choose the time-t reasonable belief set  $Q_R(t) = f(t)\bar{q}^E(t) + (1 - f(t))q$ ,  $q \in Q_R(0)$ , where  $Q_R(0)$  is the initial reasonable belief set used in the one-period setting, and f is an increasing function of time that determines the weight put on empirical observations by the policy maker. With such an approach, the reasonable belief set shrinks over time. We view the question of how a policy maker should update the reasonable belief set over time as an interesting topic for future research, and focus our discussion on novel considerations that arise in such a setting.

Previous literature has shown that market selection in financial markets in general punishes agents with incorrect beliefs, and in the long term completely wipes them out, see Sandroni (2000, 2005) and Blume and Easley (1992, 2006). Although differences in preferences may reverse this effect, as discussed in Cvitanic et al. (2012), in markets with large state spaces — which much of our analysis focuses on — the market selection force is often overwhelming, see Fedyk et al. (2013). Over time, such market selection tends to allocate the economy's resources to the agents with (the most) correct beliefs, wiping out the rest of the population, and thus mitigating speculative opportunities, and the distortions and inefficiencies they induce. From this perspective, one may view the need for policy in the multi-period extension, with its self-correcting market selection mechanism, to be lower than in the one-period economy. On the flip side, a policy maker who favors an egalitarian outcome with a somewhat equal distribution of consumption across agents will view the need for policy to be even higher in the multi-period setting, because of the punishing effect market selection has on irrational agents in the long run.

Finally, we note that our previous discussion about the multi-period extension is highly stylized in having a fixed production technology over time. In practice, new, untested, technologies evolve dynamically, and are gradually introduced and replace old technologies. In such a dynamic setting, the possibilities for the policy maker to draw inferences about new technologies from historical return realizations may be limited, leading to a setting closer to the one-period economy that we have focused on in this paper. Such a setting with evolving production technologies may provide a good characterization of the creative destruction process that drives an economy's growth (see Schumpeter 1942, and also Aghion et al. 2014). From our previous analysis it is clear that for a policy maker who is not willing to overrule individual beliefs, our framework, although useful in avoiding investment distortions driven by speculation, would provide limited guidance on how to ensure that resources are allocated where they maximize productivity and ensure optimal growth.

In conclusion, we note that the fact that our approach does not rely on objective knowledge of "true" probabilities by a policy maker is of practical importance. For example, during a period of disagreement about whether new technologies and innovations have created a bubble in the market — as during our previously discussed period in the late 1990s, when some argued that there was a bubble whereas others argued that the high valuations of Internet startups and other companies were well justified — a policy maker may be justified in taking action without taking a stand on the future prospects of these new technologies and innovations.

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# A Proofs

#### Proof of Proposition 1: Existence:

We take a similar approach as Basak 2005 (exchange economy) and Baker et al. 2016 (production economy) to characterize equilibrium as a solution to a "planner's" optimization problem. Specifically, we study

$$U = \max_{\substack{I, c^1, \dots, c^N \\ N}} \sum_{n=1}^N \lambda^n U(c_0^n, \tilde{c}_1^n | q^n), \qquad \text{s.t.},$$
(21)

$$\sum_{n=1}^{N} c_0^n = 1 - I, \qquad (22)$$

$$\sum_{n=1}^{N} c_{1m}^{n} = R_{m}I, m = 1, \dots, M.$$
(23)

Here,  $\lambda \in S^N$  is a vector of positive weights (similar to the weights in a social planner's problem, although our interpretation is not that of a social planner's problem). The maximization problem is strictly concave, so a solution to the first-order conditions (FOC) is also a unique global optimum to the maximization problem. From the FOC for individual consumption, we have

$$\begin{split} \lambda^{n} \left( c_{0}^{n} \right)^{-\gamma} &= \lambda^{n'} \left( c_{0}^{n'} \right)^{-\gamma}, \\ \lambda^{n} q_{m}^{n} \left( c_{1m}^{n} \right)^{-\gamma} &= \lambda^{n'} q_{m}^{n'} \left( c_{1m}^{n'} \right)^{-\gamma}. \end{split}$$

Using (22,23), we solve for the optimal consumption share of agent n

$$c_{0}^{n} = \frac{(\lambda^{n})^{\frac{1}{\gamma}}}{\sum_{n'=1}^{N} (\lambda^{n'})^{\frac{1}{\gamma}}} (1-I),$$
  

$$c_{1m}^{n} = \frac{(\lambda^{n})^{\frac{1}{\gamma}} (q_{m}^{n})^{1/\gamma}}{\sum_{n'=1}^{N} (\lambda^{n'})^{\frac{1}{\gamma}} (q_{m'}^{n'})^{1/\gamma}} R_{m}I.$$

Rather than working with the  $\lambda$ -weights, it is convenient to define the adjusted weights  $\hat{\lambda}^n = (\lambda^n)^{\frac{1}{\gamma}}$ . In addition, we define  $a_{mn} = (q_m^n)^{\frac{1}{\gamma}}$ . We can then write the consumption shares as

$$c_0^n = \frac{\hat{\lambda}^n}{\sum_{n'}^N \hat{\lambda}^{n'}} (1 - I), \qquad (24)$$

$$c_{1m}^{n} = \frac{\lambda^{n} a_{mn}}{\sum_{n'=1}^{N} \hat{\lambda}_{m}^{n'} a_{mn'}} R_{m} I.$$
(25)

Let  $z_m = R_m^{1-\gamma}$ ,  $T_m = \sum_n \hat{\lambda}^n a_{mn}$  and  $\alpha(\hat{\lambda}) = \sum_{m=1}^M z_m T_m^{\gamma}$ . Plugging the optimal consumption

share into the maximization problem (21), we get

$$U = \max_{I} \left( \frac{\left(\sum_{n=1}^{N} \hat{\lambda}^n\right)^{\gamma}}{1-\gamma} \left(1-I\right)^{1-\gamma} + \alpha(\hat{\lambda}) \frac{1}{1-\gamma} I^{1-\gamma} \right).$$

The FOC condition for I is then

$$-\left(\sum_{n=1}^{N}\hat{\lambda}^{n}\right)^{\gamma}(1-I)^{-\gamma}+\alpha\left(\hat{\lambda}\right)I^{-\gamma}=0,$$

and solving for the optimal I gives

$$I = \frac{Z}{1+Z},\tag{26}$$

where Z is the investment-to-consumption ratio given by

$$Z = \frac{\alpha \left(\hat{\lambda}\right)^{\frac{1}{\gamma}}}{\sum_{n'=1}^{N} \hat{\lambda}^{n'}}.$$
(27)

Equations (24-27) thus describe the unique solution to (21-23).

It follows from a standard argument that each solution to (21-23) corresponds to an equilibrium (in terms of Definition 1), for some initial allocations  $K \in S^N$ . Specifically, an agent's Euler equations when solving (5,6) can be formulated using the stochastic discount factor formulation:

$$K^{n} = c_{0}^{n} + E\left[\xi^{n}\tilde{c}_{1}^{n}|q^{n}\right].$$
(28)

Here,  $\xi^n = \left(\frac{c_m^n}{c_0^n}\right)^{-\gamma}$  is agent *n*'s stochastic discount factor (which is agent-dependent because of heterogeneous beliefs, see Basak 2005).

Using the consumption of agent n described by (24-27), (28) then implies

$$\begin{split} K^{n} &= c_{0}^{n} + E\left[\tilde{\xi}^{n}\tilde{c}_{1}^{n}|q^{n}\right] \\ &= \frac{\hat{\lambda}^{n}}{\sum_{n'}^{N}\hat{\lambda}^{n'}}(1-I(\hat{\lambda})) + \sum_{m=1}^{M}q_{m}^{n}\left(\frac{\frac{\hat{\lambda}^{n}a_{mn}}{\sum_{n'=1}^{N}\hat{\lambda}^{n'}a_{mn'}}R_{m}I(\hat{\lambda})}{\frac{\hat{\lambda}^{n}}{\sum_{n'}^{N}\hat{\lambda}^{n'}}\left(1-I(\hat{\lambda})\right)}\right)^{-\gamma}\frac{\hat{\lambda}^{n}a_{mn}}{\sum_{n'=1}^{N}\hat{\lambda}^{n'}a_{mn'}}R_{m}I(\hat{\lambda}) \\ &= \frac{\hat{\lambda}^{n}}{\sum_{n'}^{N}\hat{\lambda}^{n'}}(1-I(\hat{\lambda})) + I(\hat{\lambda})\frac{\hat{\lambda}^{n}}{\alpha\left(\hat{\lambda}\right)}\sum_{m=1}^{M}z_{m}a_{mn}T_{m}^{\gamma-1} \\ &= \frac{\hat{\lambda}^{n}}{\sum_{n'}\hat{\lambda}^{n'}}(1-I(\hat{\lambda}) + I(\hat{\lambda})F^{n}(\hat{\lambda})) \\ &= Y^{n}(\hat{\lambda}), \end{split}$$

$$(29)$$

where we have defined  $F^n(\hat{\lambda}) = \frac{L}{\alpha} G^n(\hat{\lambda}), G^n(\hat{\lambda}) = \sum_m z_m a_{mn} T_m^{\gamma-1}$  and  $L = \sum_{n'} \hat{\lambda}^{n'}$ . The equilibrium state price,  $p_m$ , is then

$$p_m = E\left[\tilde{\xi}^n \delta_m | q^n\right] = \left(\frac{c_m^n}{c_0^n}\right)^{-\gamma} q_m^n, \qquad m = 1, \dots, M,$$

where  $\delta_m$  is the random variable, such that  $\delta_m(\omega_w) = 1$ , and  $\delta_m(\omega_{m'}) = 0$ ,  $m' \neq m$ , representing the payout of the *m*:th AD security, and any  $n = 1, \ldots, N$  can be chosen. We have characterized the mapping from weights to equilibria:  $\lambda \mapsto (K, p)$ .

We next show that the mapping  $\lambda \mapsto K = Y(\lambda)$  is onto, that is, that for each  $\hat{K} \in S^N$ , there exists a  $\lambda \in S^N$ , such that  $\hat{K} = \hat{Y}(\lambda)$ , where  $\hat{Y}(\lambda) = (\hat{Y}^1(\lambda), \dots, \hat{Y}^N(\lambda))' = (Y^1(\hat{\lambda}), \dots, Y^N(\hat{\lambda}))'$  is defined by (29). This then implies that any initial allocation has an associated equilibrium, as generated by a corresponding  $\lambda$ . Note that the function  $\hat{Y}$  can be defined over the closure  $\bar{S}^N$ , providing a continuous mapping  $Y : \bar{S}^N \to \bar{S}^N$ , such that  $\hat{Y}^n(\lambda) > 0$  if and only if  $\lambda^n > 0$ . For an arbitrary  $\hat{K} \in S^N$ , define the continuous mapping  $L : \bar{S}^N \to \bar{S}^N$ , by

$$(L(\lambda))_n = \frac{\lambda^n + \max(K^n - Y^n(\lambda), 0)}{1 + \sum_n \max(\hat{K}^n - \hat{Y}^n(\lambda), 0)}$$

Also, define  $C = \sum_{n} \max(\hat{K}^n - \hat{Y}^n(\lambda), 0) \ge 0$ . By Brouwer's theorem, L has a fixed point, i.e., there exists a  $\lambda_*$ , such that  $\lambda_* = L(\lambda_*)$ . We note that since  $\hat{K} \in S^N$ , that is, since  $\hat{K}$  lies in the interior of the unit simplex, it follows that  $\lambda_*$  also does,  $\lambda_* \in S^N$ . Otherwise, if  $\lambda_*^n = 0$  for some n, we would arrive at the contradiction

$$0 = \lambda_*^n = \frac{0 + \max(\hat{K}^n - 0, 0)}{1 + C} = \frac{\hat{K}^n}{1 + C} > 0.$$

Consider the fixed point,  $\lambda_*$ , and assume that  $\hat{K}^n - \hat{Y}^n(\lambda_*) \neq 0$  for some n. Since both  $\hat{K}$  and  $\hat{Y}$  lie in the unit simplex,  $\sum_n (\hat{K}^n - \hat{Y}^n(\lambda_*)) = 0$ , and thus  $\hat{K}^n - \hat{Y}^n(\lambda_*) < 0$  for some n, and  $\hat{K}^{n'} - \hat{Y}^{n'}(\lambda_*) > 0$  for some n'. It follows that C > 0, and also that

$$\lambda_*^n = \frac{\lambda_*^n + 0}{1 + C} < \lambda_*^n,$$

which again leads to a contradiction, since  $\lambda_*^n > 0$ . Thus, it must be that  $\hat{K}^n = \hat{Y}^n(\lambda_*)$  for all n, i.e.,  $\hat{K}$  is the equilibrium allocation associated with the optimization problem (21-23) under weights  $\lambda_*$ . Since  $\hat{K} \in S^N$  was arbitrary, we have shown that the mapping  $\hat{Y}$  is onto.

We also show that the inverse mapping,  $(K, p) \mapsto \lambda$  can be derived from the individual agents' Euler conditions, i.e., that for each equilibrium outcome there is an associated optimization problem (21-23), a fact we use in Section 2.3. Specifically, consider the agent's optimization problem (5-6) associated with an equilibrium. From the FOC we have

$$(c_0^n)^{-\gamma} = \kappa^n \tag{30}$$

$$q_m^n \left(c_m^n\right)^{-\gamma} = \kappa^n p_m \tag{31}$$

where  $\kappa^n$  is the Lagrange multiplier of the optimization problem for agent *n*. Using the above we can immediately identify  $q_m^n \left(\frac{c_m^n}{c_0^n}\right)^{-\gamma} = p_m$ . Moreover, defining  $\lambda^n = \frac{1}{\kappa^n}$  we have

$$\begin{split} \lambda^n \left( c_0^n \right)^{-\gamma} &=& \lambda^{n'} \left( c_0^{n'} \right)^{-\gamma}, \\ \lambda^n q_m^n \left( c_{1m}^n \right)^{-\gamma} &=& \lambda^{n'} q_m^{n'} \left( c_{1m}^{n'} \right)^{-\gamma}. \end{split}$$

which is the same as for the maximization problem in (21-23). Hence we arrive the same consumption shares in the two problems (21-23) and (5-6). Next, using the budget conditions in (6) we get

$$\sum_{n=1}^{N} K^{n} = \sum_{n=1}^{N} c_{0}^{n} + \sum_{n=1}^{N} \sum_{m=1}^{M} p_{m} c_{m}^{n}$$

$$1 = 1 - I + \sum_{n=1}^{N} \sum_{m=1}^{M} p_{m} c_{m}^{n}$$

$$I = \sum_{m=1}^{M} p_{m} R_{m} I$$

$$1 = \sum_{m=1}^{M} \left( \frac{\sum_{n=1}^{N} \hat{\lambda}^{n} (q_{m}^{n})^{\frac{1}{\gamma}}}{\sum_{n=1}^{N} \hat{\lambda}^{n}} \right)^{\gamma} R_{m}^{1 - \gamma} Z^{-\gamma}$$

$$Z = \frac{\alpha \left( \hat{\lambda} \right)^{\frac{1}{\gamma}}}{\sum_{n'=1}^{N} \hat{\lambda}^{n'}}$$
(32)

where (32) is the same as in (27). Therefore, the aggregate investment in problem (21-23) with weight  $\lambda$  is the same as the equilibrium investment, completing the mapping from the equilibrium outcome (K, p) to the optimization problem with weight  $\lambda$ .

Uniqueness: The argument so far does not imply uniqueness. Specifically, there may potentially be multiple equilibrium price vectors consistent with an initial allocation, K, leading to different equilibria  $(K, p^1)$  and  $(K, p^2)$ ,  $p^1 \neq p^2$ . These equilibria would necessarily correspond to different weights  $\lambda^1 \neq \lambda^2$  in the corresponding optimization problems (21-23), because of the (unique) mapping  $\lambda \mapsto (K, p)$  we previously derived. So, by showing that an initial allocation K leads to a unique  $\lambda$ , i.e., that there is a bijection,  $\lambda \leftrightarrow K$ , uniqueness of equilibrium follows.

We first study the case  $\gamma \geq 1$ . Define

$$G^{nj} = \sum_{m} z_{m} a_{mn} a_{mj} T_{m}^{\gamma-2},$$
  
$$F^{nj} = \frac{L^{2}}{\alpha} G^{nj}.$$

The mapping from weights to capital,  $K = Y(\hat{\lambda})$ , is given in Equation (29). The partial derivatives

are

$$\frac{\partial I}{\partial \hat{\lambda}^{j}} = \frac{1}{\sum_{n'} \hat{\lambda}^{n'}} I(1-I)(F^{j}-1),$$
  
$$\frac{\partial F}{\partial \hat{\lambda}^{j}} = \frac{1}{\sum_{n'} \hat{\lambda}^{n'}} \left(F^{n} + (\gamma-1)F^{nj} - \gamma F^{n}F^{j}\right)$$

We define the matrix Q, with elements

$$[Q]_{nj} = -(1-I)^2 - I(1-I)(F^n + F^j) - (\gamma - 1)IF^{nj} - (\gamma + I - 1)IF^nF^j, \qquad 1 \le n, j \le N,$$

and then get

$$\frac{\partial Y^n}{\partial \hat{\lambda}^j} = \frac{\hat{\lambda}^n}{L^2} Q_{nj}, \qquad j \neq n.$$

Moreover, we get

$$\frac{\partial Y^n}{\partial \hat{\lambda}^n} = \frac{1}{L}(1 - I + IF^n) + \frac{\hat{\lambda}^n}{L^2}Q_{nn}.$$

We next do a change of coordinate transformation to y, where  $y_n = \log(\hat{\lambda}^n)$ , i.e.,  $\hat{\lambda}^n = e^{y_n}$ , i.e., we write

$$K^{n} = \hat{Y}(y) = \frac{e^{y_{n}}}{L} (1 - I(e^{y}) + I(e^{y})F^{n}(e^{y})).$$
(33)

It follows from the chain rule that partial derivatives w.r.t. y will be similar as w.r.t.  $\hat{\lambda}$ , but with extra  $e^{y_j}$  inner derivative terms:

$$\begin{aligned} \frac{\partial \hat{Y}^n}{\partial y_j} &= \frac{e^{y_n} e^{y_j}}{L^2} Q_{nj}, \qquad j \neq n, \\ \frac{\partial \hat{Y}^n}{\partial y_n} &= \frac{e^{y_n}}{L} (1 - I + IF^n) + \frac{e^{y_n} e^{y_n}}{L^2} Q_{nn}. \end{aligned}$$

Now we rewrite this on matrix form, defining the matrix  $H(e^y)$ , with elements  $[H]_{nj} = \frac{\partial \hat{Y}^n}{\partial y_j}$ , to get:

$$H(\hat{\lambda}) = \frac{1}{L^2} \Lambda_{\hat{\lambda}} Q \Lambda_{\hat{\lambda}} + \frac{1}{L} \Lambda_{\hat{\lambda}}^{1/2} \Lambda_w \Lambda_{\hat{\lambda}}^{1/2}.$$
 (34)

Here, for a general vector, x, we use the notation  $\Lambda_v = diag(x)$ , and we define the vector

$$w = (1 - I)\mathbf{1} + IF,$$

treating F as a vector. Moreover,

$$Q = -(1-I)^{2}\mathbf{11'} - I(1-I)(F\mathbf{1'} + \mathbf{1}F') + (\gamma - 1)IR - (\gamma + I - 1)IFF'$$
  
$$= -((1-I)^{2}\mathbf{11'} + I(1-I)(F\mathbf{1'} + \mathbf{1}F') + I^{2}F') + (\gamma - 1)I(R - FF')$$
  
$$= -ww' + (\gamma - 1)I(R - FF'), \qquad (35)$$

so we have

$$H(\hat{\lambda}) = \underbrace{\frac{1}{L}\Lambda_{\hat{\lambda}}^{1/2}\Lambda_{w}\Lambda_{\hat{\lambda}}^{1/2} - \frac{1}{L^{2}}\Lambda_{\hat{\lambda}}(ww')\Lambda_{\hat{\lambda}}}_{C} + (\gamma - 1)I\frac{1}{L^{2}}\Lambda_{\hat{\lambda}}(R - FF')\Lambda_{\hat{\lambda}}.$$
(36)

Here, the matrix R is the matrix with elements  $[R]_{nj} = F^{nj}$ , and we note that by defining the matrix  $a \in \mathbb{R}^{M \times N}_+$  with elements  $a_{mn}$ , it can be written on self adjoint matrix form as  $R = \frac{L^2}{\alpha} a' \Lambda_v a$ , where  $v_m = T_m^{\gamma-2} z_m$ . Thus, R is positive semidefinite. Note that homogeneity of Y implies that

$$H(\hat{\lambda})\mathbf{1} = \mathbf{0}.\tag{37}$$

We have  $F = \frac{L}{\alpha}a'w$ , where  $w_m = T^{\gamma-1}z_m$ , and thus  $FF' = \frac{L^2}{\alpha^2}a'ww'a$ , leading to

$$R - FF' = \frac{L^2}{\alpha} a' \left( \Lambda_v - \frac{1}{\alpha} w w' \right) a$$
  
=  $\frac{L^2}{\alpha} a' \Lambda_v^{1/2} \left( E - \frac{1}{\alpha} \Lambda_v^{-1/2} w w' \Lambda_v^{-1/2} \right) \Lambda_v^{1/2} a,$ 

Note that  $\Lambda_v^{-1/2}w = (T_1^{\gamma/2}z_1^{1/2}, \ldots, T_M^{1/2}z_m^{1/2})$ . Also, recall that the eigenvalues of the 1-rank perturbation of the identity matrix E - xx' are 1 with multiplicity N - 1, and 1 - (x'x) with multiplicity 1, where x is the vector with eigenvalue 1 - (x'x). Now,  $(\Lambda_v^{-1/2}w)'(\Lambda_v^{-1/2}w) = \sum_m (T_m^{\gamma/2}z_m^{1/2})^2 = \alpha$ , and thus the eigenvalues of  $E - \frac{1}{\alpha}\Lambda_v^{-1/2}ww'\Lambda_v^{-1/2}$  are 1 with multiplicity N - 1, and 0 with multiplicity 1. So, Sylvester's law of inertia (on general rectangular form, see Higham and Cheng (1998)), implies that R - FF' is also positive semidefinite, with number of 0 eigenvalues dependent on N, M, and the rank of a.

It is easy to verify that  $R\Lambda_{\hat{\lambda}}\mathbf{1} = R\hat{\lambda} = LF$ , which in turn implies that the generic zero eigenvalue is generated by  $\mathbf{1}$ , since  $FF'\Lambda_{\hat{\lambda}}\mathbf{1} = FF'\hat{\lambda} = FL$ , so

$$(R - FF')\Lambda_{\hat{\lambda}}\mathbf{1} = \mathbf{0}.$$

Next, consider the remaining term

$$C = \frac{1}{L} \Lambda_{\hat{\lambda}}^{1/2} \Lambda_w \Lambda_{\hat{\lambda}}^{1/2} - \frac{1}{L^2} \Lambda_{\hat{\lambda}} w w' \Lambda_{\hat{\lambda}}$$
$$= \frac{1}{\sqrt{L}} \Lambda_{\hat{\lambda}}^{1/2} \Lambda_w^{1/2} \left( E - gg' \right) \Lambda_w^{1/2} \Lambda_{\hat{\lambda}}^{1/2} \frac{1}{\sqrt{L}}$$

where

$$g_n = \sqrt{\frac{\hat{\lambda}_n w_n}{L}}.$$

It immediately follows that  $g'g = \frac{1}{L} \sum_n \hat{\lambda}_n((1-I) + IF_n) = 1$ , and therefore a similar argument as for R - FF' implies that there is one eigenvalue of C which is 0, and all the other eigenvalues are 1. It is also easy to check that **1** is the eigenvector that corresponds to the eigenvalue 0.

Thus, altogether, since q'(A+B)q = q'Aq + q'Bq for general matrices A and B, it follows that H is positive semidefinite, with exactly one zero eigenvalue and the corresponding eigenvector **1**, for all  $\hat{\lambda}$ .

Now, assume that  $Y(\hat{\lambda}_1) = Y(\hat{\lambda}_2)$  for distinct  $\hat{\lambda}_1$  and  $\hat{\lambda}_2$  (i.e., such that it is not the case that  $\hat{\lambda}_2$  is proportional to  $\hat{\lambda}_1$ ). Define  $y_1 = \log(\hat{\lambda}_1)$  and  $y_2 = \log(\hat{\lambda}_2)$ . It then follows that  $\hat{Y}(y_2) - \hat{Y}(y_1) = \mathbf{0}$ , and thus that  $(y_2 - y_1)'(\hat{Y}(y_2) - \hat{Y}(y_1)) = 0$ . Define  $\Delta y = y_2 - y_1$ , to get

$$\hat{Y}(y_2) - \hat{Y}(y_1) = \int_{s=0}^{s=1} H(e^{y_1 + s\Delta y})(\Delta y) ds = 0,$$

and therefore

$$\int_{s=0}^{s=1} (\Delta y)' H(e^{y_1 + s\Delta y})(\Delta y) ds = 0.$$

However, since H is positive semidefinite for all y, with eigenvector **1**, this is only possible if  $y_2 - y_1 = c\mathbf{1}$ , for some constant c, i.e., only in the proportional case,  $\frac{\hat{\lambda}_2^n}{\hat{\lambda}_1^1} = e^c$  for all n.

For  $\gamma < 1$ , it follows immediately from the definition of Q that as long as  $\gamma - 1 + I > 0$ ,  $\partial Y^n / \partial \hat{\lambda}^j < 0$ , which means that the Gross Substitution property holds, and thereby that equilibrium is unique. Since I > 0, there is thus always a  $\gamma < 1$ , such that this property holds for all  $\gamma > \gamma$ . This completes the second part of the Proposition.

Proof of Proposition 2: See derivation in the proof of Proposition 1.

*Proof of Proposition 3*: See derivation in the proof of Proposition 1.

*Proof of Proposition* 4: In an economy with homogeneous beliefs q, it follows from the first welfare theorem that any equilibrium allocation is efficient.

In the heterogeneous beliefs economy,  $q_m^n > q_m^{n'}$  and  $q_{m'}^n < q_{m'}^{n'}$  for some n, n', m, m', and the equilibrium condition:

$$\lambda^{n} q_{m}^{n} (c_{1m}^{n})^{-\gamma} = \lambda^{n'} q_{m}^{n'} (c_{1m}^{n'})^{-\gamma}, \qquad m = 1, \dots, M$$

implies that consumption shares are not constant across states, and hence the allocations are not efficient.

# Online Appendix: "Distortions and Efficiency in Production Economies with Heterogeneous Beliefs"

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# Proofs omitted from the paper

*Proof of Proposition 5*: To prove (i), note that if there at least two agents that disagree about the optimal investment level, then it follows that there is no investment level that is at least as good for all reasonable q and hence there are no belief-neutral efficient allocations. Parts (ii) and (iii) follow directly from Propositions 6 and 7 in Heyerdahl-Larsen and Walden (2018).

Proof of Proposition 6: Efficient allocations satisfy the FOC of the social planner problem with the same  $q \in Q_R$  for all agents, leading to consumption shares that are constant across states. Hence, any economy with speculative distortions — in which consumption shares are not constant across agents — will be inefficient. Since aggregate consumption is exogenous in an exchange economy, this is the only source of inefficiency, and absence of speculative distortions is a sufficient condition for an efficient allocation.

In the production economy, the social planner's problem needs to satisfy the same FOC as in the exchange economy, so again speculative distortions lead to inefficient allocations. Moreover, the FOC for aggregate consumption across dates for some  $q \in Q_R$  needs to be satisfied. If there are investment distortions, an allocation is therefore inefficient.

The FOC w.r.t. consumption shares and aggregate investments in the production economy jointly determine efficiency of an allocation, so absence of speculative distortions and investment distortions are also jointly sufficient conditions for an allocation to be efficient.

Proof of Proposition 7: Define  $\bar{\lambda}^n = \frac{\hat{\lambda}^n}{\sum_{n'=1}^N \hat{\lambda}^{n'}}$ . The equilibrium investment-to-consumption ratio, Z, is given by

$$Z = \left(\sum_{m=1}^{M} R_m^{1-\gamma} \left(\sum_{n=1}^{N} \bar{\lambda}^n \left(q_m^n\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)^{\frac{1}{\gamma}}.$$
(38)

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By Jensen's inequality we have that

$$Z > \left(\sum_{m=1}^{M} R_m^{1-\gamma} \left(\sum_{n=1}^{N} \bar{\lambda}^n \left(q_m^n\right)\right)\right)^{\frac{1}{\gamma}}, \quad \gamma < 1$$
$$Z < \left(\sum_{m=1}^{M} R_m^{1-\gamma} \left(\sum_{n=1}^{N} \bar{\lambda}^n \left(q_m^n\right)\right)\right)^{\frac{1}{\gamma}}, \quad \gamma > 1.$$

As agents agree on aggregate risk we have that  $\left(\sum_{m=1}^{M} R_m^{1-\gamma}\left(\sum_{n=1}^{N} \bar{\lambda}^n\left(q_m^n\right)\right)\right)^{\frac{1}{\gamma}} = \left(E[\tilde{R}^{1-\gamma}|q^n]\right)^{\frac{1}{\gamma}}$ for all n = 1, .., N.

*Proof of Proposition 8*: First, note that the derivative of Z with respect to agent n's belief about state m is

$$\frac{\partial Z}{\partial q_m^n} = R^{1-\gamma} Z^{1-\gamma} \bar{\lambda}^n \hat{q}_m^{1-\frac{1}{\gamma}} \left( q_m^n \right)^{\frac{1}{\gamma}-1}.$$

Now consider the effect on Z by increasing the belief of agent  $n \in \Gamma_{m,m'}$  about state m by a small amount,  $\Delta$ , and decreasing the agent's belief about state m' by  $\Delta$ . The total derivative is

$$F = \left(\frac{\partial Z}{\partial q_m^n} - \frac{\partial Z}{\partial q_{m'}^n}\right) \Delta = R^{1-\gamma} Z^{1-\gamma} \bar{\lambda}^n \hat{q}_{m'}^{1-\frac{1}{\gamma}} (q_{m'}^n)^{\frac{1}{\gamma}-1} \left( \left(\frac{\hat{q}_{m'}}{\hat{q}_m} \frac{q_m^n}{q_{m'}^n}\right)^{\frac{1}{\gamma}-1} - 1 \right).$$
(39)

The sign of equation (39) is determined by the sign of  $(q_{m'}^n)^{\frac{1}{\gamma}-1} \left( \left( \frac{\hat{q}_{m'}}{\hat{q}_m} \frac{q_m^n}{q_{m'}^n} \right)^{\frac{1}{\gamma}-1} - 1 \right)$ . We have that  $\frac{\hat{q}_{m'}}{\hat{q}_m} \frac{q_m^n}{q_{m'}^n} > 1$  if the likelihood ratio of agent *n* between state *m* and *m'* constitutes a disagreement increasing spread. For  $\gamma > 1$  ( $\gamma < 1$ ) we have that F < 0 (F > 0) and consequently investment distortions are increasing for a disagreement increasing spread. An identical argument can be made for a whole set of agents in  $\Gamma_{m,m'}$ , leading to the proposition.

Proof of Proposition 9: Let  $f^P = f_0^P$  and  $f^R = f_0^R$  denote the consumption share of a poor and rich agent respectively. We have that  $Sf_0^P + (N - S)f_0^R = N(xf_0^P + (1 - x)f_0^R) = 1$ . In this economy it follows that the investment-consumption ratio, Z, is

$$Z = \left(S\hat{Q}^P + (N-S)\hat{Q^R}\right)^{\frac{1}{\gamma}},$$

where

$$\hat{Q}^i = \left(p_1 + f^i \Delta p\right)^{\gamma}$$

for  $i \in \{P, R\}$  and  $p_1 = \frac{1-\Delta}{N}$  and  $\Delta p = p_2^{\frac{1}{\gamma}} - p_1^{\frac{1}{\gamma}}$  and  $p_2 = \frac{1}{N} + \frac{N-1}{N}\Delta$ . We rank the consumption inequality by S and  $f^P$ . Let  $S_2 \ge S$  and  $S_2 f_2^P \le S f^P$  with at least one

inequality being strict. First consider changing only the consumption share  $f^P$  to  $f_2^P$  with  $f_2^P < f^P$ and  $S_2 = S$ . Note that we have from the clearing in the commodity market

$$\frac{\partial f^R}{\partial f^P} = -\frac{S}{N-S} = -\frac{x}{1-x}, \qquad \text{where } x = S/N.$$

Next, consider the derivative of Z w.r.t.  $f^P$ 

$$\frac{\partial Z}{\partial f^P} = Z^{1-\gamma} \Delta p S\left(\left(\hat{Q}^P\right)^{1-\frac{1}{\gamma}} - \left(\hat{Q}^R\right)^{1-\frac{1}{\gamma}}\right),$$

and since  $\hat{Q}^P < \hat{Q}^R$  this is negative for  $\gamma > 1$  and positive for  $\gamma < 1$ . Consequently, the distortions are decreasing when the poor becomes poorer as there is over-consumption when  $\gamma > 1$  and over-investment when  $\gamma < 1$ .

Next, consider the case when  $S_2 > S_1$  and  $f_2^P S_2 = f^P S = A$ , i.e., the first inequality is strict. We have the following derivatives

$$\begin{array}{lll} \displaystyle \frac{\partial f^P}{\partial S} & = & \displaystyle -\frac{A}{S^2} \\ \displaystyle \frac{\partial f^R}{\partial S} & = & \displaystyle \frac{1-A}{\left(1-S\right)^2} \end{array}$$

Next, consider the derivative of Z

$$\begin{split} \frac{\partial Z}{\partial S} &= Z^{1-\gamma} \left( \frac{1}{\gamma} \left( \hat{Q}^P - \hat{Q}^R \right) + \Delta p \left( S \left( \hat{Q}^P \right)^{1-\frac{1}{\gamma}} \frac{-A}{S^2} + (N-S) \left( \hat{Q}^R \right)^{1-\frac{1}{\gamma}} \frac{1-A}{(1-S)^2} \right) \right) \\ &= Z^{1-\gamma} \left( \frac{1}{\gamma} \left( \hat{Q}^P - \hat{Q}^R \right) + \Delta p \left( \left( \hat{Q}^R \right)^{1-\frac{1}{\gamma}} f^R - \left( \hat{Q}^P \right)^{1-\frac{1}{\gamma}} f^P \right) \right) \\ &= h \left( f^P \right) - h \left( f^R \right), \end{split}$$

where

$$h(x) = \frac{1}{\gamma} \left( p_1^{\frac{1}{\gamma}} + x\Delta p \right)^{\gamma} + \left( p_1^{\frac{1}{\gamma}} + x\Delta p \right)^{\gamma-1} x\Delta p.$$

We have

$$h'(x) = (\gamma - 1) \left( p_1^{\frac{1}{\gamma}} + x \Delta p \right)^{\gamma - 2},$$

and therefor h'(x) > 0 for  $\gamma > 1$  and h'(x) < 0 for  $\gamma < 1$ . Since  $f^P < f^R$ , we have that  $\frac{\partial Z}{\partial S} < 0$  for  $\gamma > 1$  and  $\frac{\partial Z}{\partial S} > 0$  for  $\gamma < 1$ . Following the same argument as for the derivative of Z w.r.t.  $f^P$ , savings distortions are decreasing in S. Combining the two results, we have that distortions are decreasing in consumption inequality, as measured by  $f^P$  and S.

Proof of Proposition 10: Assume that 1 < S < N of the large markets are closed, and define x = $\frac{S}{N} \in (0,1)$ . An agent who is optimistic about a market that is open is called an *o*-agent. There are N-S such agents. There are also S, in total, s-agents, who are optimistic about markets that are shut down. The prices of AD securities in open markets are all the same, P, because of symmetry.

An s-agent chooses investments,  $I^s$  and demand  $d^s$  to maximize:

$$U^{s} = \frac{\left(\frac{1}{N} - I^{s} - (N - S)d^{s}P\right)^{1 - \gamma}}{1 - \gamma} + \frac{(I^{s})^{1 - \gamma}}{1 - \gamma}Q^{s} + \frac{(I^{s} + \hat{d}^{s})^{1 - \gamma}}{1 - \gamma}(1 - Q^{s}),$$

where  $Q^s = (S-1)(1-\Delta)/N + 1/N + \Delta(N-1)/N$ , leading to the FOCs:

$$(1 - \kappa^{s} - (N - S)\hat{d}^{s})^{-\gamma} = \frac{P\frac{\Delta}{1 - \Delta}N + S}{1 - P(N - S)}$$
$$P(1 - \kappa^{s} - (N - S)\hat{d}^{s})^{-\gamma} = (1 + \hat{d}^{s})^{-\gamma}.$$

Here,  $\hat{d}^s = d^s/I^s$ , and  $\kappa^1 = \frac{K^n}{I^s} = \frac{1}{NI^s}$ . From (40,40) we get

$$(1+\hat{d}^{s})^{-\gamma} = \frac{P(\Delta N + (1-\Delta)S)}{(1-\Delta)(1-P(N-S))}$$
  

$$\kappa^{s} = 1+\hat{d}^{s}(N-S)P + (\alpha^{s})^{-1/\gamma},$$

where

$$\alpha^s = \frac{\Delta N + (1 - \Delta)S}{N(1 - P(N - S))}$$

Using a similar argument for o-agents, we define their demand in the open market they are optimistic about,  $d^m$ , and in the other N - S - 1 open markets,  $d^o$ , as well as their investments  $I^o$ , and corresponding normalized variables  $\hat{d}^m = d^m/I^o$ ,  $\hat{d}^o = d^o/I^o$ , and  $\kappa^o = \frac{1}{NI^o}$ .

The first order conditions then imply:

$$(1 + \hat{d}^{o})^{-\gamma} = \frac{PS}{1 - P(N - S)},$$
  

$$(1 + \hat{d}^{m})^{-\gamma} = \frac{P(1 - \Delta)S}{(1 + \Delta(N - 1))(1 - P(N - S))},$$
  

$$\kappa^{0} = 1 + \hat{d}^{m}P + (N - S - 1)\hat{d}^{o}P + \alpha_{2}^{-1/\gamma},$$

where

$$\alpha_2 = \frac{(1-\Delta)S}{N(1-P(N-S))}$$

By inspection one sees that

$$(1+\hat{d}^m)^{-\gamma} < (1+\hat{d}^o)^{-\gamma} < (1+\hat{d}^s)^{-\gamma},$$

and thus

$$\hat{d}^m > \hat{d}^o > \hat{d}^s.$$

The excess demand for an AD security is:

$$e = S\frac{\hat{d}^{s}}{\kappa^{s}} + \frac{\hat{d}^{m} + (N - S - 1)\hat{d}^{o}}{\kappa^{o}} = 0,$$

and the market clearing condition is then

$$e = 0.$$

The form off the excess demand function follows from noting that in each open market, the number of s-agents is S, the number of pessimistic o-agents is N - S - 1, and there is exactly one optimistic o-agent in the market.

We study the excess demand function for the price P = 1/N (which is the equilibrium price in the complete market), which leads to

$$(1+\hat{d}^{s})^{-\gamma} = 1 + \frac{\Delta}{1-\Delta} \frac{N}{S} > 1$$

$$(1+\hat{d}^{o})^{-\gamma} = 1$$

$$(1+\hat{d}^{m})^{-\gamma} = \frac{1}{1+N\frac{\Delta}{1-\Delta}} < 1$$

$$\alpha_{1} = 1 - \Delta + \Delta \frac{N}{S}$$

$$\alpha_{2} = 1 - \Delta,$$

$$\kappa_{1} = 1 + \hat{d}^{s} (1-x) + \alpha_{1}^{-1/\gamma},$$

$$\kappa_{2} = 1 + \hat{d}^{m}P + \alpha_{2}^{-1/\gamma},$$

so that  $\hat{d}^s < 0 = \hat{d}^o < \hat{d}^m$ , and

$$e = S \frac{\hat{d}^{s}}{1 + \hat{d}^{s} \left(1 - \frac{S}{N}\right) + \alpha_{1}^{-1/\gamma}} + \frac{\hat{d}^{m}}{1 + \hat{d}^{m} \frac{1}{N} + \alpha_{2}^{-1/\gamma}}$$
$$= S \frac{1}{\frac{1 + \alpha_{1}^{-1/\gamma}}{\hat{d}^{s}} + \left(1 - \frac{S}{N}\right)} + \frac{1}{\frac{1 + \alpha_{2}^{-1/\gamma}}{\hat{d}^{m}} + \frac{1}{N}}$$

For large N, with x = S/N fixed, we rewrite this as :

$$e = x \frac{1}{\frac{1+(1-\Delta+\Delta/x)^{-1/\gamma}}{(1+\Delta/((1-\Delta)x))^{-1/\gamma}-1} + (1-x)} + \frac{1}{N} \frac{1}{\frac{1+(1-\Delta)^{-1/\gamma}}{\left(\frac{1}{1+N\frac{\Delta}{1-\Delta}}\right)^{-1/\gamma}} + \frac{1}{N}}$$

The first term is negative and independent of N, whereas the second term tends to zero for large N at the rate  $1/N^{1-1/\gamma}$ . Thus, e < 0 for large N. As a consequence, in equilibrium  $P < \frac{1}{N}$ , and  $\hat{d}^o > 0$ .

Define the normalized price,  $\hat{P}(x) = \frac{P}{N}$ . For large N, the above FOC, described as functions of x = S/N, become:

$$(1 + \hat{d}^{s}(x))^{-\gamma} = \frac{\hat{P}(x)(\Delta + (1 - \Delta)x)}{(1 - \Delta)(1 - \hat{P}(x)(1 - x))} (1 + \hat{d}^{o}(x))^{-\gamma} = \frac{\hat{P}(x)x}{1 - \hat{P}(x)(1 - x)}, (1 + \hat{d}^{m}(x))^{-\gamma} = \frac{\hat{P}(x)(1 - \Delta)x}{(1 + \Delta(1 - x))(1 - \hat{P}(x)(1 - x))} \kappa_{1}(x) = 1 + \hat{d}(x)(1 - x)\hat{P}(x) + \alpha_{1}(x)^{-1/\gamma}, \kappa_{2}(x) = 1 + d^{m}(x)\hat{P}/N + (1 - x - 1/N)\hat{d}^{o}(x)\hat{P}(x) + \alpha_{2}(x)^{-1/\gamma}, \alpha_{1}(x) = \frac{\Delta + (1 - \Delta)x}{1 - \hat{P}(x)(1 - x)} \alpha_{2}(x) = \frac{(1 - \Delta)x}{1 - \hat{P}(x)(1 - x)}.$$

and the market clearing condition becomes

$$e = x\frac{\hat{d}^s}{\kappa_1} + \frac{\hat{d}^m/N + (1 - x - 1/N)\hat{d}^o}{\kappa_2} = 0.$$

For large N, the  $\hat{d}^m/N$  and  $-\hat{d}^o/N$  terms become arbitrarily small, leading to the large economy market clearing condition

$$e = x\frac{\hat{d}^s}{\kappa^s} + (1-x)\frac{\hat{d}^o}{\kappa^o} = 0.$$
 (40)

One verifies that when  $\Delta = 0$ , the equilibrium price is the same as the complete market price regardless of the fraction of closed markets,  $\hat{P}(x) \equiv 1$ . A Taylor expansion of (40) around  $\Delta = 0$  and  $\hat{P} = 1$ , and application of the implicit function theorem for the differential

$$\frac{de}{d\Delta} = \frac{\partial e}{\partial \Delta} + \frac{\partial e}{\partial \hat{P}} \frac{d\hat{P}}{d\Delta} = 0$$

leads to

$$\frac{d\hat{P}}{d\Delta} = -x\frac{2\gamma}{2\gamma + \Delta(1-3x)},$$

and thus  $\hat{P} = 1 - \Delta x \frac{2\gamma}{2\gamma + \Delta(1-3x)} + O(\Delta^2)$  for moderate  $\Delta$ .

Total investments are

$$I = SI^{s} + (N - S)I^{o} = x\frac{1}{\kappa^{s}} + (1 - x)\frac{1}{\kappa^{o}},$$

which when plugging in the formulas for  $\kappa^s$ ,  $\kappa^o$ , and  $\hat{P}$  leads to

$$I=\frac{1}{2}+\frac{x-1}{4\gamma}\Delta+O(\Delta^2),$$

It follows that I is increasing in x, and equals the undistorted value  $I = \frac{1}{2}$  when x = 1. Thus, distortions are higher the higher x is, i.e., the higher the fraction of closed markets. We are done.

Proof of Proposition 11: The Arrow-Debreu price,  $p_m$ , of state m is  $p_m = q_m R_m^{-\gamma} Z^{-\gamma}$ . Since  $q_m^n = q_m$  for all n and  $R_m$  is exogenous, if follows that mispricing of this asset only occurs if investments are distorted. Moreover, it follows that Z is too high (low) if investments are too high (low) and hence the price,  $p_m$  is too low (high) as it is inversely related to Z.

Proof of Proposition 12: To prove (i) note that the value of the total investment at time 0 is I by no-arbitrage, i.e., the price is  $P_0 = I$ . The value at time 1 in state m is  $P_{1,m} = R_m I$  and therefore the expected return as perceived by agent n is  $E^n(R_m) = \sum_{m=1}^M q_m^n R_m$ . Note that the expected return as perceived by agent n in the economy with heterogeneous beliefs is the same as the expected return in the homogeneous agent economy with belief  $q^n$ . Hence, the expected return on the market cannot be distorted in the production economy.

To prove *(ii)*, note that we assume that idiosyncratic risk is independent of aggregate risk and all agents agree on this. The risk-free rate is distorted if and only if the bond price,  $B_0$ , is distorted. Consider the bond price:

$$B_{0} = Z^{-\gamma} \sum_{m=1}^{M} \hat{q}_{m} R_{m}^{-\gamma}$$

$$= Z^{-\gamma} \sum_{m \in \mathcal{A}} q_{m} R_{m}^{-\gamma} \sum_{m \in \mathcal{I}} \hat{q}_{m}$$

$$= \frac{1}{\left(\sum_{m \in \mathcal{A}} q_{m} R_{m}^{1-\gamma}\right) \left(\sum_{m \in \mathcal{I}} \hat{q}_{m}\right)} \left(\sum_{m \in \mathcal{A}} q_{m} R_{m}^{-\gamma}\right) \left(\sum_{m \in \mathcal{I}} \hat{q}_{m}\right)$$

$$= \frac{\sum_{m \in \mathcal{A}} q_{m} R_{m}^{-\gamma}}{\sum_{m \in \mathcal{A}} q_{m} R_{m}^{1-\gamma}}$$

$$= B_{0}^{U}$$

where  $B_0^U$  is the bond price in the homogeneous agent economy, and where  $m \in \mathcal{A}$  denotes the summation over the aggregate states and  $m \in \mathcal{I}$  is the summation over the idiosyncratic states. The decomposition is possible due to the independence assumption. Note that since there is only disagreement about the idiosyncratic state, the set of bond prices in the homogeneous agent economies is a single value,  $B_0^U$ . Hence, the risk-free rate is not distorted in the production economy when agents agree on aggregate risk.

Proof of Proposition 13: To prove (i) note that the Arrow-Debreu price,  $p_m$ , of state m in the exchange economy is is  $p_m = q_m \left(\frac{C_m}{C_0}\right)^{-\gamma}$  for a state in which agents agree on the probability. Since  $q_m^n = q_m$  for all n and  $\left(\frac{C_m}{C_0}\right)$  is exogenous, the Arrow-Debreu price of a state which there is no disagreement about cannot be mispriced in the exchange economy.

(*ii*) is proved by counter-example. Consider an economy with M = 4 states and N = 2 agents. Define  $G_m = \begin{pmatrix} C_m \\ C_0 \end{pmatrix}$  and assume that R = (0.9, 1, 1.05, 1.1). Assume that agent 1 and 2 have beliefs  $q^1 = (0.1, 0.4, 0.4, 0.1)$  and  $q^1 = (0.1150.30.540.045)$ , respectively. It can be shown that  $E^1(G) = E^2(G)$  and  $E^1(G^{1-\gamma}) = E^2(G^{1-\gamma})$ , i.e, they agree on the expected aggregate consumption growth and the  $1 - \gamma$  uncentered moment of the consumption growth. It follows that the expected return in the homogeneous agent economies are the same

$$E^{1}(R) = \frac{E^{1}(C_{1})}{C_{0}^{\gamma}E^{1}(G^{1-\gamma})} = \frac{E^{2}(C_{1})}{C_{0}^{\gamma}E^{2}(G^{1-\gamma})} = E^{2}(R) = 1.0377$$

The expected return in the disagreement economy is

$$E^{1}(R) = \frac{E^{1}(C_{1})}{C_{0}^{\gamma} \sum_{m=1}^{4} \hat{q}_{m} G_{m}^{1-\gamma}} = \frac{E^{2}(C_{1})}{C_{0}^{\gamma} \sum_{m=1}^{4} \hat{q}_{m} G_{m}^{1-\gamma}} = 1.045,$$

and hence the aggregate market is mispriced.

To prove *(iii)*, note that we assume that idiosyncratic risk is independent of aggregate risk and all agents agree on this. The risk-free rate is distorted if and only if the bond price,  $B_0$ , is distorted. Consider the bond price:

$$B_0 = \sum_{m=1}^{M} \hat{q}_m G_m^{-\gamma}$$
$$= \sum_{m \in \mathcal{A}} q_m G_m^{-\gamma} \sum_{m \in \mathcal{I}} \hat{q}_m$$
$$= B_0^U \sum_{m \in \mathcal{I}} \hat{q}_m$$

where  $B_0^U$  is the bond price in the homogeneous agent economy and where  $m \in \mathcal{A}$  denotes the summation over the aggregate states and  $m \in \mathcal{I}$  is the summation over the idiosyncratic states. It

follows by Jensen's inequality (as we assume  $\gamma \neq 1$ ) that  $\sum_{m \in \mathcal{I}} \hat{q}_m \neq 1$  and hence the risk-free rate is always distorted in the disagreement exchange economy when there is agreement about aggregate risk.

# References

Heyerdahl-Larsen, C., and J. Walden. 2018. Welfare in Economies with Production and Heterogeneous Beliefs. Working Paper, UC Berkeley.